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University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Exam for 2nd cycle studies of MACHINE LEARNING

June 30, 2022

Solving time: 150 minutes

In each of the 30 problems there are three variants: (a), (b), and (c). For each variant you should answer if it is true, writing **YES** *or* **NO** *in the box to the left of it. In the case of an error you should cross out the box and write the correct word on its left side.*

Example of a correctly solved problem

4. Every integer of the form $10^n - 1$, where n is integer and positive,

   YES (a) is divisible by 9;

   NO (b) is prime;

   YES (c) is odd.

You can write only in the indicated places and only the words **YES** *and* **NO**. *Use pen.*

Scoring

You get "big" points (0 – 30) and "small" points (0 – 90):

- *one "big" point for each problem, in which you correctly solved all three variants;*
- *one "small" point for each correctly solved variant. So 3 "small" points in a single problem give one "big" point.*

The final result of the exam is the number

$$W = \min(30, D + m/100),$$

where D is the number of "big" points, and m is the number of "small" points, e.g. score 5.50 means that a candidate correctly solved 50 variants in the whole test, but gave correct answers to all three variants in a set for some five problems.

"Big" points are more important. "Small" points are just to increase resolution in the case when many candidates get the same number of "big" points.

Good luck!

1. A series of positive numbers $\sum_{n=1}^{\infty} a_n$ converges. It follows that

- [ ] (a) $\lim_{n\to\infty} a_n = 0$;
- [ ] (b) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \leq 1$;
- [ ] (c) there exists a positive number s, such that the sequence $(n^s a_n)$ is bounded.

2. Function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + 3xy + 2y^2$ has at point $(0, 0)$

- [ ] (a) a local minimum;
- [ ] (b) the global minimum;
- [ ] (c) a vanishing gradient.

3. Let a_1, a_2, a_3 be three vectors in a linear space X of dimension 3 over the field $\mathbb{R}$. In the linear space X^* (dual to X) there exist three linear forms f_1, f_2, f_3, such that

$$f_i(a_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

It follows that

- [ ] (a) the set of vectors a_1, a_2, a_3 is linearly independent;
- [ ] (b) the set of vectors a_1, a_2, a_3 spans X;
- [ ] (c) the set of linear forms f_1, f_2, f_3 is a basis of X^*.

4. An $n \times n$ matrix A with real entries is symmetric. It follows that

- [ ] (a) matrix A is nonsingular;
- [ ] (b) matrix A is similar to some diagonal matrix;
- [ ] (c) matrix A is nonnegative definite.

5. Suppose that matrix $A \in \mathbb{R}^{n,n}$ ($n \in \mathbb{N}$, $n > 0$) has the following property: for every $b \in \mathbb{R}^n$ equation $Ax = b$ has exactly one solution $x \in \mathbb{R}^n$. It follows that

- [ ] (a) the rank of A^{T} equals n;
- [ ] (b) there is a vector $x \in \mathbb{R}^n \setminus \{0\}$, such that $Ax = 0$;
- [ ] (c) A is a matrix of an epimorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$.

6. Let X, X', U, U', V, V' be linear spaces, such that $X = U \oplus V$ and $X' = U' \oplus V'$. It follows that

- [ ] (a) for any linear maps $f\colon U \to U'$ and $g\colon V \to V'$ there exist linear maps $h_1, h_2\colon X \to X'$, such that $h_i|_U = f$, $h_i|_V = g$ for $i = 1, 2$ and $h_1 \neq h_2$;
- [ ] (b) if linear spaces X and X' are isomorphic then U is isomorphic to U' or U is isomorphic to V';
- [ ] (c) if U is isomorphic to U' and V is isomorphic to V' then the spaces X and X' are isomorphic.

7. Let r be an equivalence relation in $\mathbb{R}$ that is not the full relation. It follows that

- [ ] (a) every equivalence class of r is countable;
- [ ] (b) there exists an infinite relation $q \subseteq \mathbb{R} \times \mathbb{R}$, such that $q \cap r = \emptyset$;
- [ ] (c) for every $x \in \mathbb{R}$ and for every $y \in \mathbb{R}$, such that $\langle x, y \rangle \in r$, there exists $z \in \mathbb{R}$ satisfying $\langle x, z \rangle \notin r$.

8. Intersection of a nonempty countable family of subsets of $\mathrm{P}(\mathbb{N})$

- [ ] (a) is countable;
- [ ] (b) is the greatest lower bound of this family in the partially ordered set $\langle \mathrm{P}(\mathrm{P}(\mathbb{N})), \subseteq \rangle$;
- [ ] (c) is nonempty.

9. The number of words of length $n \geq 2$ consisting of letters A, B, C, D, such that each of the letters A, B occurs an odd number of times, is equal to

- [ ] (a) 4^{n-1};
- [ ] (b) $4^{n-1} - 2^{n-1}$;
- [ ] (c) $\frac{1}{4}\binom{n+2}{3}$ if n is even.

10. Let G be the graph whose vertices represent squares of a chessboard of size $n \times n$, and edges connect squares such that a knight can jump from one square to the other in a single move. It follows that

- [ ] (a) G is planar for some $n \geq 3$;
- [ ] (b) G is bipartite for every $n \geq 3$;
- [ ] (c) G contains an Eulerian circuit for some $n \geq 3$.

11. Let $a_n = n$ for $n \geq 0$. It follows that

- [ ] (a) $\frac{1}{(1-x)^2}$ is the ordinary generating function of $\langle a_n \rangle_{n \geq 0}$;
- [ ] (b) xe^x is the exponential generating function of $\langle a_n \rangle_{n \geq 0}$;
- [ ] (c) $\frac{x^2}{(1-x)^4}$ is the ordinary generating function of $\langle \sum_{k=0}^{n} a_k a_{n-k} \rangle_{n \geq 0}$.

12. The chromatic number of graph G is 2. It follows that

- [ ] (a) G is planar;
- [ ] (b) the length of each cycle in G is even;
- [ ] (c) if G has 100 vertices then it has at most 2000 edges.

13. A sufficient condition for the inequality $P(|X| \geq 10) > \frac{1}{10}$ to hold is that the random variable X

- [ ] (a) has variance equal to 50;
- [ ] (b) is nonnegative and its expected value is equal to 20;
- [ ] (c) has expected value equal to 1000.

14. Let X_1, X_2, X_3 be a simple random sample from a Poisson distribution with unknown parameter $\lambda > 0$. Consider the following estimators for λ:

$$\hat{\lambda}_1 = \frac{X_1 + X_2 + X_3}{3}, \qquad \hat{\lambda}_2 = \frac{2X_1 + 2X_2 + X_3}{5}.$$

☐ (a) Both estimators are biased.

☐ (b) The variance of either estimator does not depend on parameter λ.

☐ (c) $\mathrm{Var}(\hat{\lambda}_1) < \mathrm{Var}(\hat{\lambda}_2)$ for each $\lambda > 0$.

15. Random variables X and Y have finite expected values and Y assumes only positive values. It follows that

☐ (a) $E(X + Y) = EX + EY$;

☐ (b) $E(X \cdot Y) = EX \cdot EY$;

☐ (c) $E(X/Y) = EX/EY$.

16. Random variables X and Y have normal distributions and are independent. It follows that

☐ (a) e^X has normal distribution;

☐ (b) the probability that $X > Y$ is positive;

☐ (c) $X + Y$ has normal distribution.

17. A Markov chain has 4 states: 1, 2, 3, 4. The probability of changing state from i to j is $1/i$ if $j \leq i$ and 0 otherwise. It follows that

☐ (a) this Markov chain has two transient classes;

☐ (b) this Markov chain has two recurrent classes;

☐ (c) the expected hitting time from state 2 to state 1 is 2.

18. Consider the iteration

$$y_n = Ax_n, \qquad x_{n+1} = \frac{y_n}{\|y_n\|_2}, \qquad n = 0, 1, \ldots,$$

where

$$x_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} -2 & -3 \\ -3 & 6 \end{pmatrix}.$$

It follows that

☐ (a) $x_{20} = \frac{1}{113988951853731 43} \cdot \begin{pmatrix} 104 \\ 259 \end{pmatrix}$;

☐ (b) $v = \lim_{n\to\infty} x_n$ exists and $\|v\|_2 = 1$;

☐ (c) $v = \lim_{n\to\infty} x_n$ exists and $v^T \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 0$.

19. Consider matrices

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \\ 1 & -1 \\ -1 & -3 \end{pmatrix}, \qquad Q = \frac{1}{2} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ -1 & -1 \end{pmatrix}, \qquad R = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}.$$

☐ (a) Matrices Q and R form a QR decomposition of the matrix A.

☐ (b) The expression $\|(1,1,1,1)^{\mathrm{T}} - Ax\|_2$ reaches its minimum at $x = (\frac{1}{2}, -\frac{1}{4})^{\mathrm{T}}$.

☐ (c) $\|A\|_2 = \sqrt{8}$.

20. Consider an algorithm that operates on nonnegative integers in binary representation:

```
j = n;
while (j > 0) j = j - 1;
```

☐ (a) There are $O(1)$ bit changes in j during a single j = j − 1 operation.

☐ (b) There are $O(k)$ bit changes in j during the first k loop iterations for each $k = 1, \ldots, n$.

☐ (c) There are $O(k)$ bit changes in j during the last k loop iterations for each $k = 1, \ldots, n$.

21. Consider a binary heap (of type MIN) represented in an array. We insert n zeros and then n ones into the heap.

☐ (a) The next operation ExtractMin takes time $O(1)$.

☐ (b) The next operation Insert(0) takes time $O(1)$.

☐ (c) The next operation Insert(1) takes time $O(1)$.

22. In an Ethernet-based IP subnet, a host wants to send a DNS query to a server that is on a different subnet. For this purpose the host sends the query to the router and puts

☐ (a) the MAC address of the router's interface in the destination address field in the header of the Ethernet frame;

☐ (b) the IP address of the router's interface in the destination address field in the header of the IP datagram;

☐ (c) the number 22 in the destination port field in the header of the transport layer segment.

23. Assume that the only columns of each of the tables `R`, `S`, `T` are `A` and `B`, and that each of these tables contains n rows. It follows that the query

```
SELECT * FROM R, S, T WHERE R.B = S.A AND S.B = T.A AND T.B = R.A
```

returns

☐ (a) at most n rows;

☐ (b) at most n^2 rows;

☐ (c) at most n^3 rows.

24. Assume that table R has exactly 3 columns. It follows that

- (a) $\pi_1(\mathtt{R}) \cup \pi_2(\mathtt{R}) \cup \pi_3(\mathtt{R})$ has exactly 3 columns;
- (b) $\pi_1(\mathtt{R}) \times \pi_1(\mathtt{R}) \times \pi_1(\mathtt{R})$ has exactly 3 columns;
- (c) $\mathtt{R} \times \mathtt{R}$ has exactly 3 columns.

25. Consider the following program excerpt:

```
process A {
  x = 2;
  B!x;
  B?x;
}

process B {
  y = 3;
  A!y;
  A?y;
}
```

Sequential processes A and B execute concurrently. The instruction A!x means that the value of variable x is sent to process A. The instruction A?x means that a value is received from process A and stored in variable x. The operator = means assignment. Communication is synchronous in the distributed model. It follows that:

- (a) x and y are shared variables;
- (b) process A will end, and when it ends, variable x has value 3;
- (c) process B will end, and when it ends, variable y has value 2.

26. On a system with a memory arbiter, two processes A and B were started concurrently. Weak semaphores are used to synchronize them.

```
int x = 0;
semaphore S1 = 1;
semaphore S2 = 1;

process A {
  for (int i = 0; i < 5; i++) {
    P(S1);
    x++;
    V(S1);
  }
}

process B {
  for (int i = 0; i < 5; i++) {
    P(S2);
    x++;
    V(S2);
  }
}
```

After the processes have finished

- (a) the value of variable x is not greater than 10;
- (b) the value of variable x is not less than 5;
- (c) the value of variable x is equal to 10.

27. In a processor management strategy that uses the multilevel feedback queue method

(a) scheduling strategies are specified for each queue;

(b) rules for moving processes between queues are specified;

(c) the queue of processes with the lowest priority is scheduled without preemption.

28. Consider the following program in Python

```
somelist = {"a", "b", "c"}
s = ""
for i in somelist:
    if len(s) > 0:
     s += ":" + i
    else:
       s += i
```

The intention behind this code is that after its execution the variable s holds the string a:b:c. However, some of its executions give a different result. Every its execution will give the intended result after

(a) proper corrections to code indentation are done;

(b) somelist is initialized to a list of elements "a", "b", "c";

(c) both occurrences of += are replaced with :=.

29. Consider the following snippet of Python code

```
    class Fruit:
        def __init__(self, kind):
            self.kind = kind

    class Apple(Fruit):
        def __init__(self, kind, color):
            self.color = color

    x = Apple('rennet', 'red')
```

After this snippet is executed the instruction

(a) **print**(x.color) will print 'red';

(b) **print**(x.kind) will print 'rennet';

(c) **print**(x.kind) will print 'red'.

30. Consider three ways to represent a container in Python

```
container1 = list(range(1, 202200))
container2 = set(range(1, 202200))
container3 = tuple(range(1, 202200))
```

The longest time of iteration over the container is in the case of

(a) container1;

(b) container2;

(c) container3.