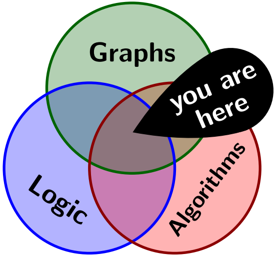
Monadically Stable and Monadically Dependent Graph Classes

Characterizations and Algorithmic Meta-Theorems

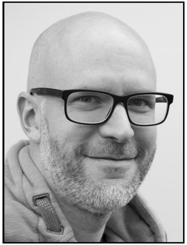
Nikolas Mählmann
previously: University of Bremen, now: University of Warsaw

CSL 2026, Ackermann Award



Acknowledgements

For their great mentorship and collaboration I want to especially thank:



Sebastian Siebertz

Jan Dreier

Szymon Toruńczyk

Michał Pilipczuk

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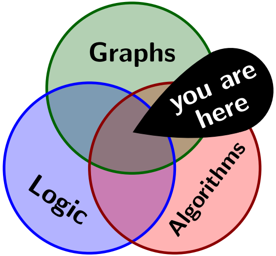
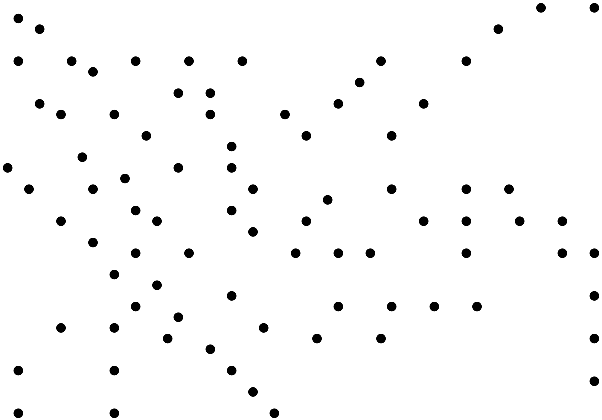


Graphs



A *graph* consists of *vertices* connected by *edges*.

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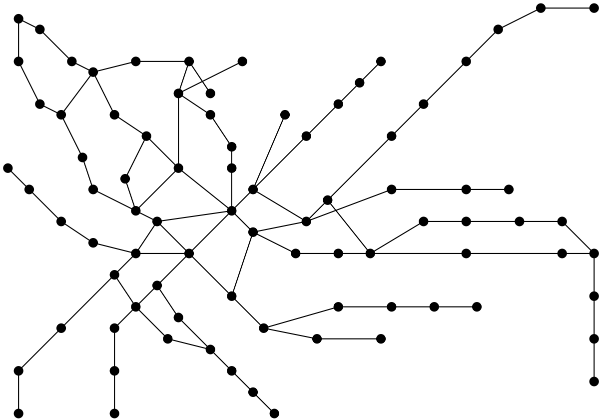

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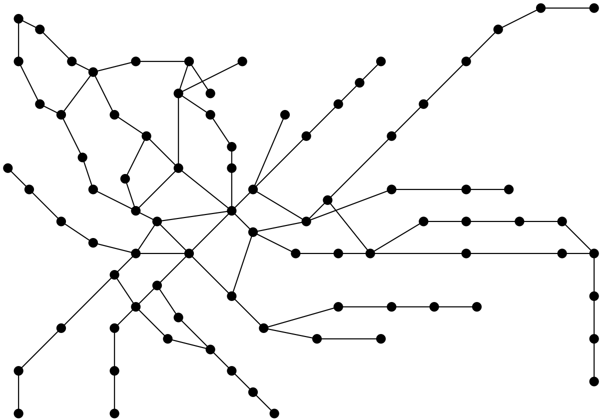

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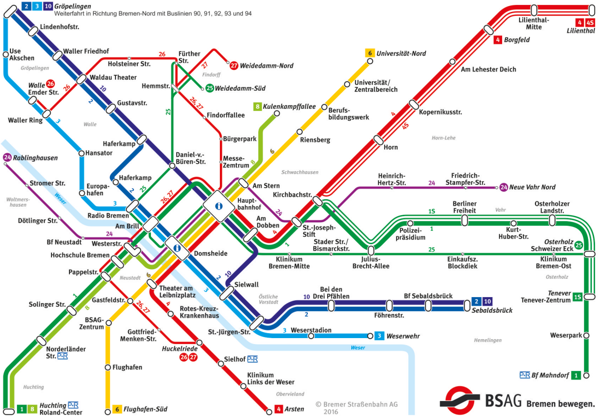

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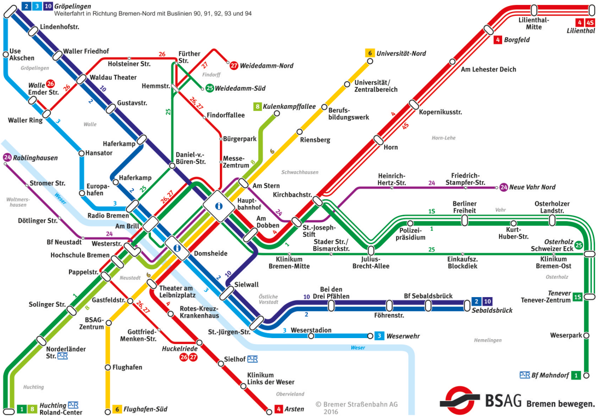

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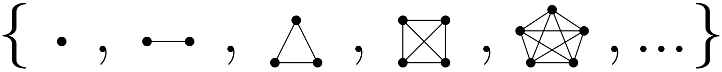
A *graph class* is a (usually infinite) set of graphs. Example:



The FO Model Checking Problem

Problem: Given a graph G and an FO sentence φ, decide whether

$$G \models \varphi.$$

Example: G contains a dominating set of size k iff.

$$G \models \exists x_1 \ldots \exists x_k \forall y : \bigvee_{i \in [k]} \big(y = x_i \vee \mathrm{Edge}(y, x_i)\big).$$

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Question: On which classes is model checking fpt, i.e., solvable in time $f(|\varphi|) \cdot n^c$?

Nowhere Dense Classes of Graphs

For sparse graph classes, we know the exact limits of tractability.

Theorem [Grohe, Kreutzer, Siebertz, 2014]

Let $\mathcal{C}$ be a *monotone* graph class.

- If $\mathcal{C}$ is *nowhere dense*, then model checking is fixed-parameter tractable on $\mathcal{C}$.
- Otherwise model checking is AW[$*$]-hard on $\mathcal{C}$.

Nowhere denseness generalizes: bounded degree, bounded tree-width, planarity, excluding a minor, ...

Monotone and Hereditary Graph Classes

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Cliques are not *monotone*: closed under taking subgraphs.
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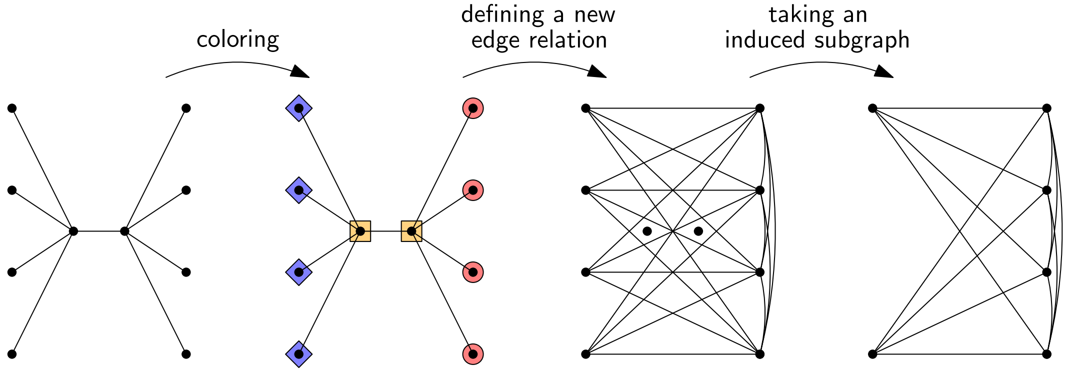
But cliques are *hereditary*: closed under taking induced subgraphs.
(i.e., deleting vertices)

To go beyond sparse classes, we need to shift from monotone to hereditary classes.

Transductions

Transductions are graph transformations defined by FO logic.

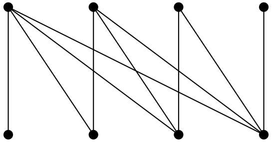
Example: $\varphi(x, y) = (\mathrm{dist}(x, y) = 3) \vee (\mathrm{Red}(x) \wedge \mathrm{Red}(y))$



Monadic Stability and Monadic Dependence

Definition

A class is *monadically stable*, if it does not transduce the class of all half graphs.



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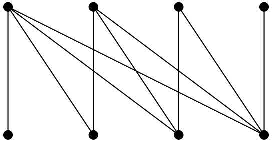

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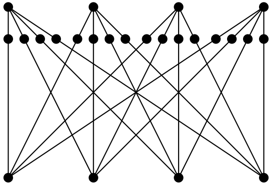

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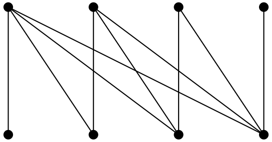

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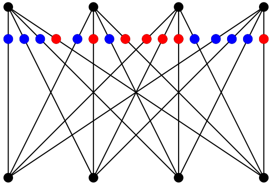

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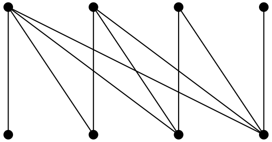

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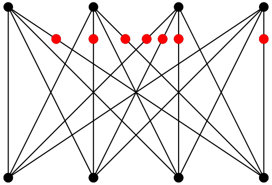

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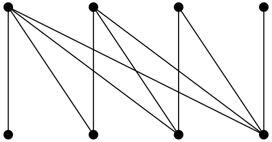

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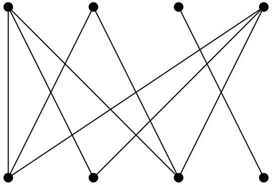

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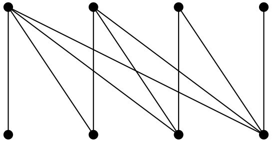

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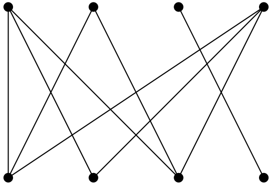

nowhere dense $\subsetneq$ monadically stable $\subsetneq$ monadically dependent

Algorithmic Results

The Model Checking Conjecture

Let $\mathcal{C}$ be a hereditary graph class.

- $\mathcal{C}$ is monadically dependent $\Rightarrow$ model checking is efficient on $\mathcal{C}$.
- $\mathcal{C}$ is monadically independent $\Rightarrow$ model checking is hard on $\mathcal{C}$.

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We show:

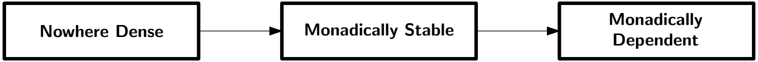
Theorem [Dreier, **NM**, Siebertz, 2023], [Dreier, Eleftheriadis, **NM**, McCarty, Pilipczuk, Toruńczyk, 2024]

$\mathcal{C}$ is monadically stable $\Rightarrow$ model checking is solvable in time $f(|\varphi|) \cdot n^{6.0001}$ on $\mathcal{C}$.

Theorem [Dreier, **NM**, Toruńczyk, 2024]

$\mathcal{C}$ is hereditary and monadically independent $\Rightarrow$ model checking is $\mathrm{AW}[*]$-hard on $\mathcal{C}$.

Model Checking in Hereditary Graph Classes



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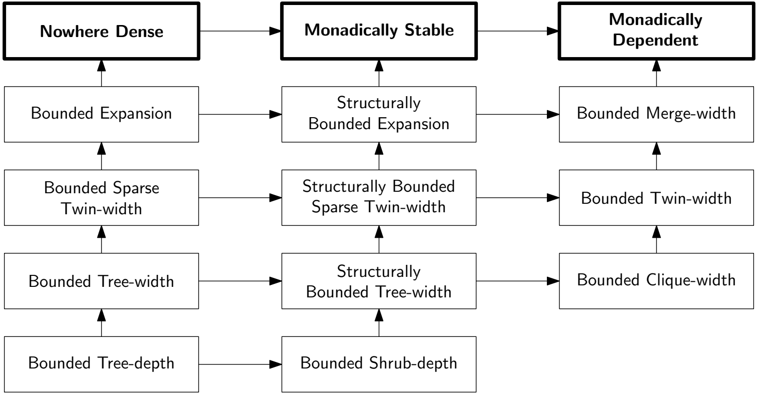

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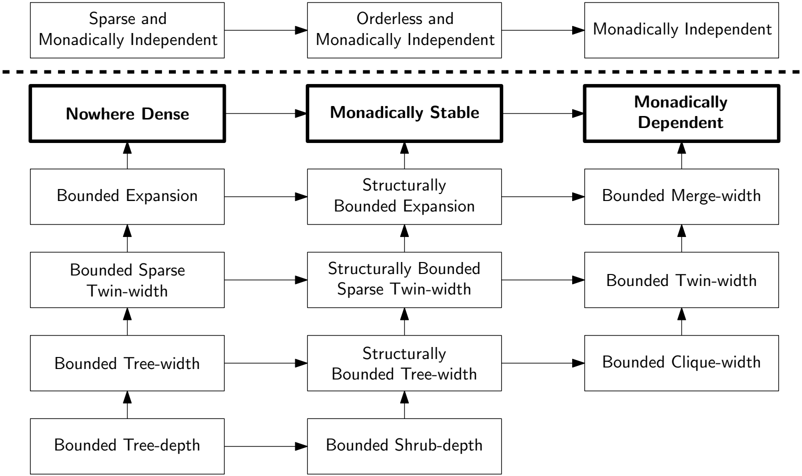

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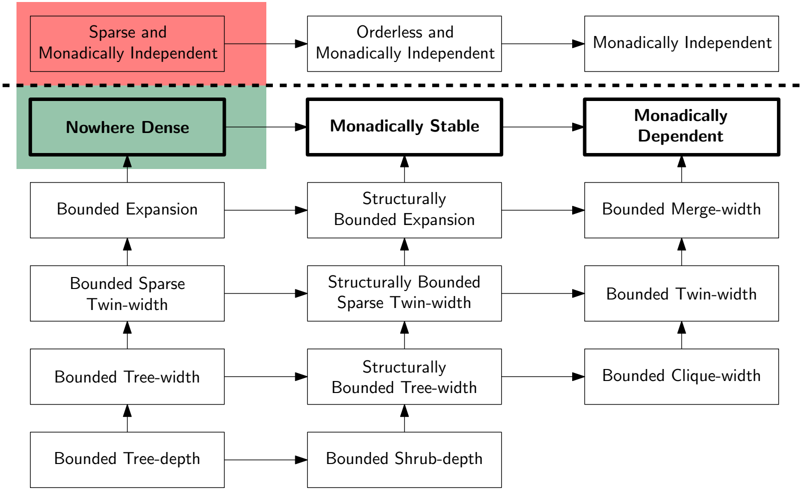

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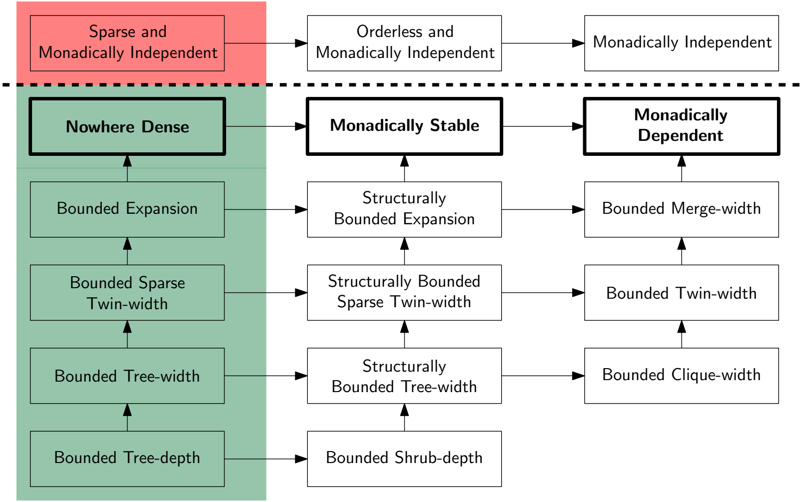

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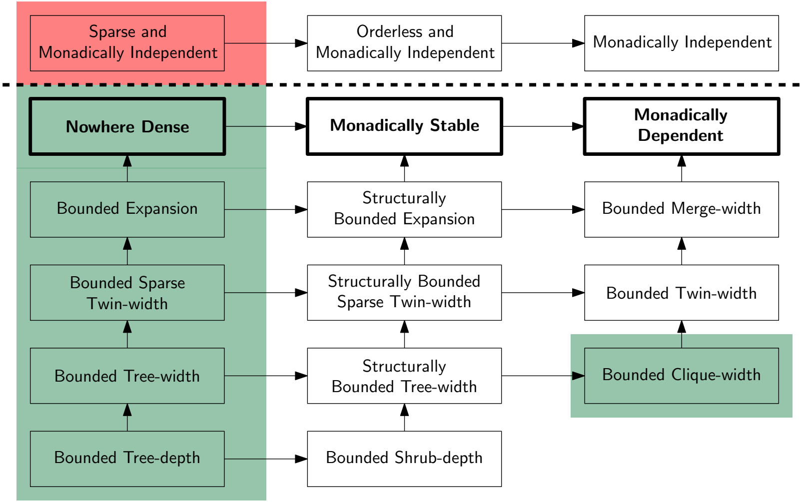

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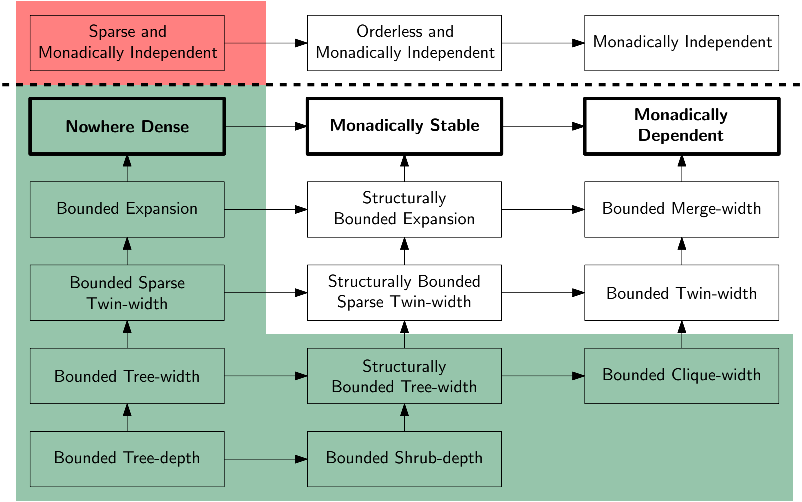

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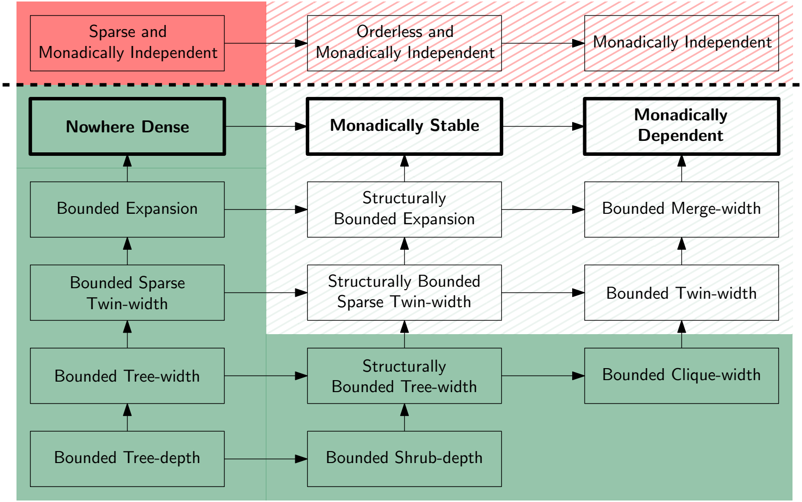
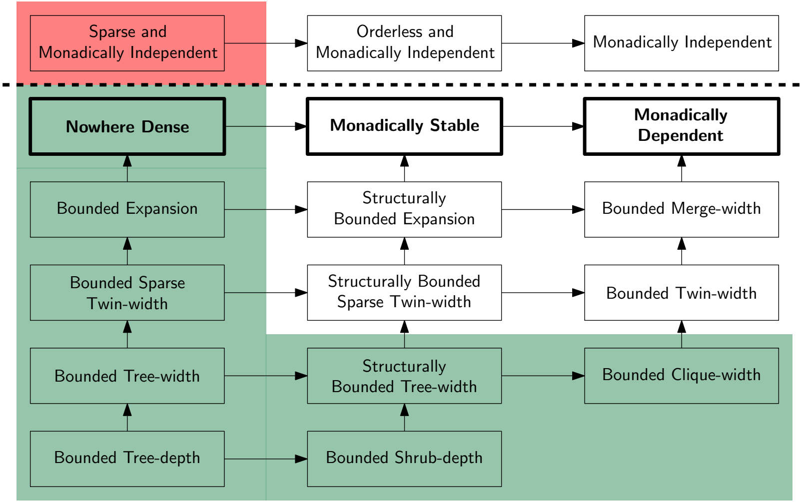
Conjectured Tractability Limits



Our Results ★



Sparse and Monadically Independent
Orderless and Monadically Independent
Monadically Independent
Nowhere Dense
Monadically Stable
Monadically Dependent
Bounded Expansion
Structurally Bounded Expansion
Bounded Merge-width
Bounded Sparse Twin-width
Structurally Bounded Sparse Twin-width
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Bounded Tree-width
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Bounded Clique-width
Bounded Tree-depth
Bounded Shrub-depth

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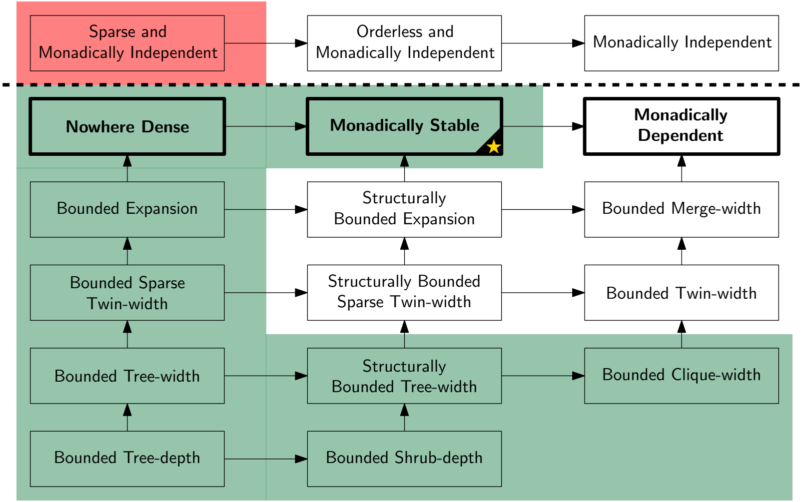

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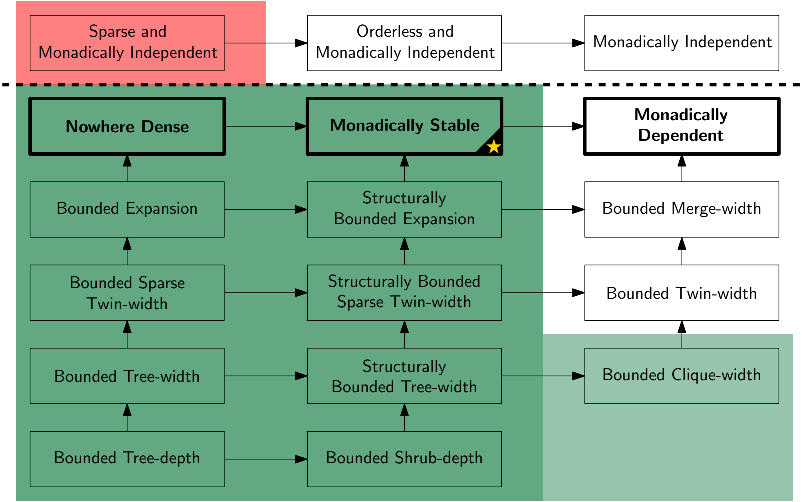

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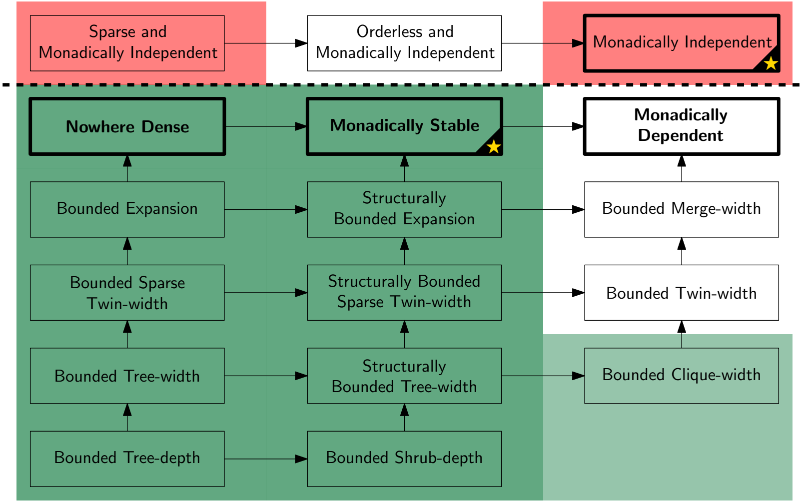

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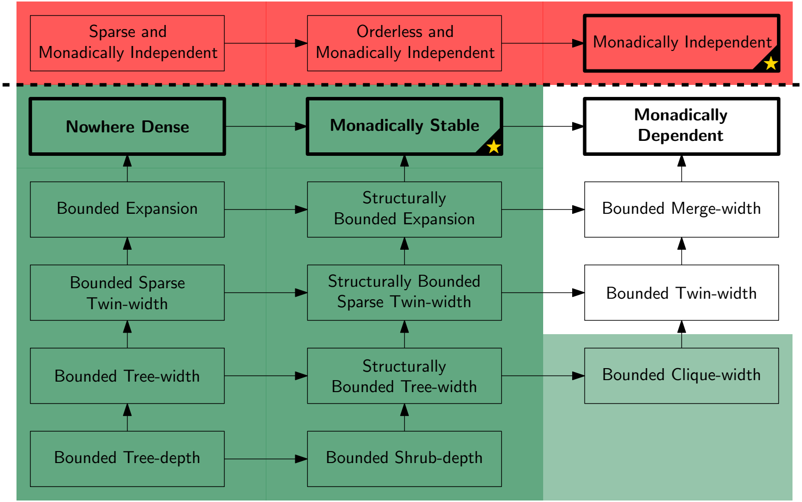

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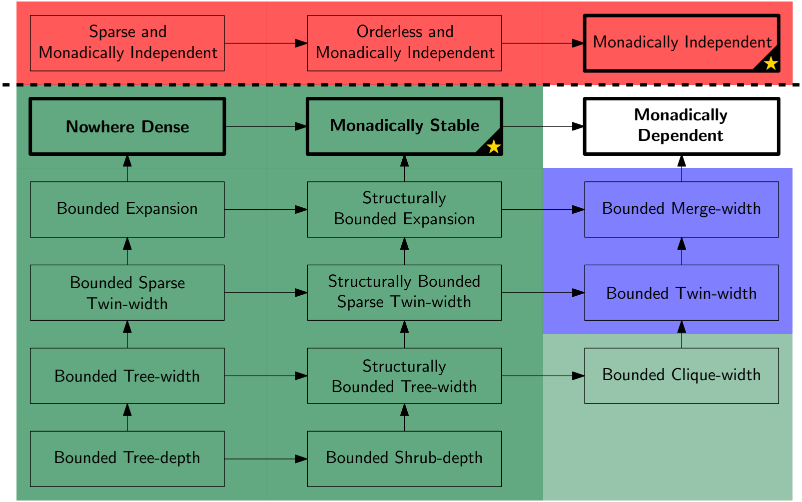
Conditional Results



Sources: twin-width [Bonnet, Kim, Thomassé, Watrigant, 2020], merge-width [Dreier, Toruńczyk, 2025]

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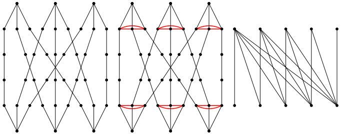

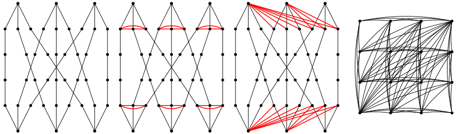

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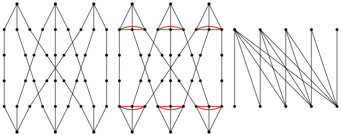
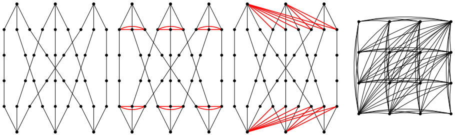
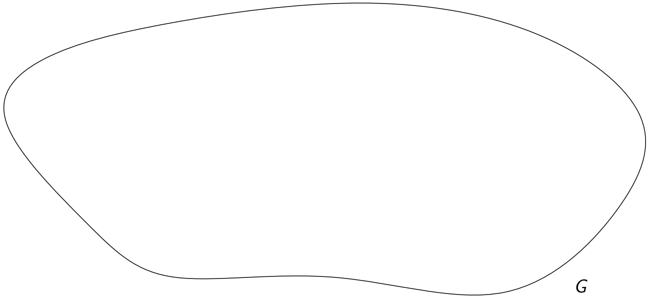




3. **The flipper game** (only for monadic stability)

Characterizing Nowhere Denseness: Uniform Quasi-Wideness



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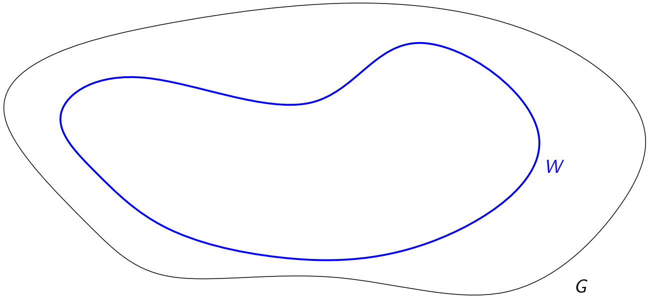

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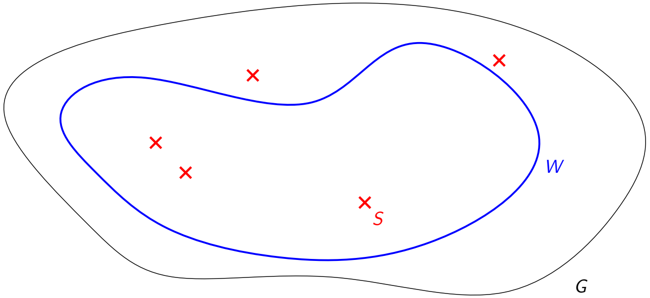

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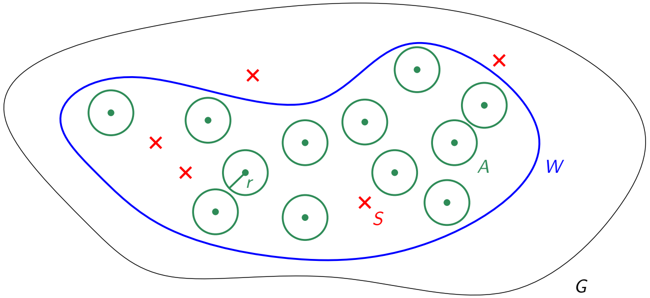

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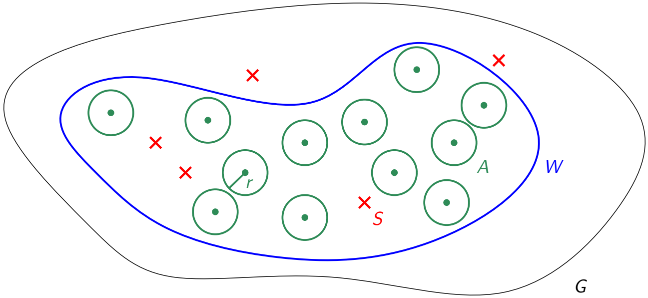

Uniform Quasi-Wideness (slightly informal)

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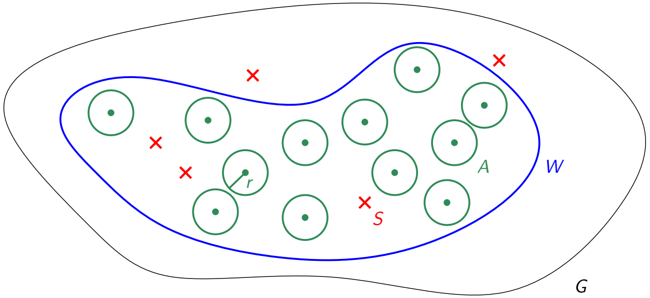

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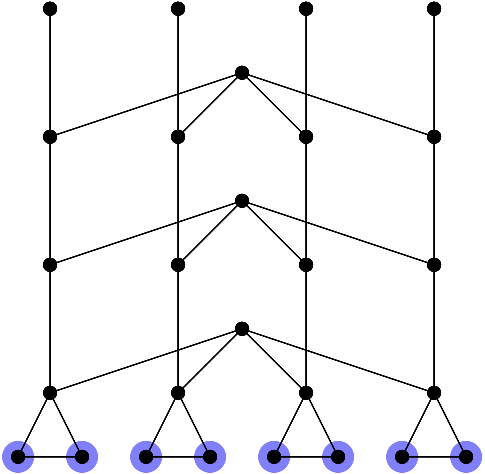
Theorem [Něsetřil, Ossona de Mendez, 2011]

A class $\mathcal{C}$ is uniformly quasi-wide if and only if it is nowhere dense.

Uniform Quasi-Wideness: Example



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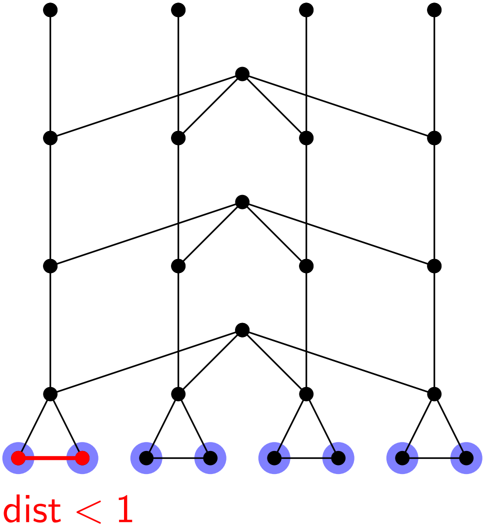

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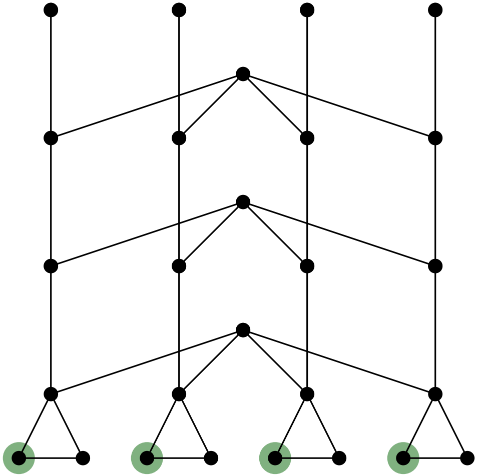

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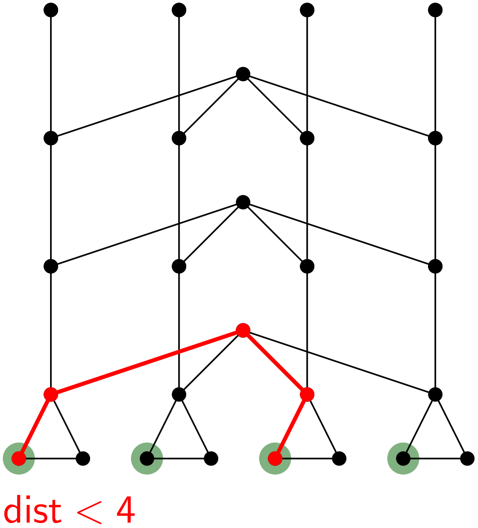

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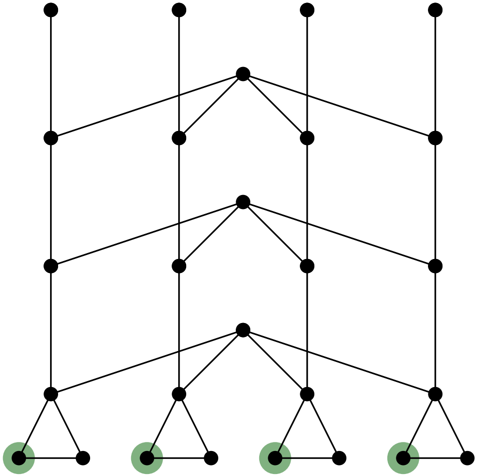

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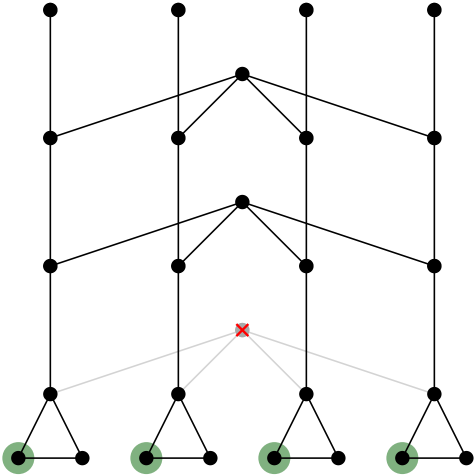

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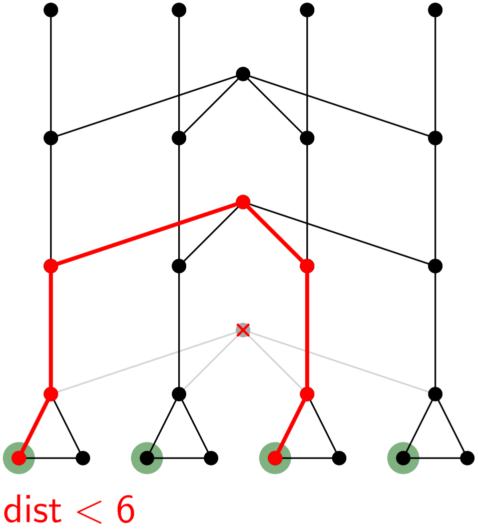

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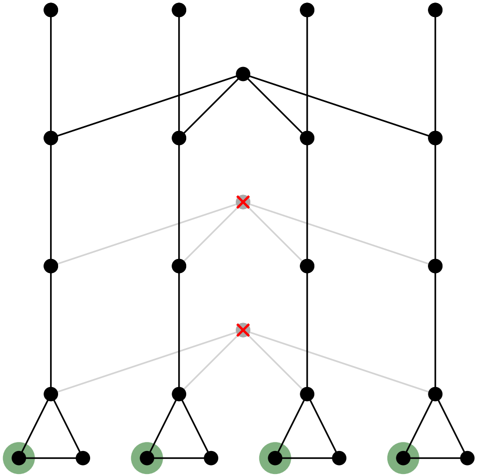
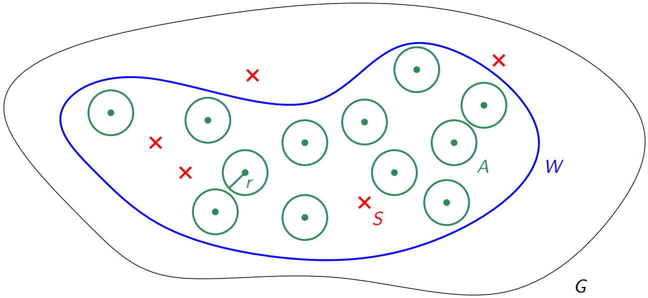


Characterizing Nowhere Denseness: Uniform Quasi-Wideness



Uniform Quasi-Wideness (slightly informal)

A class $\mathcal{C}$ is *uniformly quasi-wide* if for every radius r, in every large set W we find a still large set A that is r-independent after removing a set S of constantly many vertices.

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Towards Dense Graphs

Question: Is there a similar characterization for monadic stability/dependence?

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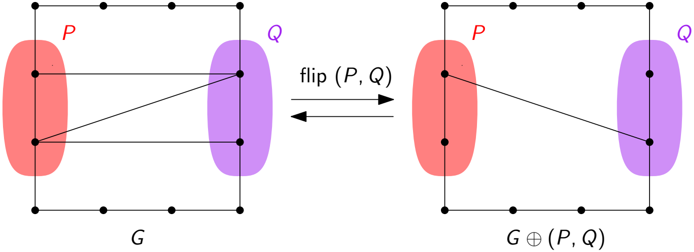
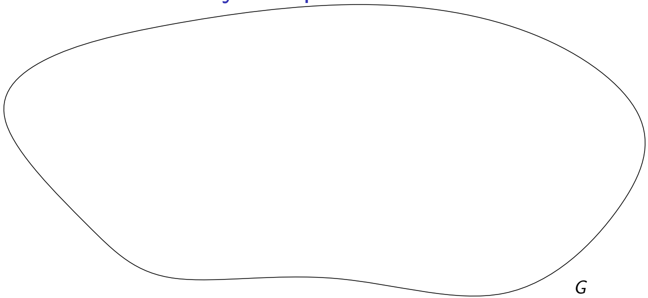
Denote by $G \oplus (P, Q)$ the graph obtained from G by complementing edges between pairs of vertices from $P \times Q$.



Characterizing Monadic Stability: Flip-Flatness



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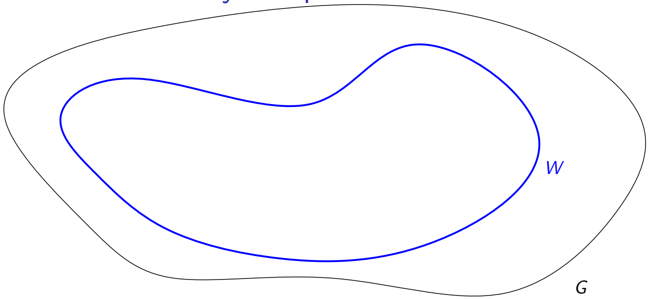

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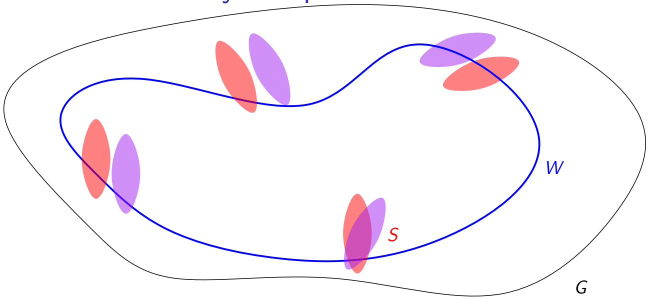

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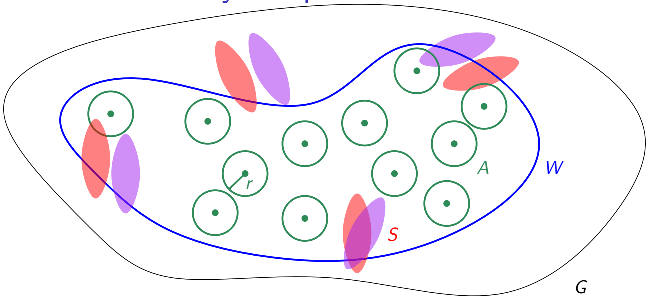

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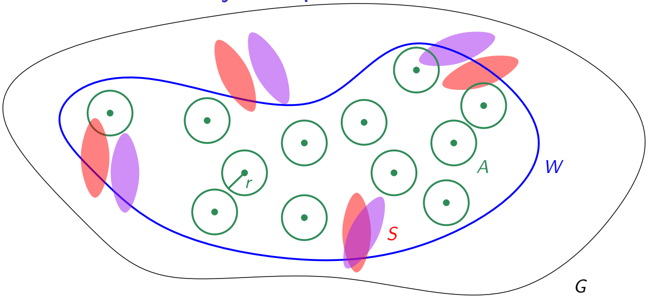

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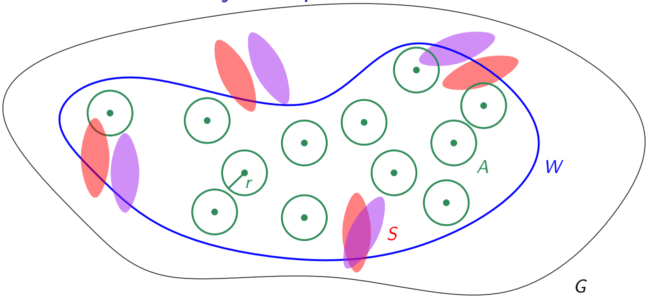

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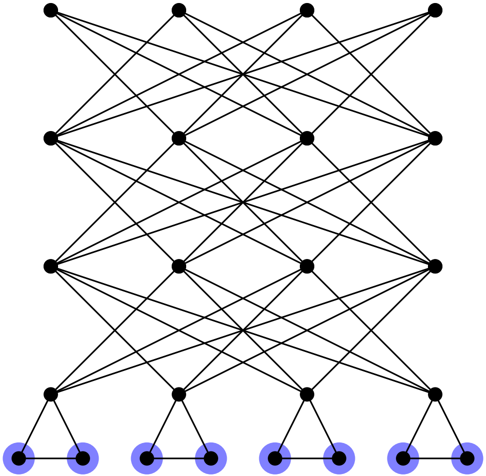

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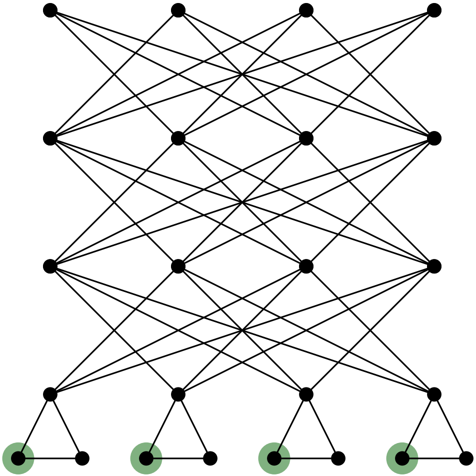

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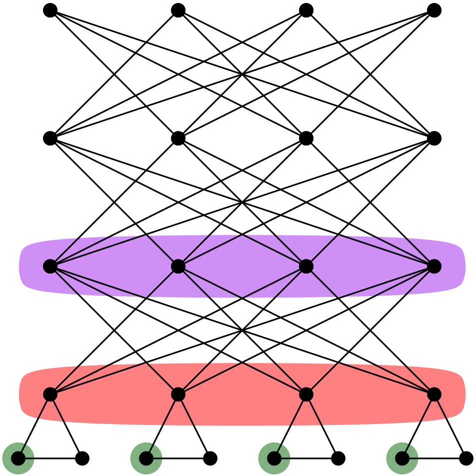

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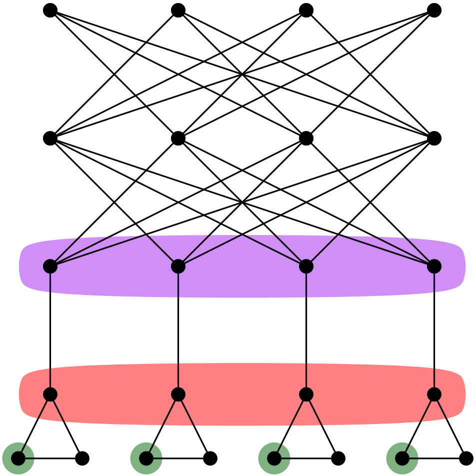

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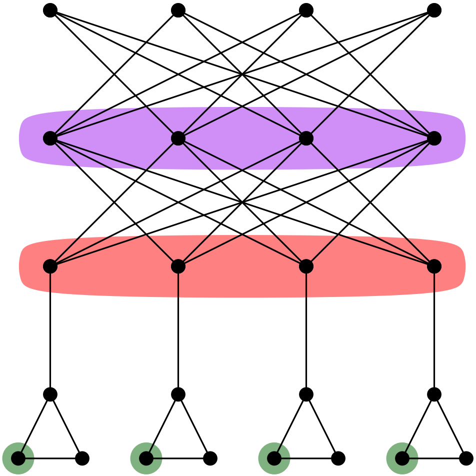

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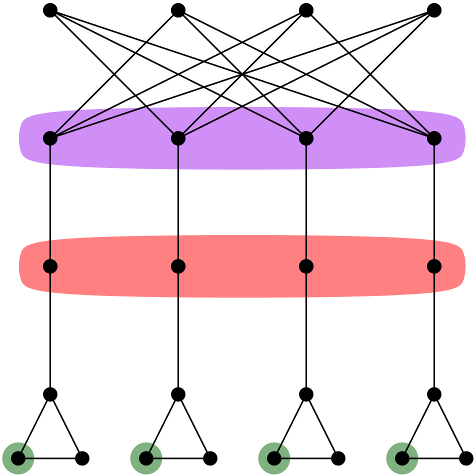

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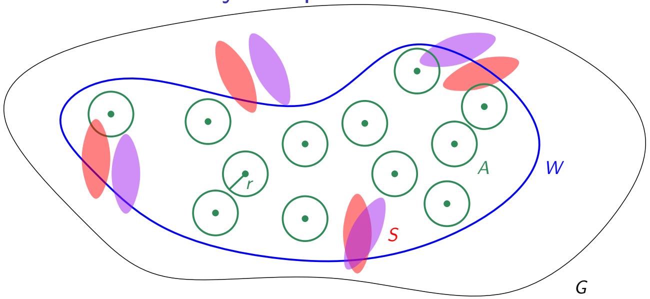

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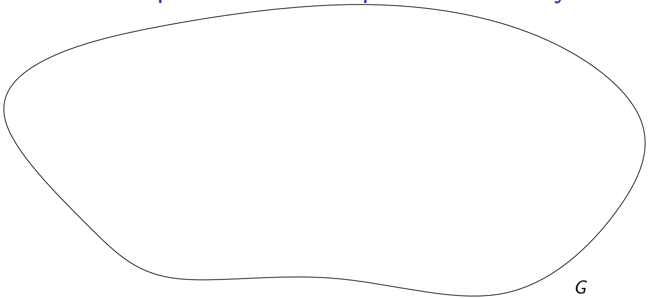

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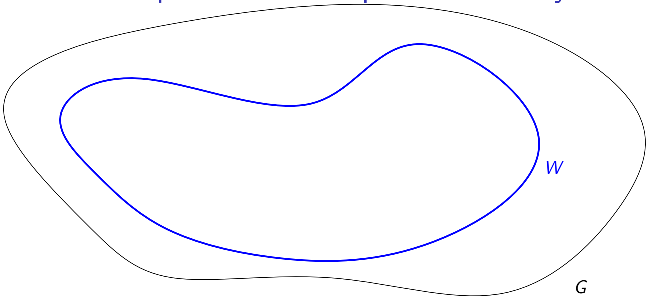

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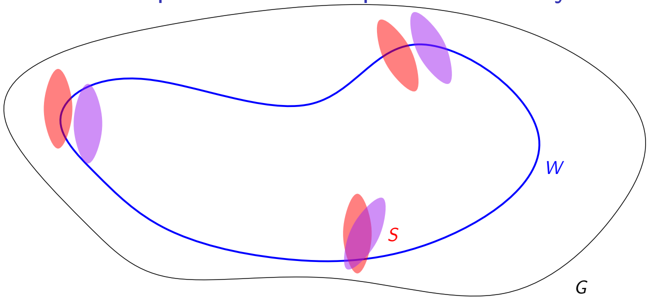

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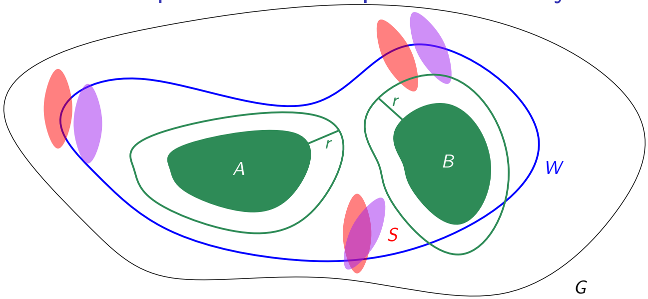

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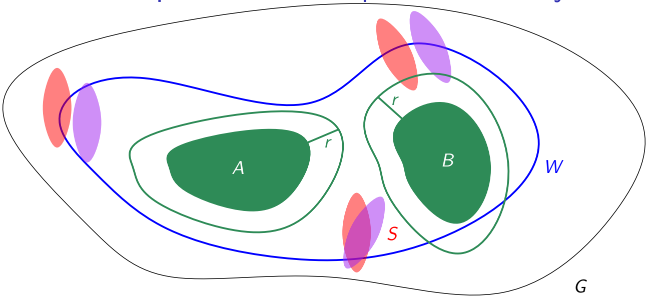

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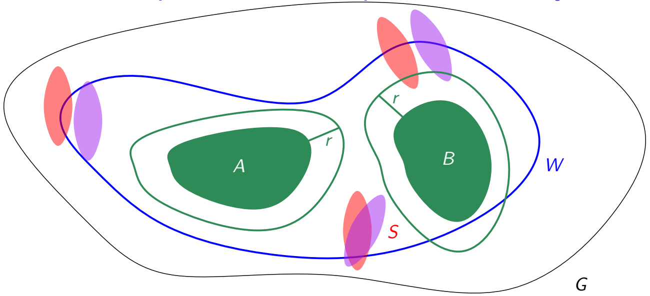

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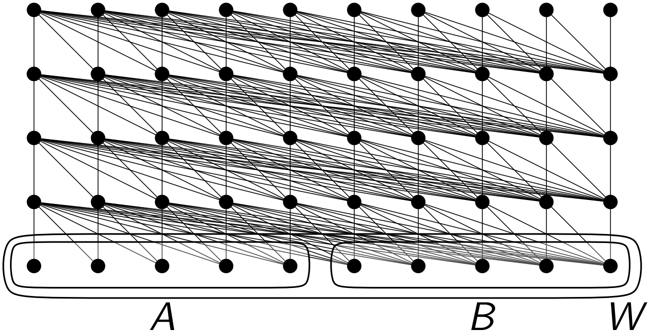

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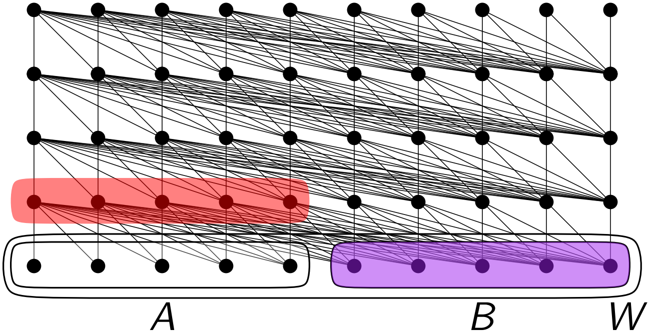

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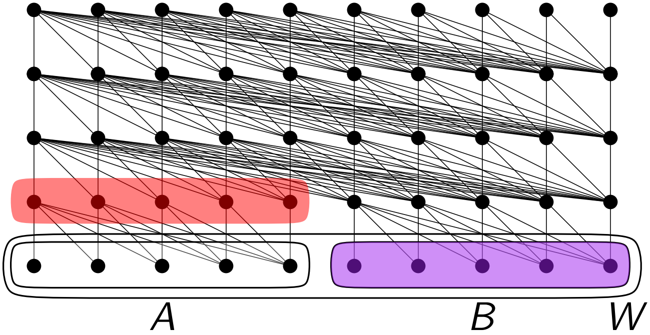

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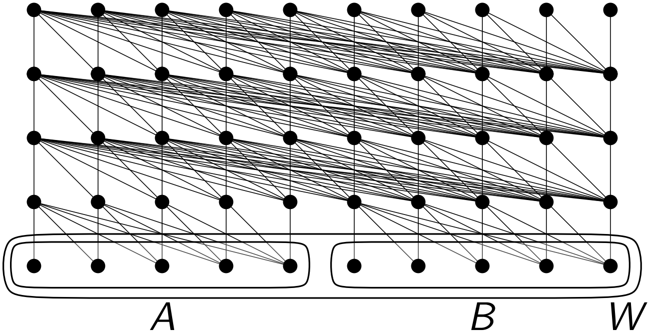

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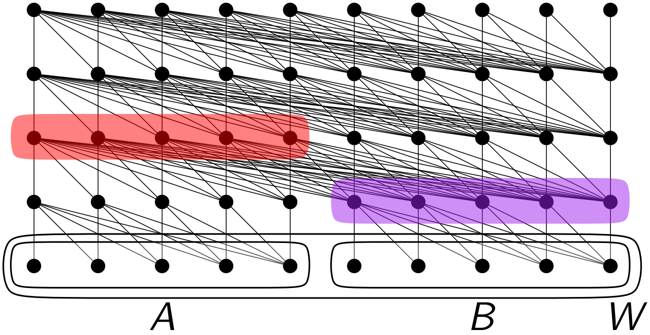

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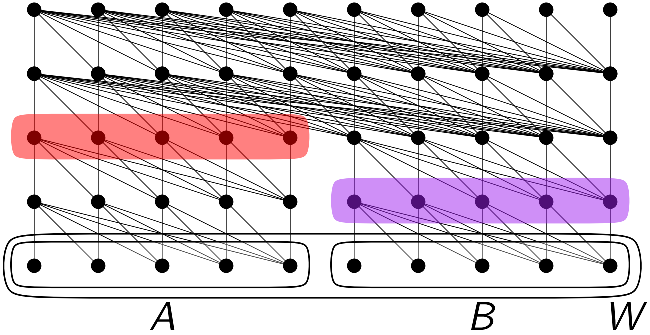

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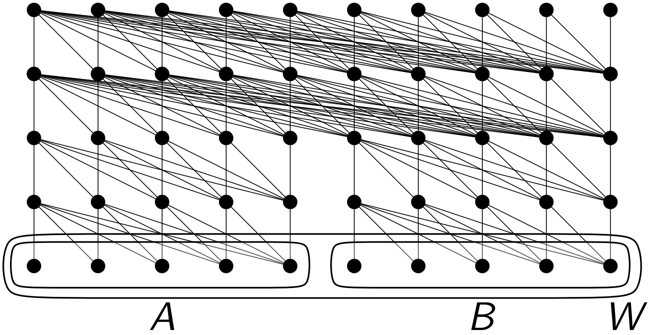

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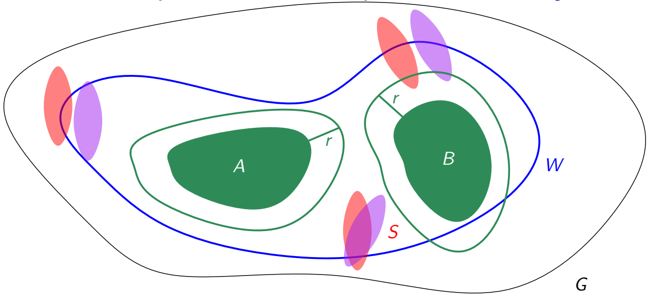

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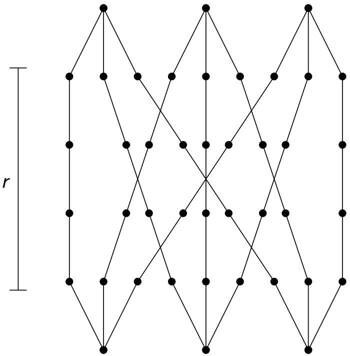
Ramsey-theoretic characterization ✓ next up: forbidden induced subgraphs

Characterizing Monadic Dependence by Forbidden Induced Subgraphs



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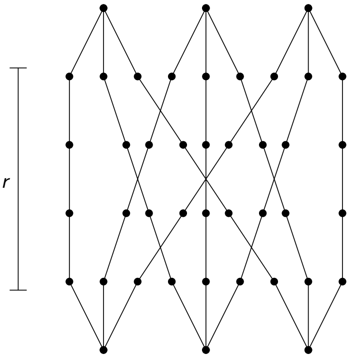

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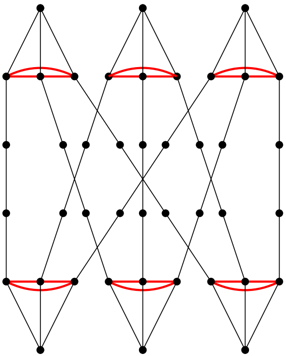

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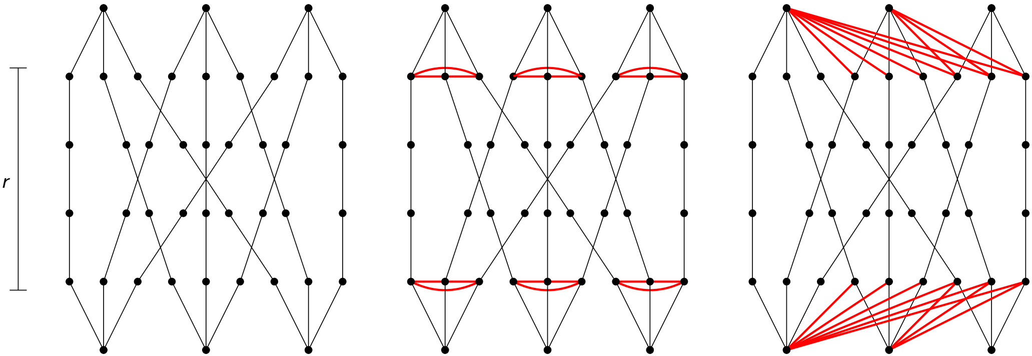

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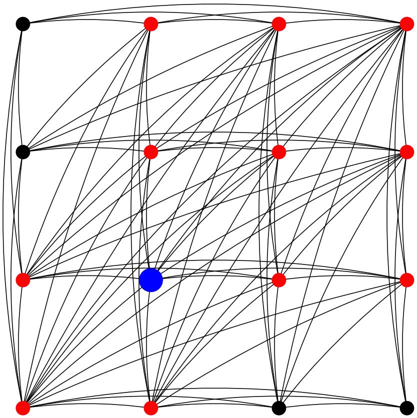

comparability grid

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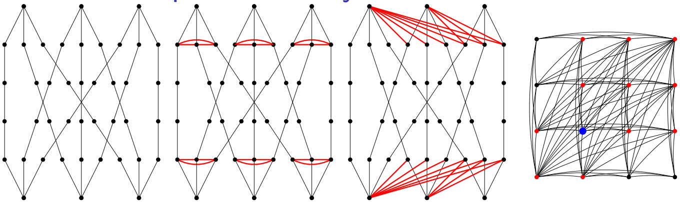

Theorem [Dreier, **NM**, Toruńczyk, 2024]

Let $\mathcal{C}$ be a graph class. Then $\mathcal{C}$ is monadically dependent if and only if for every $r \geq 1$ there exists $k \in \mathbb{N}$ such $\mathcal{C}$ excludes as induced subgraphs

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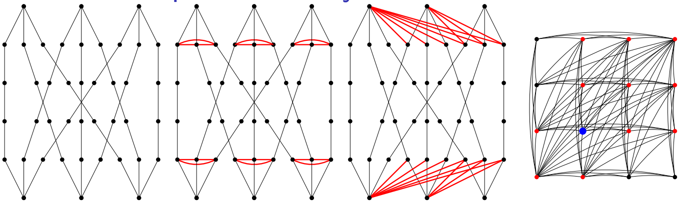

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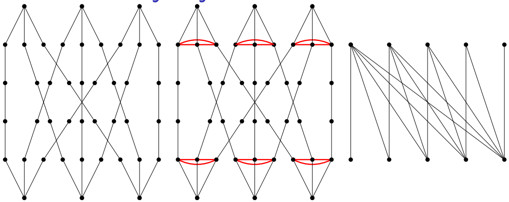
$\Rightarrow$ Model checking is hard on every hereditary, monadically independent graph class.

Characterizing Monadic Stability by Forbidden Induced Subgraphs



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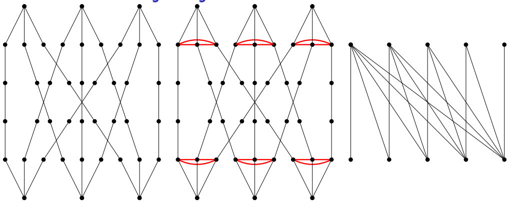

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Characterizations: ramsey-theoretic ✓ forbidden induced subgraphs ✓

Next up: a game characterization for monadic stability

The Flipper Game

The radius-r Flipper game is played on a graph G_1. In round i

1. Flipper chooses a flip F
2. Localizer chooses G_{i+1} as a radius-r ball in $G_i \oplus F$.

Flipper wins once G_i has size 1.

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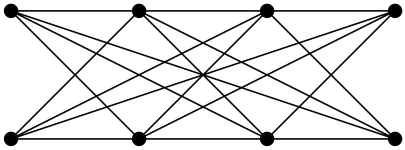

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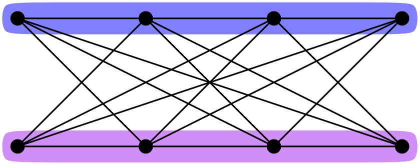

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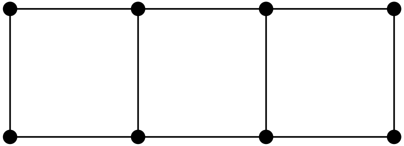

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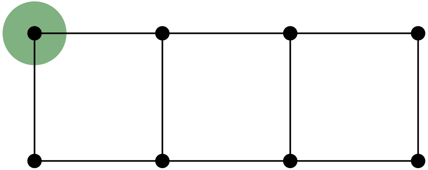

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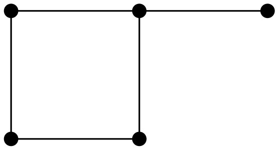

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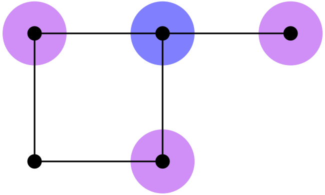

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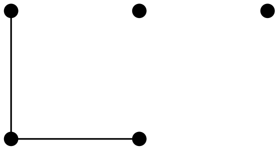

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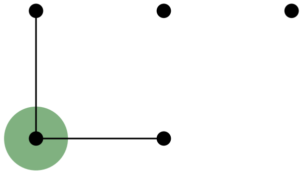

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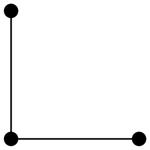

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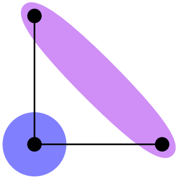

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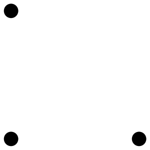

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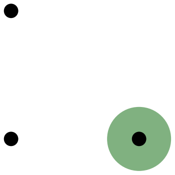

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The Flipper Game in Monadically Stable Classes

Theorem [Gajarský, **NM**, McCarty, Ohlmann, Pilipczuk, Przybyszewski, Siebertz, Sokołowski, Toruńczyk]

A graph class $\mathcal{C}$ is monadically stable $\Leftrightarrow$

$\forall r \exists \ell$ such that Flipper wins the radius-r game on all graphs from $\mathcal{C}$ in ℓ rounds.

Proof builds on flip-flatness. Flippers moves are computable in time $\mathcal{O}_{\mathcal{C},r}(n^2)$.

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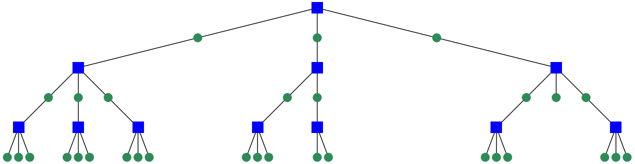
The game tree is a **bounded depth** decomposition of a graph into r-neighborhoods.



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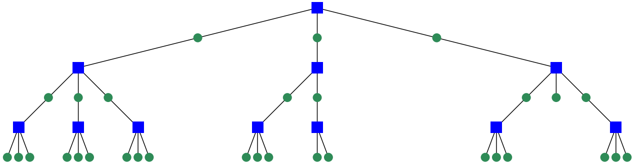
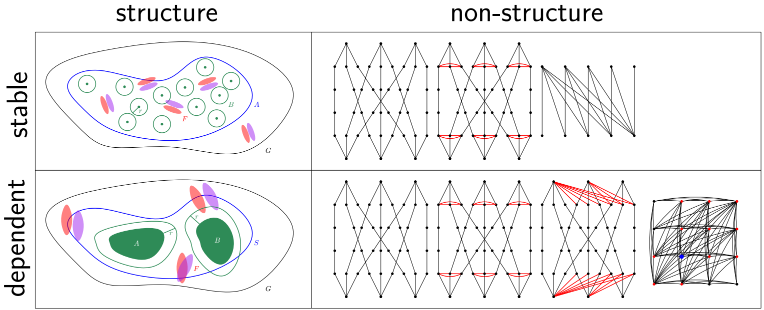


The decomposition can be further compressed by clustering neighborhoods.

Dynamic programming on the compressed tree gives **fpt model checking**.

Summary

We have initiated the development of a combinatorial structure theory for monadically stable and dependent graph classes:



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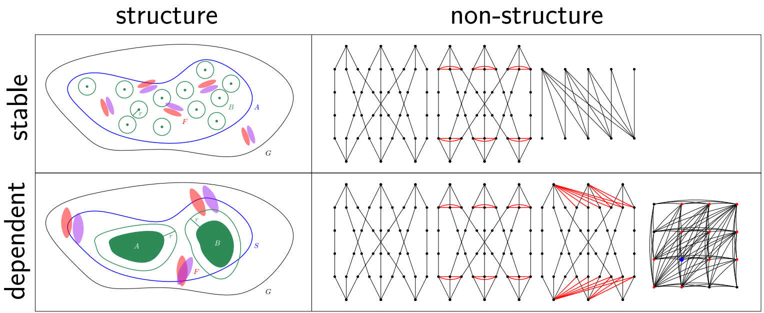

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Thanks! Vielen Dank! Dziękuję bardzo! Merci!

Backup slides

Publications 1/2

1. *Indiscernibles and Flatness in Monadically Stable and Monadically NIP Classes*
   joint work with Jan Dreier, Sebastian Siebertz, Szymon Toruńczyk
   presented at ICALP 2023
2. *Flipper Games for Monadically Stable Graph Classes*
   joint work with Jakub Gajarský, Rose McCarty, Pierre Ohlmann, Michał Pilipczuk, Wojciech Przybyszewski, Sebastian Siebertz, Marek Sokołowski, Szymon Toruńczyk
   presented at ICALP 2023
3. *First-Order Model Checking on Structurally Sparse Graph Classes*
   joint work with Jan Dreier, Sebastian Siebertz
   presented at STOC 2023

Publications 2/2

4. *First-Order Model Checking on Monadically Stable Graph Classes*
   joint work with Jan Dreier, Ioannis Eleftheriadis, Rose McCarty, Michał Pilipczuk, Szymon Toruńczyk
   accepted at FOCS 2024

5. *Flip-Breakability: A Combinatorial Dichotomy for Monadically Dependent Graph Classes*
   joint work with Jan Dreier, Szymon Toruńczyk
   presented at STOC 2024

Flip-Flatness

Theorem

A graph class $\mathcal{C}$ is *flip-flat* if for every radius $r \in \mathbb{N}$ there exists a function $N_r : \mathbb{N} \to \mathbb{N}$ and a constant $k_r \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $G \in \mathcal{C}$ and $W \subseteq V(G)$ with $|W| \geq N_r(m)$ there exist a subset $A \subset W$ with $|A| \geq m$ and a k_r-flip H of G such that for every two distinct vertices $u, v \in A$:

$$\mathrm{dist}_H(u, v) > r.$$

Flip-Breakability

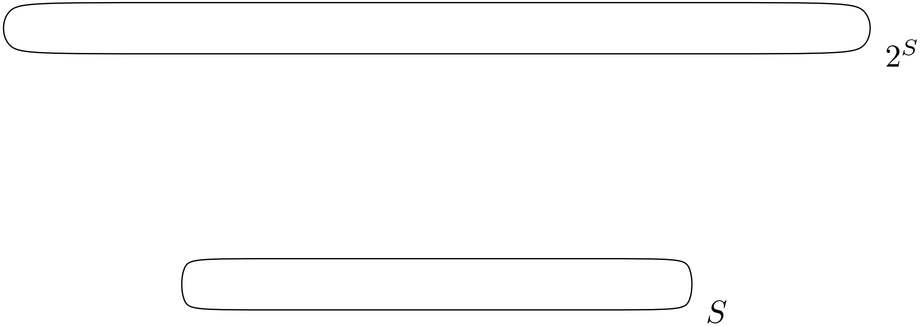
Theorem

A graph class $\mathcal{C}$ is *flip-breakable* if for every radius $r \in \mathbb{N}$ there exists a function $N_r : \mathbb{N} \to \mathbb{N}$ and a constant $k_r \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $G \in \mathcal{C}$ and $W \subseteq V(G)$ with $|W| \geq N_r(m)$ there exist subsets $A, B \subset W$ with $|A|, |B| \geq m$ and a k_r-flip H of G such that:

$$\mathrm{dist}_H(A, B) > r.$$

Flip-Breakability ⇒ Monadic Dependence

Assume towards a contradiction a class $\mathcal{C}$ is not monadically dependent but flip-breakable.



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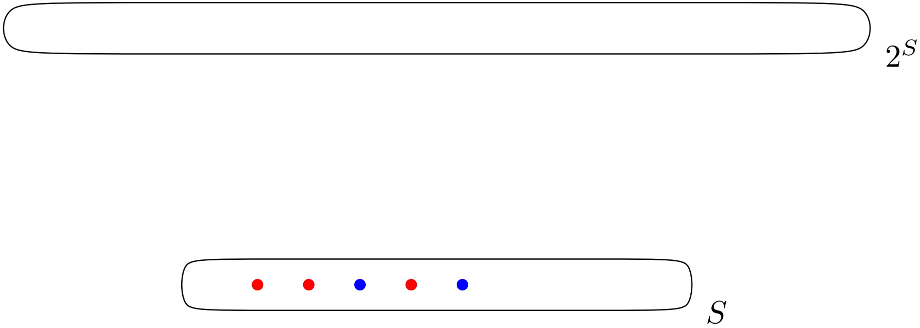

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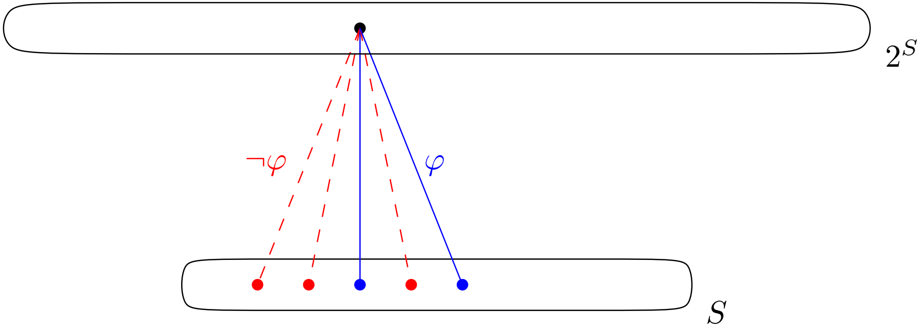

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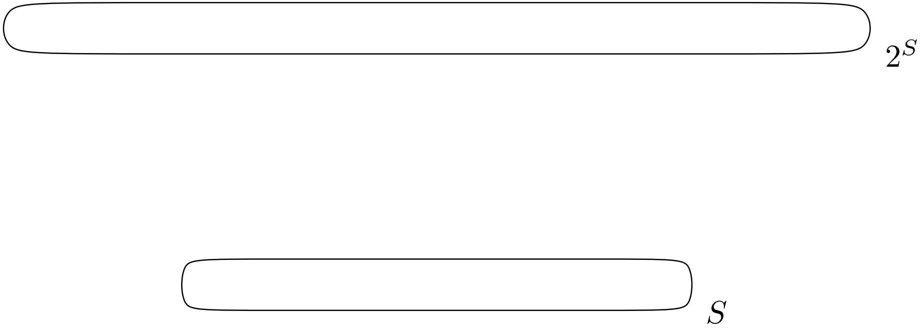

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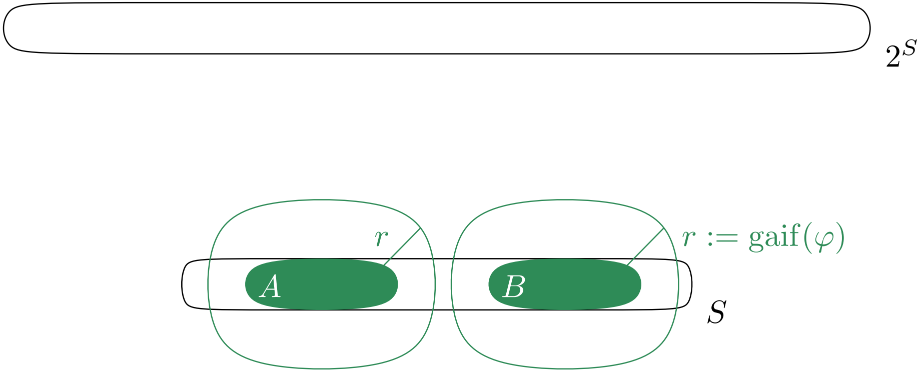

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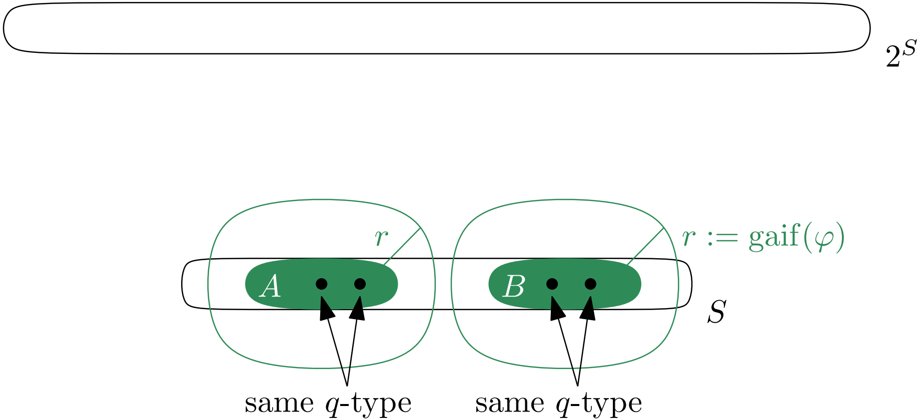

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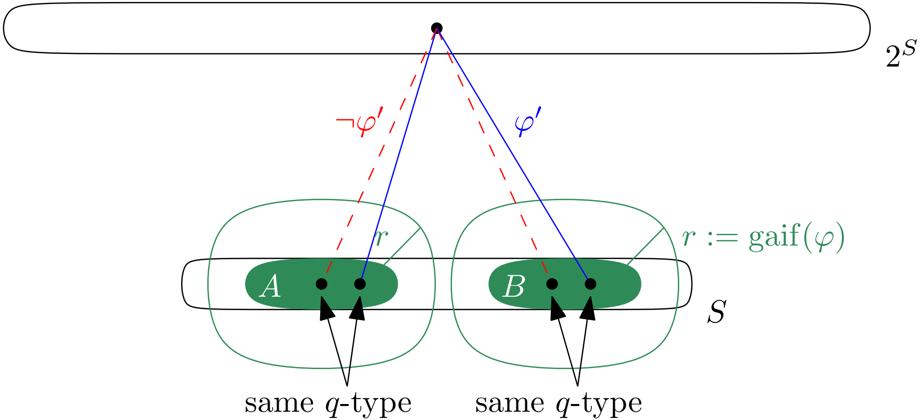

Stability and Dependence in Model Theory

On a class $\mathcal{C}$, a formula $\varphi(\bar{x}, \bar{y})$ has

- the *order property* if for every $k \in \mathbb{N}$ there are $G \in \mathcal{C}$ and two sequences $(\bar{a}_i)_{i\in[k]}$, $(\bar{b}_j)_{j\in[k]}$ of tuples in G, such that for all $i, j \in [k]$: $G \models \varphi(\bar{a}_i, \bar{b}_j) \Leftrightarrow i \leq j$.
- the *independence property* if for every $k \in \mathbb{N}$ there are $G \in \mathcal{C}$, a size k set $A \subseteq V(G)^{|\bar{x}|}$ and a sequence $(\bar{b}_J)_{J\subseteq A}$ of tuples in G such that for all $\bar{a} \in A, J \subseteq A$

$$G \models \varphi(\bar{a}, \bar{b}_J) \quad \Leftrightarrow \quad \bar{a} \in J.$$

A graph class is *stable* if it does not have the order property.
It is *monadically stable* if the class of colored graphs from $\mathcal{C}$ is stable.

A graph class is *dependent* if it does not have the independence property.
It is *monadically dependent* if the class of colored graphs from $\mathcal{C}$ is dependent.

Approximation Algorithms

Distance-r dominating set:

- constant factor approximation in bounded expansion classes [Dvořák 2013]
- $O(d \cdot \log(d \cdot OPT))$ approximation of the distance-1 case on graphs with VC dimension $\leq d$ [Brönnimann, Goodrich, 1995]

Distance-r independent set:

- constant factor approximation in bounded expansion classes [Dvořák 2013]
- n^{ε} approximation in nowhere dense classes [Dvořák 2019]
- n^{ε} approximation in bounded twin-width classes [Bergé et al. 2022]