A Coinductive Confluence Proof for Infinitary Lambda-Calculus

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Presentation plan

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Presentation plan

1. Coinduction.

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Presentation plan

1. Coinduction.
2. Infinitary lambda-calculus.

Coincution

Coinductive definitions

Coinduction

Coinductive definitions

$$\mathbb{T} \quad ::= \quad V \parallel A(\mathbb{T}) \parallel B(\mathbb{T}, \mathbb{T})$$

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The set $\mathbb{T}$ consists of all *finite and infinite* terms built up from variables and the constructors A and B.

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Coinductive definitions

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The set $\mathbb{T}$ consists of all *finite and infinite* terms built up from variables and the constructors A and B.

The set of all possibly infinite labelled trees with labels specified by the grammar.

Coindution

Guarded corecursion

For $t \in \mathbb{T}$, $x \in V$, $\mathtt{subst}_x^t : \mathbb{T} \to \mathbb{T}$.

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Guarded corecursion

For $t \in \mathbb{T}$, $x \in V$, $\mathtt{subst}^t_x : \mathbb{T} \to \mathbb{T}$.

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\mathtt{subst}^t_x(x) & = & t \\
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\mathtt{subst}^t_x(B(s_1, s_2)) & = & B(\mathtt{subst}^t_x(s_1), \mathtt{subst}^t_x(s_2))
\end{array}$$

Each (co)recursive call of $\mathtt{subst}^t_x$ occurs *directly* inside a constructor for $\mathbb{T}$.

Coinduction

Coinductive definitions of relations

$$\frac{}{x \to x}\;(1) \qquad \frac{t \to t'}{A(t) \to A(t')}\;(2)$$

$$\frac{s \to s' \quad t \to t'}{B(s,t) \to B(s',t')}\;(3) \qquad \frac{t \to t'}{A(t) \to B(t',t')}\;(4)$$

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The relation $\to$ is the greatest fixpoint νF of a function $F : \mathcal{P}(\mathbb{T} \times \mathbb{T}) \to \mathcal{P}(\mathbb{T} \times \mathbb{T})$ defined as follows.

$$\begin{array}{rcl} F(R) & = & \{\langle t_1, t_2\rangle \mid (t_1 \equiv t_2 \equiv x) \vee \\ & & \exists t, t' \,(t_1 \equiv A(t) \wedge t_2 \equiv A(t') \wedge R(t,t')) \vee \\ & & \exists s, t, s', t' (t_1 \equiv B(s,t) \wedge t_2 \equiv B(s',t') \wedge \\ & & \qquad R(s,s') \wedge R(t,t')) \vee \\ & & \exists t, t' \,(t_1 \equiv A(t) \wedge t_2 \equiv B(t',t') \wedge R(t,t'))\} \end{array}$$

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& \qquad\qquad R(s,s') \wedge R(t,t')) \vee \\
& \exists t, t' \, (t_1 \equiv A(t) \wedge t_2 \equiv B(t',t') \wedge R(t,t'))\}
\end{aligned}$$

F is monotone, i.e., $F(R) \subseteq F(S)$ for $R \subseteq S$.

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Sample coinductive proof

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$\square$

Coindiction

Usual coinduction principle

Monotone $F : \mathcal{P}(A) \to \mathcal{P}(A)$ for some set A.

Coindution

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Monotone $F : \mathcal{P}(A) \to \mathcal{P}(A)$ for some set A.

By the Knaster-Tarski fixpoint theorem:

$$\mu F = \bigcap \{X \in \mathcal{P}(A) \mid F(X) \subseteq X\}$$

$$\nu F = \bigcup \{X \in \mathcal{P}(A) \mid X \subseteq F(X)\}.$$

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$$\nu F = \bigcup \{X \in \mathcal{P}(A) \mid X \subseteq F(X)\}.$$

This yields the following proof principles

$$\frac{F(X) \subseteq X}{\mu F \subseteq X}\ (\mathrm{IND}) \qquad \frac{X \subseteq F(X)}{X \subseteq \nu F}\ (\mathrm{COIND})$$

where $X \in \mathcal{P}(A)$.

Coinduction

Alternative characterisation of νF

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- $\nu^0 F = A$,
- $\nu^{\alpha+1} F = F(\nu^\alpha F)$,
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There exists an ordinal ζ such that $\nu^\zeta F = \nu F$.

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There exists an ordinal ζ such that $\nu^\zeta F = \nu F$.

(In all definitions in the paper we actually have $\zeta = \omega$)

Coinduction

Coinductive definitions of relations

$$\frac{}{x \to^{\alpha+1} x}\ (1) \qquad \frac{t \to^{\alpha} t'}{A(t) \to^{\alpha+1} A(t')}\ (2)$$

$$\frac{s \to^{\alpha} s' \quad t \to^{\alpha} t'}{B(s,t) \to^{\alpha+1} B(s',t')}\ (3) \qquad \frac{t \to^{\alpha} t'}{A(t) \to^{\alpha+1} B(t',t')}\ (4)$$

Where $\to^{\alpha} = \nu^{\alpha} F$.

We show: $\forall x \in A(\varphi(x) \to x \in \nu F)$.

Instead show: $\forall x \in A(\varphi(x) \to x \in \nu^{\alpha} F)$ by transfinite induction on α.

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- $\alpha = 0$: $\forall x \in A(\varphi(x) \to x \in A)$ holds trivially.

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Instead show: $\forall x \in A(\varphi(x) \to x \in \nu^\alpha F)$ by transfinite induction on α.

- $\alpha = 0$: $\forall x \in A(\varphi(x) \to x \in A)$ holds trivially.
- α a limit ordinal: $\forall x \in A(\varphi(x) \to x \in \bigcap_{\beta<\alpha} \nu^\beta F)$ is equivalent to the inductive hypothesis $\forall \beta < \alpha \forall x \in A(\varphi(x) \to x \in \nu^\beta F)$.

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- α a limit ordinal: $\forall x \in A(\varphi(x) \to x \in \bigcap_{\beta<\alpha} \nu^\beta F)$ is equivalent to the inductive hypothesis $\forall \beta < \alpha \forall x \in A(\varphi(x) \to x \in \nu^\beta F)$.
- So it remains to show the inductive step for α a successor ordinal.

We show: for all $t \in \mathbb{T}$, $t \to t$.

If $t \equiv x$ then this follows by rule (1).

If $t \equiv A(t')$ then $t' \to t'$ by the coinductive hypothesis, so $t \equiv A(t') \to A(t') \equiv t$ by rule (2).

If $t \equiv B(t_1, t_2)$ then $t_1 \to t_1$ and $t_2 \to t_2$ by the coinductive hypothesis, so $t \to t$ by rule (3).

$\square$

Coindcution

Sample coinductive proof

We show: for all $t \in \mathbb{T}$, $t \to^{\alpha} t$ (just the inductive step for $\alpha + 1$).

If $t \equiv x$ then this follows by rule (1).

If $t \equiv A(t')$ then $t' \to^{\alpha} t'$ by the coinductive hypothesis, so $t \equiv A(t') \to^{\alpha+1} A(t') \equiv t$ by rule (2).

If $t \equiv B(t_1, t_2)$ then $t_1 \to^{\alpha} t_1$ and $t_2 \to^{\alpha} t_2$ by the coinductive hypothesis, so $t \to^{\alpha+1} t$ by rule (3).

$\square$

Coindaction

Existentials

Monotone $F : \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$ for some set A.

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Monotone $F : \mathcal{P}(A \times A) \to \mathcal{P}(A \times A)$ for some set A.

How to show statements of the following form?

$$\forall x, y, z \in A(\varphi(x, y, z) \to \exists a \in A(\langle x, a\rangle \in \nu F \wedge \langle y, a\rangle \in \nu F))$$

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Existentials

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$$\forall x, y, z \in A(\varphi(x, y, z) \to \exists a \in A(\langle x, a \rangle \in \nu F \wedge \langle y, a \rangle \in \nu F))$$

Instead show

$$\forall x, y, z \in A(\varphi(x, y, z) \to (\langle x, f(x, y, z)\rangle \in \nu F \wedge \langle y, f(x, y, z)\rangle \in \nu F))$$

for some function $f : A^3 \to A$.

Coindcution

Sample coinductive proof with existentials

We show: for all $s, t, t' \in \mathbb{T}$, if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

Coindution

Sample coinductive proof with existentials

We show: for all $s, t, t' \in \mathbb{T}$, if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

It suffices to find a function $f : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ such that:

($\star$) if $s \to t$ and $s \to t'$ then $t \to f(t, t')$ and $t' \to f(t, t')$.

Coinduction

Sample coinductive proof with existentials

The rules for $\to$ suggest a definition of f:

$$
\begin{array}{rcl}
f(x,x) & = & x \\
f(A(t),A(t')) & = & A(f(t,t')) \\
f(A(t),B(t',t')) & = & B(f(t,t'),f(t,t')) \\
f(B(t,t),A(t')) & = & B(f(t,t'),f(t,t')) \\
f(B(t,t),B(t',t')) & = & B(f(t,t'),f(t,t')) \\
f(B(t_1,t_2),B(t_1',t_2')) & = & B(f(t_1,t_1'),f(t_2,t_2')) \\
f(t,t') & = & \text{some arbitrary term} \\
 & & \text{if none of the above matches}
\end{array}
$$

The definition is guarded, so f is well-defined.

Coindution

Sample coinductive proof with existentials

We show one case of a coinductive proof of:

$(\star)$ if $s \to t$ and $s \to t'$ then $t \to f(t, t')$ and $t' \to f(t, t')$.

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Sample coinductive proof with existentials

We show one case of a coinductive proof of:

($\star$) if $s \to t$ and $s \to t'$ then $t \to f(t,t')$ and $t' \to f(t,t')$.

If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$,

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If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis $t_1 \to f(s_1, t_1, t_1')$ and $t_1' \to f(s_1, t_1, t_1')$.

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Sample coinductive proof with existentials

We show one case of a coinductive proof of:

$(\star)$ if $s \to t$ and $s \to t'$ then $t \to f(t, t')$ and $t' \to f(t, t')$.

If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t'_1)$ with $s_1 \to t_1$ and $s_1 \to t'_1$, then by the coinductive hypothesis $t_1 \to f(s_1, t_1, t'_1)$ and $t'_1 \to f(s_1, t_1, t'_1)$. We have $f(s, t, t') \equiv A(f(s_1, t_1, t'_1))$.

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Sample coinductive proof with existentials

We show one case of a coinductive proof of:

($\star$) if $s \to t$ and $s \to t'$ then $t \to f(t, t')$ and $t' \to f(t, t')$.

If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis $t_1 \to f(s_1, t_1, t_1')$ and $t_1' \to f(s_1, t_1, t_1')$. We have $f(s, t, t') \equiv A(f(s_1, t_1, t_1'))$. Hence $t \equiv A(t_1) \to f(s, t, t')$ and $t \equiv A(t_1') \to f(s, t, t')$, by rule (2).

Coinduction

Sample coinductive proof with existentials – shorter formulation

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We show one case of a coinductive proof of: if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

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We show one case of a coinductive proof of: if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

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If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t'_1)$ with $s_1 \to t_1$ and $s_1 \to t'_1$, then by the coinductive hypothesis we obtain s'_1 such that $t_1 \to s'_1$ and $t'_1 \to s'_1$.

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Sample coinductive proof with existentials – shorter formulation

We show one case of a coinductive proof of: if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis we obtain s_1' such that $t_1 \to s_1'$ and $t_1' \to s_1'$. We have $t \equiv A(t_1) \to A(s_1')$ and $t \equiv A(t_1') \to A(s_1')$, by rule (2).

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We show one case of a coinductive proof of: if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t'_1)$ with $s_1 \to t_1$ and $s_1 \to t'_1$, then by the coinductive hypothesis we obtain s'_1 such that $t_1 \to s'_1$ and $t'_1 \to s'_1$. We have $t \equiv A(t_1) \to A(s'_1)$ and $t \equiv A(t'_1) \to A(s'_1)$, by rule (2). So we may take $s' \equiv A(s'_1)$.

We show one case of a coinductive proof of: if $s \to t$ and $s \to t'$ then there exists $s' \in \mathbb{T}$ with $t \to s'$ and $t' \to s'$.

If $s \equiv A(s_1)$, $t \equiv A(t_1)$ and $t' \equiv A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis we obtain s_1' such that $t_1 \to s_1'$ and $t_1' \to s_1'$. We have $t \equiv A(t_1) \to A(s_1')$ and $t \equiv A(t_1') \to A(s_1')$, by rule (2). So we may take $s' \equiv A(s_1')$.

The definition of $f : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ is left implicit and follows straightforwardly from the given proof.

Definition (Terms)

$$\Lambda^\infty \quad ::= \quad V \parallel \lambda V.\Lambda^\infty \parallel \Lambda^\infty\Lambda^\infty$$

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Definition (Infinitary β-reduction)

$$\frac{s \to_\beta^* x}{s \to_\beta^\infty x}$$

$$\frac{s \to_\beta^* t_1 t_2 \quad t_1 \to_\beta^\infty t_1' \quad t_2 \to_\beta^\infty t_2'}{s \to_\beta^\infty t_1' t_2'}$$

$$\frac{s \to_\beta^* \lambda x.r \quad r \to_\beta^\infty r'}{s \to_\beta^\infty \lambda x.r'}$$

Theorem (Endrullis, Polonsky. TYPES 2011)

$s \to_{\beta}^{\infty} t$ iff there exists a strongly convergent β-reduction from s to t

Definition (Root-stable)

t is *root-stable* if either

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- $t \equiv x$ with $x \not\equiv \bot$,

Definition (Root-stable)

t is *root-stable* if either

- $t \equiv x$ with $x \not\equiv \perp$, or
- $t \equiv \lambda x.t'$,

Definition (Root-stable)

t is *root-stable* if either

- $t \equiv x$ with $x \not\equiv \bot$, or
- $t \equiv \lambda x.t'$, or
- $t \equiv t_1 t_2$ and there does not exist s such that $t_1 \to_\beta^\infty \lambda x.s$.

Definition (Root-stable)

t is *root-stable* if either

- $t \equiv x$ with $x \not\equiv \bot$, or
- $t \equiv \lambda x.t'$, or
- $t \equiv t_1 t_2$ and there does not exist s such that $t_1 \to_\beta^\infty \lambda x.s$.

Definition (Root-active)

t is *root-active* if there does not exist a root-stable s such that $t \to_\beta^\infty s$

Definition (Equivalence of root-active subterms)

$$\frac{t, s \text{ are root-active}}{t \sim s} \qquad \frac{}{x \sim x}$$

$$\frac{t \sim s}{\lambda x.t \sim \lambda x.s} \qquad \frac{t_1 \sim s_1 \quad t_2 \sim s_2}{t_1 t_2 \sim s_1 s_2}$$

Infinitary lambda-calculus

Main result

Theorem (Confluence of infinitary β-reduction up to equivalence of root-active subterms)

If $t \sim t'$, $t \to_\beta^\infty s$ and $t' \to_\beta^\infty s'$, then there exist r, r' such that $s \to_\beta^\infty r$, $s' \to_\beta^\infty r'$ and $r \sim r'$.

Theorem (Confluence of infinitary β-reduction up to equivalence of root-active subterms)

If $t \sim t'$, $t \to_\beta^\infty s$ and $t' \to_\beta^\infty s'$, then there exist r, r' such that $s \to_\beta^\infty r$, $s' \to_\beta^\infty r'$ and $r \sim r'$.

This was already obtained by Kennaway, Klop, Sleep and de Vries in 1995, but we give a different coinductive proof.

Infinitary lambda-calculus

Böhm reduction – definitions

Definition ($\perp$-rules)

$$t \quad \to \quad \perp \qquad \text{if } t \text{ is root-active and } t \not\equiv \perp$$

Infinitary lambda-calculus

Böhm reduction – definitions

Definition ($\bot$-rules)

$$t \quad \to \quad \bot \qquad\qquad \text{if } t \text{ is root-active and } t \not\equiv \bot$$

Definition (Infinitary Böhm reduction)

$$\frac{s \to^{*}_{\beta\bot} x}{s \to^{\infty}_{\beta\bot} x}$$

$$\frac{s \to^{*}_{\beta\bot} t_1 t_2 \quad t_1 \to^{\infty}_{\beta\bot} t_1' \quad t_2 \to^{\infty}_{\beta\bot} t_2'}{s \to^{\infty}_{\beta\bot} t_1' t_2'}$$

$$\frac{s \to^{*}_{\beta\bot} \lambda x.r \quad r \to^{\infty}_{\beta\bot} r'}{s \to^{\infty}_{\beta\bot} \lambda x.r'}$$

Theorem (Confluence of infinitary Böhm reduction)

If $t \to^{\infty}_{\beta\perp} t_1$ and $t \to^{\infty}_{\beta\perp} t_2$ then there exists t_3 such that $t_1 \to^{\infty}_{\beta\perp} t_3$ and $t_2 \to^{\infty}_{\beta\perp} t_3$.

Theorem (Confluence of infinitary Böhm reduction)

If $t \to^{\infty}_{\beta\bot} t_1$ and $t \to^{\infty}_{\beta\bot} t_2$ then there exists t_3 such that $t_1 \to^{\infty}_{\beta\bot} t_3$ and $t_2 \to^{\infty}_{\beta\bot} t_3$.

This was also obtained by Kennaway et al., but our proof uses coinduction.

To prove confluence of infinitary β-reduction up to equivalence of root-active subterms, we introduce ϵ-calculus (similar to the ϵ-calculus in Kennaway et al.).

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Definition (Terms of ϵ-calculus)

$$\Lambda^{\epsilon} \quad ::= \quad V \parallel \lambda V.\Lambda^{\epsilon} \parallel \Lambda^{\epsilon}\Lambda^{\epsilon} \parallel \epsilon(\Lambda^{\epsilon})$$

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Definition (ϵ-contraction)

$$\epsilon^n(\lambda x.s)t \to_\epsilon \epsilon(s[t/x])$$

Definition (Parallel ϵ-reduction)

$$\frac{}{x \to_1 x} \qquad \frac{s \to_1 s'}{\lambda x.s \to_1 \lambda x.s'}$$

$$\frac{s \to_1 s' \quad t \to_1 t'}{st \to_1 s't'} \qquad \frac{t \to_1 t'}{\epsilon(t) \to_1 \epsilon(t')}$$

$$\frac{t_1[t_2/x] \to_1 t'}{\epsilon^n(\lambda x.t_1)t_2 \to_1 \epsilon(t')}$$

Definition (Infinitary ϵ-reduction)

$$\frac{}{x \to_{\epsilon}^{\infty} x} \qquad \frac{s \to_{\epsilon}^{\infty} s'}{\lambda x.s \to_{\epsilon}^{\infty} \lambda x.s'}$$

$$\frac{s \to_{\epsilon}^{\infty} s' \quad t \to_{\epsilon}^{\infty} t'}{st \to_{\epsilon}^{\infty} s't'} \qquad \frac{s \to_{1}^{*} \epsilon(t) \quad t \to_{\epsilon}^{\infty} t'}{s \to_{\epsilon}^{\infty} \epsilon(t')}$$

Infinitary lambda-calculus

Confluence proof

Definition (Erasure)

$$\frac{}{\epsilon^n(x) \succ x} \qquad \frac{s' \text{ is root-active}}{\epsilon^\omega \succ s'}$$

$$\frac{s \succ s'}{\epsilon^n(\lambda x.s) \succ \lambda x.s'} \qquad \frac{s \succ s' \quad t \succ t'}{\epsilon^n(st) \succ s't'}$$

Infinitary lambda-calculus

Confluence proof

$$
\begin{tikzcd}
t \arrow[r, "1"'] \arrow[d, "1"'] & t_1 \arrow[d, dashed, "1"'] \\
t_2 \arrow[r, dashed, "1"'] & t_3
\end{tikzcd}
$$

Infinitary lambda-calculus

Confluence proof

$$
\begin{tikzcd}
t \arrow[r, "1"'] \arrow[d, "1"'] & t_1 \arrow[d, dashed, "1"'] \\
t_2 \arrow[r, dashed, "1"'] & t_3
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[r, "\infty", "\epsilon"'] \arrow[d, "1"'] & t_1 \arrow[d, dashed, "1"'] \\
t_2 \arrow[r, dashed, "\infty", "\epsilon"'] & t_3
\end{tikzcd}
$$

Infinitary lambda-calculus

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$$
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\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[r, "\infty", "\epsilon"'] \arrow[d, "1"'] & t_1 \arrow[d, dashed, "1"'] \\
t_2 \arrow[r, dashed, "\infty", "\epsilon"'] & t_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
t \arrow[r, "\infty", "\epsilon"'] \arrow[d, "\infty", "\epsilon"'] & t_1 \arrow[d, dashed, "\infty", "\epsilon"'] \\
t_2 \arrow[r, dashed, "\infty", "\epsilon"'] & t_3
\end{tikzcd}
$$

Infinitary lambda-calculus

Confluence proof

$$\begin{tikzcd} t \arrow[r, "1"'] \arrow[d, "1"'] & t_1 \arrow[d, dashed, "1"'] \\ t_2 \arrow[r, dashed, "1"'] & t_3 \end{tikzcd} \qquad \begin{tikzcd} t \arrow[r, "\infty", "\epsilon"'] \arrow[d, "1"'] & t_1 \arrow[d, dashed, "1"'] \\ t_2 \arrow[r, dashed, "\infty", "\epsilon"'] & t_3 \end{tikzcd}$$

$$\begin{tikzcd} t \arrow[r, "\infty", "\epsilon"'] \arrow[d, "\infty", "\epsilon"'] & t_1 \arrow[d, dashed, "\infty", "\epsilon"'] \\ t_2 \arrow[r, dashed, "\infty", "\epsilon"'] & t_3 \end{tikzcd}$$

$t_1 \sim t_2$ iff there exists s with $s \succ t_1$ and $s \succ t_2$

Infinitary lambda-calculus

Confluence proof

$$
\begin{tikzcd}
 & t \arrow[dl, "\beta"'] \arrow[r, phantom, "\prec"] & t_0 \arrow[dl, "\epsilon"'] \arrow[dr, "\epsilon"] \arrow[r, phantom, "\succ"] & t' \arrow[dr, "\beta"] & \\
s \arrow[dr, "\beta"'] \arrow[r, phantom, "\prec"] & s_0 \arrow[dr, "\epsilon"'] & & s_0' \arrow[dl, "\epsilon"] \arrow[r, phantom, "\succ"] & s' \arrow[dl, "\beta"] \\
 & r \arrow[r, phantom, "\prec"] & r_0 \arrow[r, phantom, "\succ"] & r' &
\end{tikzcd}
$$