Needed rewriting

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Abstract

We provide a simple proof of an old result of Huet and Lévy that needed reduction is hypernormalizing in orthogonal term rewriting systems. The proof does not require any notion of labeling, permutation equivalence or standardization.

1 Introduction

In this note we present a simple proof of a landmark theorem of Huet and Lévy [4] for orthogonal TRSs: (1) every term not in normal form contains a needed redex and (2) needed reduction is hypernormalizing: if a term t has a normal form then no reduction starting from t contains infinitely many needed steps. A redex r is needed in t if in every reduction from t to normal form some descendant of r is contracted. In other words, contracting a descendant of a needed redex cannot be avoided in a reduction to normal form. A reduction step is needed if it contracts a needed redex. The theorem of Huet and Lévy forms a basis for optimal reductions in orthogonal TRSs (see e.g. [7, Chapter 9]).

There are many proofs that needed reduction is hypernormalizing in orthogonal TRSs [4, 3, 6], [2, Section 8], [7, Section 9.2]. Most of them rely on the notions of Lévy labeling, permutation equivalence or standardization. The proof in the present paper does not use any of these. The paper [3] proves a generalization of the theorem of Huet and Lévy to regular stable sets of results instead of normal forms. The proof is simple and does not employ any labeling or the Standardization Theorem. The method is somewhat different from ours. The proof in the present paper does not easily generalize to the situation in [3]. Our methods bear some similarity to the proof of another generalization of the theorem of Huet and Lévy in [6], where almost orthogonal TRSs with sets of necessary redexes are considered. See also [5].

We show both parts (1) and (2) of the theorem of Huet and Lévy by induction on the length of a shortest reduction to normal form. For part (1) we show that if $t' \leftarrow t$ and the expansion occurs in the redex pattern of a needed redex in t', then the new redex is needed in t. For part (2) we prove that expansion preserves the property of being needed-hypernormalizing (Lemma 4.6). We show that if $t' \leftarrow t$ and t' is needed-hypernormalizing then any reduction from t with infinitely many needed steps may be simulated by a reduction from t' with infinitely many needed steps. For this purpose, we need a variant of the parallel moves lemma which ensures that the projection of a needed step is either empty or contains a needed step (Lemma 4.5).

2 Preliminaries

In this section we fix notation and recall some notions of term rewriting. We assume basic familiarity with term rewriting [1, 7] and define only a few of the less common or standard notions.

By $\to^*$ we denote the reflexive-transitive closure of the relation $\to$, and by $\to^=$ the reflexive closure. By $\to^!$ we denote reduction to normal form, i.e., $t \to^! s$ if $t \to^* s$ and s is in normal form. By $\cdot$ we denote composition of relations, e.g. $t \to \cdot \leftarrow t'$ holds iff there exists s such that $t \to s$ and $s \leftarrow t'$. We use $\to_R$ for the one-step reduction relation of the TRS R, omitting R when clear from the context. A TRS is *orthogonal* if it is non-ambiguous and left-linear. Orthogonal TRSs are known to be confluent [1], i.e., ${}^*\!\leftarrow \cdot \to^* \subseteq \to^* \cdot {}^*\!\leftarrow$ holds for the reduction relation of an orthogonal TRS.

In the rest of this paper we assume a fixed orthogonal TRS R.

A *context* $C[\Box_1, \ldots, \Box_n]$, also denoted just by C, is a term with exactly one occurence of each *hole* $\Box_1, \ldots, \Box_n$. By $C[t_1, \ldots, t_n]$ we denote the term obtained from C by replacing the holes with the terms $t_1, \ldots, t_n$.

A *position* is a finite sequence of natural numbers identifying a subterm occurence in a term. By $\mathrm{Pos}(t)$ we denote the set of positions in a term t, and by $\mathrm{Pos}_V(t)$ the set of variable positions in t. A position p is *above* q, denoted $p \le q$, if there is a position r such that $pr = q$. If p is above q then q is *below* p. Two positions p, q are *parallel*, denoted $p \perp q$ if neither is below the other. We write $p < q$ if $p \le q$ and $p \ne q$. If $p \in \mathrm{Pos}(p)$ then by $t_{|p}$ we denote the subterm of t at position p.

Let $\nu : t \to t'$ be a reduction step contracting the redex at position $p \in \mathrm{Pos}(t)$ using the rewrite rule $l \to r$. Let $q \in \mathrm{Pos}(t)$, The set $q\backslash\nu$ of *descendants* of q in t' by the step q is defined as follows:

$$q\backslash\nu = \begin{cases} \{q\} & \text{if } q < p \text{ or } p \perp q \\ \{pp_3p_2 \mid r_{|p_3} = l_{|p_1}\} & \text{if } q = pp_1p_2 \text{ with } p_1 \in \mathrm{Pos}_V(l) \\ \emptyset & \text{otherwise} \end{cases}$$

The notion of a descendant extends to many-step reduction in an obvious way.

For the sake of readability we often confuse a term (redex) u with its specific occurence in a term t, i.e., with a specific position $p \in \mathrm{Pos}(t)$ such that $t_{|p} = u$. Hence we talk about descendants of a term or a redex. Note that in an orthogonal TRS all descendants of a redex are redexes, by the same rule.

Redexes r_1 and r_2 in t *overlap* if one occurs in the redex pattern of the other, i.e., they are neither parallel nor one occurs below a variable position of the other.

We set $R^\perp = R \cup \{\perp \to \perp\}$ where $\perp$ is a fresh constant not present in the signature of R. In what follows we consider reductions in $R^\perp$, and denote them just by $\to$. By redexes we understand redexes in $R^\perp$. Note that $R^\perp$ is still orthogonal, has the same normal forms as R, and on $\perp$-free terms the reduction relations of R and $R^\perp$ coincide.

The subterm t in $C[t]$ is *needed* if $C[\perp]$ does not have a normal form. A *needed reduction step*, denoted $\to_n$, is a reduction step which reduces a needed redex. A *needed reduction* is a reduction containing only needed reduction steps. A term t is *needed-normalizing* (resp. *needed-hypernormalizing*) if there is no infinite reduction starting at t and containing exclusively (resp. infinitely many) needed reduction steps.

Note that if t is needed-hypernormalizing then it is needed-normalizing. Note also that being needed-normalizing does not necessarily ensure that repeated contraction of needed redexes results in a normal form, because a priori a term may contain only non-needed redexes. In the next section we show this is in fact impossible: every term not in normal form contains a needed redex.

Usually, a needed redex is defined as a redex r in t such that in every reduction from t to normal form a descendant of r is contracted. We now prove that for an orthogonal TRS this is equivalent to our definition. First, we show that $\perp$ subterms cannot influence a reduction to normal form.

Lemma 2.1. *If $C[\bot] \to^! s$ then also $C[t] \to^! s$ for any term t. Moreover, in the reduction $C[t] \to^! s$ no contractions occur in any descendant of t.*

Proof. Because no left-hand sides of rules of R contain $\bot$, in each R-redex the $\bot$ subterms must be below variable positions. Also, the $\bot$-contractions $\bot \to \bot$ may be ignored. Hence all descendants of $\bot$ must be ultimately erased in $C[\bot] \to^* s$ and thus the reduction $C[\bot] \to^* s$ may be simulated from $C[t]$ to yield $C[t] \to^* s$. In the reduction $C[t] \to^! s$ no contractions occur in any descendant of t, because none occured in $C[\bot] \to^* s$ in any descendant of $\bot$. □

Corollary 2.2. *If s is not needed in t then neither is any subterm of s.*

Lemma 2.3. *A redex r in t is needed iff in every reduction from t to normal form a descendant of r is contracted.*

Proof. We show that a redex r in $t = C[r]$ is not needed iff in some reduction from t to normal form no descendant of r is contracted. Assume r is not needed in $C[r]$, i.e., $C[\bot] \to^! s$. Then by Lemma 2.1 there is a reduction $C[r] \to^! s$ in which no descendant of r is contracted. For the other direction, assume $C[r] \to^! s$ is a reduction to normal form in which no descendant of r is contracted. Then all descendants of r are redexes, because the TRS is orthogonal. Hence, all descendants of r must be erased in the reduction $C[r] \to^! s$, and they cannot overlap with any of the contracted redexes. Thus, we may simulate this reduction from $C[\bot]$ to yield $C[\bot] \to^! s$. □

3 Existence of needed redexes

In this section we prove that every term not in normal form contains a needed redex. First, we need an auxiliary lemma.

Lemma 3.1. *If r is a needed redex in t and u occurs in the redex pattern of r, then u is also needed in t.*

Proof. We may assume u occurs in r below the root. We have $t = C[r]$ and $r = C'[u]$ with C' non-empty. Suppose $C[C'[\bot]] \to^! s$. Because the TRS $R^\bot$ is orthogonal, all redexes occuring in $C'[\bot]$, except the $\bot$-redex, must occur below variable positions of r. This implies that any descendant of $C'[\bot]$ has the form $C''[\bot]$ where $C''[u]$ is still a redex and C'' is not empty. So let $C''[\bot]$ be a descendant of $C'[\bot]$. Suppose $C''[\bot]$ occurs in the pattern of some redex r'. Then $\bot$ must occur below a variable position of r', because C'' is non-empty and redex patterns do not contain $\bot$ except possibly at the root. Hence, the redex $C''[u]$ overlaps with r'. Contradiction. We have thus shown that no descendants of $C'[\bot]$ overlap with a redex and all of them contain $\bot$. This implies that all descendants of $C'[\bot]$ must be ultimately erased in the reduction $C[C'[\bot]] \to^* s$. Therefore, like in Lemma 2.1 the reduction $C[C'[\bot]] \to^* s$ may be simulated from $C[\bot]$ to yield $C[\bot] \to^* s$. □

Theorem 3.2. *In an orthogonal TRS, if t contains a redex then it contains a needed redex.*

Proof. If t does not have a normal form then every redex in t is needed. Indeed, if $t = C[u]$ and u is not needed in t, then $C[\bot] \to^! s$, but then $t = C[u] \to^! s$ by Lemma 2.1. So assume t has a normal form. We proceed by induction on the length of a shortest reduction from t to normal form.

If $t \to s$ with s in normal form, then the contracted redex u in t is needed. Indeed, we have $t = C[u]$, and because the TRS is orthogonal, the redex u is outermost in t. Then the $\bot$-redex is the only redex in $C[\bot]$. Thus $C[\bot]$ reduces only to itself, so it has no normal form.

The remaining case is when $t \to t' \to^! s$ and t' contains a needed redex r. Suppose the contracted redex u in $t = C[u] \to C[u'] = t'$ is not needed. Then u' is not needed in $C[u']$. Hence the needed redex r in t' cannot occur in u' by Corollary 2.2. If r occurs parallel to u' then $t = C'[r] \to C''[r] = t'$ and $C'[\bot] \to C''[\bot]$. If $C'[\bot] \to^! s'$ then $C''[\bot] \to^! s'$ by confluence, but this contradicts the fact that r is needed in t'. Thus it remains to consider the case when u' occurs in r. Then $t = C'[r']$, $t' = C'[r]$ and $r' \to r$. Then r' is needed in t. If u' occurs below a variable position of r then r' is still a redex, so it is a needed redex in t. Otherwise u' occurs in the redex pattern of r. Since r is needed in t', by Lemma 3.1 also u' is needed in $t' = C[u']$. But we already concluded that u' is not needed in $C[u']$. Contradiction. □

Corollary 3.3. *If t is needed-normalizing then $t \to_n^! s$ for some normal form s.*

4 Hypernormalization of needed reduction

In this section we show that needed reduction is hypernormalizing, i.e., every term is needed-hypernormalizing. This directly implies the (hyper)normalization of the needed reduction strategy (c.f. [7, Section 9.2.2]).

First we show that if $t \to t'$ and t' is needed-normalizing then for each needed redex u in t which does not overlap with the contracted redex, there exists at least one descendant of u in t' which is needed in t'. For this we need several auxiliary lemmas.

Lemma 4.1. *If s is a non-needed redex in t and $t \to_n t'$ then none of the descendants of s in t' is needed in t'.*

Proof. Because $R^\bot$ is orthogonal, s does not overlap with the contracted redex. We have $t = C[s] \to C'[s_1, \ldots, s_n] = t'$ where $s_1, \ldots, s_n$ are the descendants of s. Also $C[\bot] \to^= C'[\bot, \ldots, \bot]$ (we need $\to^=$ because the contracted redex may occur below a variable position of s). Because s is not needed in $C[s]$, there is s' with $C[\bot] \to^! s'$. By confluence $C'[\bot, \ldots, \bot] \to^! s'$. For $k = 1, \ldots, n$, using Lemma 2.1 repeatedly we obtain

$$C'[s_1, \ldots, s_{k-1}, \bot, s_{k+1}, \ldots, s_n] \to^* s'.$$

Hence none of the s_k is needed in t'. □

Lemma 4.2. *If $t_1, \ldots, t_n$ are non-needed redexes in $C[t_1, \ldots, t_n]$ and $C[t_1, \ldots, t_n] \to_n t$, then there is a context C' and non-needed redexes $t'_1, \ldots, t'_m$ with $t = C'[t'_1, \ldots, t'_m]$ and $C[\bot, \ldots, \bot] \to C'[\bot, \ldots, \bot]$.*

Proof. By Lemma 4.1 none of the descendants of $t_1, \ldots, t_n$ is needed in t. Hence we may take $t'_1, \ldots, t'_m$ to be the descendants of $t_1, \ldots, t_n$. Then $t = C'[t'_1, \ldots, t'_m]$ for an appropriate context C'. Note that by Corollary 2.2 the contraction $C[t_1, \ldots, t_n] \to_n t$ cannot occur inside one of the terms $t_1, \ldots, t_n$. Because the TRS $R^\bot$ is orthogonal and $t_1, \ldots, t_n$ are non-needed redexes, none of $t_1, \ldots, t_n$ overlaps with the contracted redex. Hence, the reduction $C[t_1, \ldots, t_n] \to_n C'[t'_1, \ldots, t'_m]$ occurs in the context. So we have $C[\bot, \ldots, \bot] \to C'[\bot, \ldots, \bot]$. □

Corollary 4.3. *If $t_1, \ldots, t_n$ are non-needed redexes in $C[t_1, \ldots, t_n]$ and $C[t_1, \ldots, t_n] \to_n^! s$, then $C[\bot, \ldots, \bot] \to^! s$.*

Lemma 4.4. *If u is a needed redex in t and $t \rightrightarrows t'$ and u is not one of the contracted redexes and t' is needed-normalizing, then there is a descendant of u in t' which is needed.*

Proof. We have $t = C[u] \rightrightarrows C'[u_1, \ldots, u_m] = t'$ where $u_1, \ldots, u_m$ are the descendants of u. Because u is not a contracted redex and the TRS $R^\perp$ is orthogonal, u does not overlap with any of the contracted redexes. Hence $C[\perp] \rightrightarrows C'[\perp, \ldots, \perp]$. Assume none of $u_1, \ldots, u_m$ is needed in t'. Because t' is needed-normalizing we have $t' = C'[u_1, \ldots, u_m] \to_n^! s$ by Corollary 3.3. Hence by Corollary 4.3 we obtain $C[\perp] \rightrightarrows C'[\perp, \ldots, \perp] \to^* s$. But this implies that u is not needed in t. Contradiction. □

We now prove a sharpening of a variant of the parallel moves lemma.

Lemma 4.5. *If $s \leftarrow t \rightrightarrows t'$ and t' is needed-normalizing then there is s' with $s \rightrightarrows s' \,{}^*\!\!\leftarrow t'$. Moreover, if the step $t \to s$ is needed, then either $t' \to^* s'$ contains a needed step or $t' = s'$.*

Proof. Let u be the redex contracted in $t \to s$. First assume that u is also contracted in $t \rightrightarrows t'$. Then $s \rightrightarrows t'$ because the TRS $R^\perp$ is orthogonal. So we may take $s' = t'$. Now assume u is not contracted in $t \rightrightarrows t'$. Because the TRS $R^\perp$ is orthogonal, by contracting the descendants of u in t' we obtain s' such that $t' \to^* s'$ and $s \rightrightarrows s'$. If u is needed then by Lemma 4.4 one of the descendants of u in t' is needed, so $t' \to^* s'$ contains a needed step. □

The following central lemma shows that expansion preserves needed-hypernormalization.

Lemma 4.6. *If $t \to t'$ and t' is needed-hypernormalizing, then t is needed-hypernormalizing.*

Proof. Suppose $t = t_0 \to t_1 \to t_2 \to \ldots$ is an infinite reduction with infinitely many needed steps. By induction we construct a reduction $t' = t'_0 \to^* t'_1 \to^* t'_2 \to^* \ldots$ such that it contains infinitely many needed steps and $t_n \rightrightarrows t'_n$ for each $n \in \mathbb{N}$. Suppose $t'_0, \ldots, t'_n$ have already been chosen. Hence $t_n \rightrightarrows t'_n$ and $t_n \to t_{n+1}$. Because t' is needed-hypernormalizing and $t' \to^* t'_n$, the term t'_n is also needed-hypernormalizing, hence needed-normalizing. Thus by Lemma 4.5 there exists t'_{n+1} such that $t'_n \to^* t'_{n+1}$ and $t_{n+1} \rightrightarrows t'_{n+1}$, and if $t_n \to t_{n_1}$ is a needed step then either $t'_n \to^* t'_{n+1}$ contains a needed step or $t'_{n+1} = t_{n+1}$. If $t'_{n+1} = t_{n+1}$ then we finish the construction by taking $t'_m = t_m$ for $m > n$. Otherwise we continue with $n+1$, applying Lemma 4.5 as above. Note that by construction if the step $t_n \to t_{n+1}$ is needed then $t'_n \to^* t'_{n+1}$ contains a needed step, except perhaps for a single $n \in \mathbb{N}$. Hence there is a reduction starting at t' with infinitely many needed steps, so t' is not needed-hypernormalizing. Contradiction. □

From the above lemma our main theorem directly follows.

Theorem 4.7. *In an orthogonal TRS, needed reduction is hypernormalizing, i.e., a term has a normal form iff it is needed-hypernormalizing.*

Proof. For the implication from left to right, we proceed by induction on the length of a shortest reduction of a term t to normal form. If t is in normal form then it is needed-hypernormalizing by definition. So assume $t \to t'$ and t' is needed-hypernormalizing. But then t is needed-hypernormalizing by Lemma 4.6.

For the other direction, assume t is needed-hypernormalizing. Then it is needed-normalizing, so it has a normal form by Corollary 3.3. □

Corollary 4.8. *The needed reduction strategy is normalizing, i.e., repeated contraction of needed redexes results in a normal form, whenever the considered term has one.*

Proof. Follows from Theorem 3.2 and Theorem 4.7. □

Remark 4.9. The method of the present paper does not easily adapt e.g. to hyper root-normalization of root-needed reduction [5]. For example, the proof of Corollary 4.3 would fail, because then the endpoint s' of the reduction $C[\perp, \ldots, \perp] \to^* s'$ obtained using Lemma 4.3 may be different from s: it might contain $\perp$ because s might contain non-root-needed redexes.

References

[1] Franz Baader and Tobias Nipkow. *Term Rewriting and All That.* Cambridge University Press, 1999.

[2] Inge Bethke, Jan Willem Klop, and Roel C. de Vrijer. Descendants and origins in term rewriting. *Inf. Comput.*, 159(1-2):59–124, 2000.

[3] John R. W. Glauert, Richard Kennaway, and Zurab Khasidashvili. Stable results and relative normalization. *J. Log. Comput.*, 10(3):323–348, 2000.

[4] Gérard P. Huet and Jean-Jacques Lévy. Computations in orthogonal rewriting systems, I and II. In Jean-Louis Lassez and Gordon D. Plotkin, editors, *Computational Logic - Essays in Honor of Alan Robinson*, pages 395–443. The MIT Press, 1991.

[5] Aart Middeldorp. Call by need computations to root-stable form. In Peter Lee, Fritz Henglein, and Neil D. Jones, editors, *Conference Record of POPL'97: The 24th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, Papers Presented at the Symposium, Paris, France, 15-17 January 1997*, pages 94–105. ACM Press, 1997.

[6] R. C. Sekar and I. V. Ramakrishnan. Programming in equational logic: Beyond strong sequentiality. *Inf. Comput.*, 104(1):78–109, 1993.

[7] Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.