Confluence of nearly orthogonal infinitary term rewriting systems*

Łukasz Czajka

Institute of Informatics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
lukaszcz@mimuw.edu.pl

Abstract

We give a relatively simple coinductive proof of confluence, modulo equivalence of root-active terms, of nearly orthogonal infinitary term rewriting systems. Nearly orthogonal systems allow certain root overlaps, but no non-root overlaps. Using a slightly more complicated method we also show confluence modulo equivalence of hypercollapsing terms. The condition we impose on root overlaps is similar to the condition used by Toyama in the context of finitary rewriting.

1998 ACM Subject Classification F.4.2 Grammars and Other Rewriting Systems

Keywords and phrases infinitary rewriting, confluence, coinduction

Digital Object Identifier 10.4230/LIPIcs.RTA.2015.x

1 Introduction

Infinitary term rewriting extends term rewriting by infinite terms and transfinite reductions. This enables the consideration of "limits" of terms under infinite reduction sequences. For instance, in a term rewriting system with the rule

$$f(a) \to c(f(a))$$

the term $f(a)$ "in the limit" reduces to an infinite term c^ω such that $c^\omega = c(c^\omega)$. In fact, c^ω is the normal form of $f(a)$ in an infinitary term rewriting system (iTRS) containing the above single rule.

In this paper we show confluence modulo equivalence of root-active terms of nearly orthogonal iTRSs. This implies that nearly orthogonal iTRSs have the unique normal forms property. Nearly orthogonal iTRSs allow certain root overlaps, but no non-root overlaps. More precisely, for each root critical pair $\langle t_1, t_2 \rangle$ we require that there exists s such that $t_1 \Rightarrow s$ and $t_2 \to^\infty s$, where $\to^\infty$ is strongly convergent infinitary reduction and $\Rightarrow$ is parallel reduction. Since almost orthogonal (i.e. weakly orthogonal with no non-root overlaps) iTRSs are nearly orthogonal, this shows that the failure of the unique normal forms property in weakly orthogonal iTRSs (see [11, 10]) is due to the possibility of non-root overlaps.

Our proof method is different from [21, 22] and it is relatively simple, but it does not easily generalise to confluence modulo equivalence of hypercollapsing terms. Using a bit more complicated method similar to [21] we also prove confluence modulo equivalence of hypercollapsing terms of nearly orthogonal iTRSs. Because of space limits the details of the second proof were moved to an appendix.

Actually, confluence modulo equivalence of root-active terms follows easily from confluence modulo equivalence of hypercollapsing terms. However, the method of the proof of confluence

* Partially supported by Narodowe Centrum Nauki grant 2012/07/N/ST6/03398.

26th International Conference on Rewriting Techniques and Applications (RTA'15).
Editor: Maribel Fernández; pp. 1–22
Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

modulo equivalence of root-active terms is perhaps more interesting than the result itself, and than the second method which is a coinductive adaptation of [21]. The first method is also slightly simpler. For these reasons we chose to present this method in detail, despite it being less general.

In the context of finitary rewriting, confluence of non-orthogonal TRSs was studied by Huet, Toyama, Gramlich and van Oostrom in [18, 34, 17, 35, 36]. The condition we impose on root critical pairs is similar to the conditions used by Toyama. It is not possible to use conditions similar to those from the cited papers for non-root overlaps, because the unique normal forms property fails already for weakly orthogonal iTRSs [11, 10]. As a counterexample, consider the following weakly orthogonal iTRS from [11, 10].

$$P(S(x)) \to x \qquad S(P(x)) \to x$$

Then $S(P^2(S^3(P^4(\ldots))))$ has two distinct normal forms P^ω and S^ω.

1.1 Related work

Infinitary rewriting was introduced in [22]. For an introduction and a general overview see [21, 13]. A coinductive definition of infinitary reductions which corresponds to strongly convergent reductions was introduced in [15] for the infinitary lambda-calculus. The paper [12] introduces a coinductive definition of infinitary reductions in iTRSs, capturing reductions of arbitrary ordinal length. Our coinductive definition of infinitary reductions is based on [15]. Coinductive techniques in infinitary lambda-calculus were investigated in [20]. In [7] confluence, modulo equivalence of root-active terms, of infinitary lambda-calculus was proven coinductively. A simpler proof method for confluence modulo equivalence of terms with no head normal form was later found in [8]. In this paper the proof of confluence modulo equivalence of root-active terms follows a strategy similar to [8]. It also bears some similarity to the proof of the unique normal forms property of orthogonal iTRSs in [28]. The general strategy of the proof of confluence modulo equivalence of hypercollapsing terms, as well as proofs of some lemmas, are similar to [21]. Some other papers related to the methods of the present work are [1, 2, 24, 25, 26, 27, 23, 14].

2 Coinduction

In this section we give a brief explanation of coinduction as it is used in the present paper. Our style of writing coinductive proofs is perhaps not completely standard, but it is similar to how such proofs are presented in e.g. [15, 4, 31, 30, 29]. However, in contrast to some of these papers, we do not claim that our proofs are a paper presentation of proofs formalised in a proof assistant (though they could probably be formalised in such a system).

First, we give an explanation of how our proofs of existential statements should be interpreted. This is the only part that may be non-obvious to someone already well-acquainted with coinduction. Then we shall give an elementary explanation of coinduction. A reader not familiar with coinduction should perhaps skip the following example and return to it after reading the rest of this section.

▶ **Example 1.** Let T be the set of all finite and infinite terms defined coinductively by

$$T ::= V \parallel A(T) \parallel B(T,T)$$

where V is a countable set of variables, and A, B are constructors. By $x, y, \ldots$ we denote variables, and by $t, s, \ldots$ we denote elements of T. Define a binary relation $\to$ on T

coinductively by the following rules.

$$\frac{}{x \to x}\ (1) \qquad \frac{t \to t'}{A(t) \to A(t')}\ (2) \qquad \frac{s \to s' \quad t \to t'}{B(s,t) \to B(s',t')}\ (3) \qquad \frac{t \to t'}{A(t) \to B(t',t')}\ (4)$$

We want to show: for all $s, t, t' \in T$, if $s \to t$ and $s \to t'$ then there exists $s' \in T$ with $t \to s'$ and $t' \to s'$. The idea is to skolemize this statement. So we need to find a Skolem function $f : T^3 \to T$ which will allow us to prove the Skolem normal form:

$(\star)$ if $s \to t$ and $s \to t'$ then $t \to f(s,t,t')$ and $t' \to f(s,t,t')$.

The rules for $\to$ suggest a definition of f:

$$\begin{array}{rcl} f(x,x,x) & = & x \\ f(A(s),A(t),A(t')) & = & A(f(s,t,t')) \\ f(A(s),A(t),B(t',t')) & = & B(f(s,t,t'),f(s,t,t')) \\ f(A(s),B(t,t),A(t')) & = & B(f(s,t,t'),f(s,t,t')) \\ f(A(s),B(t,t),B(t',t')) & = & B(f(s,t,t'),f(s,t,t')) \\ f(B(s_1,s_2),B(t_1,t_2),B(t_1',t_2')) & = & B(f(s_1,t_1,t_1'),f(s_2,t_2,t_2')) \\ f(s,t,t') & = & \text{some arbitrary term if none of the above matches} \end{array}$$

The definition is guarded, so f is well-defined, i.e., there exists a unique function $f : T^3 \to T$ satisfying the above equations.

We now proceed with a coinductive proof of $(\star)$. Assume $s \to t$ and $s \to t'$. If $s = t = t' = x$ then $f(s,t,t') = x$, and $x \to x$ by rule (1). If $s = A(s_1)$, $t = A(t_1)$ and $t' = A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis $t_1 \to f(s_1,t_1,t_1')$ and $t_1' \to f(s_1,t_1,t_1')$. We have $f(s,t,t') = A(f(s_1,t_1,t_1'))$. Hence $t = A(t_1) \to f(s,t,t')$ and $t = A(t_1') \to f(s,t,t')$, by rule (2). If $s = B(s_1,s_2)$, $t = B(t_1,t_2)$ and $t' = B(t_1',t_2')$, with $s_1 \to t_1$, $s_1 \to t_1'$, $s_2 \to t_2$ and $s_2 \to t_2'$, then by the coinductive hypothesis we have $t_1 \to f(s_1,t_1,t_1')$, $t_1' \to f(s_1,t_1,t_1')$, $t_2 \to f(s_2,t_2,t_2')$ and $t_2' \to f(s_2,t_2,t_2')$. Hence $t = B(t_1,t_2) \to B(f(s_1,t_1,t_1'),f(s_2,t_2,t_2')) = f(s,t,t')$ by rule (3). Analogously, $t' \to f(s,t,t')$ by rule (3). Other cases are similar.

Usually, it is inconvenient to invent the Skolem function beforehand, because the definition of the Skolem function and the coinductive proof of the Skolem normal form are typically interdependent. Therefore, we adopt a style of doing a proof by coinduction of a statement[1]

$$\begin{array}{rcl} \psi(R_1,\ldots,R_m) & = & \forall_{x_1,\ldots,x_n \in T} \,.\, \varphi(\vec{x}) \to \\ & & \quad \exists_{y \in T}.R_1(g_1(\vec{x}),\ldots,g_k(\vec{x}),y) \wedge \ldots \wedge R_m(g_1(\vec{x}),\ldots,g_k(\vec{x}),y) \end{array}$$

with an existential quantifier. We intertwine the corecursive definition of the Skolem function f with a coinductive proof of the Skolem normal form

$$\begin{array}{l} \forall_{x_1,\ldots,x_n \in T} \,.\, \varphi(\vec{x}) \to \\ \quad R_1(g_1(\vec{x}),\ldots,g_k(\vec{x}),f(\vec{x})) \wedge \ldots \wedge R_m(g_1(\vec{x}),\ldots,g_k(\vec{x}),f(\vec{x})) \end{array}$$

We pretend that the coinductive hypothesis[2] is $\psi(R_1^\alpha,\ldots,R_m^\alpha)$. Each element obtained from the existential quantifier in the coinductive hypothesis is interpreted as a corecursive

[1] Here $\varphi(\vec{x})$ is a statement/formula (whatever it means) with only $x_1,\ldots,x_n$ occuring free. We believe that for explanatory purposes it is not necessary to make this any more precise. In general, we abbreviate $x_1,\ldots,x_n$ with $\vec{x}$. The symbols $R_1,\ldots,R_m$ stand for coinductive relations on T, i.e., relations defined as greatest fixpoints of some monotone functions on the powerset of an appropriate cartesian product of T. The symbols $g_1,\ldots,g_k$ denote some functions of $\vec{x}$. The statement φ may contain $R_1,\ldots,R_m$, but their occurences in φ are not affected by substituting different relations in ψ, e.g., if $\psi(R) = \forall_{x\in T}.R(x) \to R(g(x))$ then $\psi(S) = \forall_{x\in T}.R(x) \to S(g(x))$.

[2] We use $R_1^\alpha,\ldots,R_m^\alpha$ for the α-approximants of the coinductive relations $R_1,\ldots,R_m$. A reader confused by this terminology should take a look at our explanation of coinduction after this example.

invocation of the Skolem function. When later we exhibit an element to show the existential subformula of $\psi(R_1^{\alpha+1}, \ldots, R_m^{\alpha+1})$, we interpret this as the definition of the Skolem function in the case specified by the assumptions currently active in the proof. Note that this exhibited element may (or may not) depend on some elements obtained from the existential quantifier in the coinductive hypothesis, i.e., the definition of the Skolem function may involve corecursive invocations.

To illustrate our style of doing coinductive proofs of statements with an existential quantifier, we redo the proof done above. For illustrative purposes, we indicate the arguments of the Skolem function, i.e., we write $s'_{s,t,t'}$ in place of $f(s,t,t')$. These subscripts s,t,t' are normally omitted.

We show by coinduction that if $s \to t$ and $s \to t'$ then there exists $s' \in T$ with $t \to s'$ and $t' \to s'$. Assume $s \to t$ and $s \to t'$. If $s = t = t' = x$ then take $s'_{x,x,x} = x$. If $s = A(s_1)$, $t = A(t_1)$ and $t' = A(t_1')$ with $s_1 \to t_1$ and $s_1 \to t_1'$, then by the coinductive hypothesis we obtain[3] $s'_{s_1,t_1,t_1'}$ with $t_1 \to s'_{s_1,t_1,t_1'}$ and $t_1' \to s'_{s_1,t_1,t_1'}$. Hence $t = A(t_1) \to A(s'_{s_1,t_1,t_1'})$ and $t = A(t_1') \to A(s'_{s_1,t_1,t_1'})$, by rule (2). Thus we may take $s'_{s,t,t'} = A(s'_{s_1,t_1,t_1'})$. If $s = B(s_1,s_2)$, $t = B(t_1,t_2)$ and $t' = B(t_1',t_2')$, with $s_1 \to t_1$, $s_1 \to t_1'$, $s_2 \to t_2$ and $s_2 \to t_2'$, then by the coinductive hypothesis we obtain $s'_{s_1,t_1,t_1'}$ and $s'_{s_2,t_2,t_2'}$ with $t_1 \to s'_{s_1,t_1,t_1'}$, $t_1' \to s'_{s_1,t_1,t_1'}$, $t_2 \to s'_{s_2,t_2,t_2'}$ and $t_2' \to s'_{s_2,t_2,t_2'}$. Hence $t = B(t_1,t_2) \to B(s'_{s_1,t_1,t_1'}, s'_{s_2,t_2,t_2'})$ by rule (3). Analogously, $t' \to B(s'_{s_1,t_1,t_1'}, s'_{s_2,t_2,t_2'})$ by rule (3). Thus we may take $s'_{s,t,t'} = B(s'_{s_1,t_1,t_1'}, s'_{s_2,t_2,t_2'})$. Other cases are similar.

It is quite clear that the above proof, when interpreted in the way outlined before, implicitly defines the Skolem function f. It should be kept in mind that in every case the definition of the Skolem function needs to be guarded. We do not explicitly mention this each time, but verifying this is part of verifying the proof.

At this point a foundationally minded reader might wonder what is exactly the coinduction principle employed in our proofs. The answer to this is simple: whichever you like. With enough patience one could, in principle, reformulate all proofs to directly employ the usual coinduction principle in set theory based on the Knaster-Tarski fixpoint theorem [33]. Since all our proofs and corecursive definitions are actually guarded, one could probably[4] formalise them in a proof assistant based on type theory with a syntactic guardedness check, e.g., in Coq [6, 16]. Perhaps the most straightforward, but maybe not the foundationally nicest, way of justifying our proofs is by reducing coinduction to transfinite induction, as outlined below.

Let T and $\to$ be as in Example 1. Formally, the relation $\to$ is the greatest fixpoint of a monotone $F : \mathcal{P}(T \times T) \to \mathcal{P}(T \times T)$ defined by

$$F(R) = \{\langle t_1, t_2\rangle \mid \exists_{x\in V}(t_1 = t_2 = x) \vee \exists_{t,t'\in T}(t_1 = A(t) \wedge t_2 = B(t',t') \wedge R(t,t')) \vee \ldots\}.$$

Alternatively, using the Knaster-Tarski fixpoint theorem, the relation $\to$ may be characterised as the greatest binary relation on T (i.e. the greatest subset of $T \times T$ w.r.t. set inclusion) such that $\to \subseteq F(\to)$, i.e., such that for every $t_1, t_2 \in T$ with $t_1 \to t_2$ one of the following holds:

1. $t_1 = t_2 = x$ for some variable $x \in V$,
2. $t_1 = A(t)$, $t_2 = A(t')$ with $t \to t'$,

[3] More precisely: by corecursively applying the Skolem function to s_1, t_1, t_1' we obtain $s'_{s_1,t_1,t_1'}$, and by the coinductive hypothesis we have $t_1 \to s'_{s_1,t_1,t_1'}$ and $t_1' \to s'_{s_1,t_1,t_1'}$.

[4] The author has not paid enough attention to the type theory specific details involved in such a formalisation to claim this with complete certainty.

3. $t_1 = B(s,t)$, $t_2 = B(s',t')$ with $s \to s'$ and $t \to t'$,
4. $t_1 = A(t)$, $t_2 = B(t',t')$ with $t \to t'$.

Yet another way to think about $\to$ is that $t_1 \to t_2$ holds if and only if there exists a *potentially infinite* derivation tree of $t_1 \to t_2$ built using the rules (1) – (4).

The rules (1) – (4) could also be interpreted inductively to yield the least fixpoint of F. This is the conventional interpretation, and it is indicated with a single line in each rule separating premises from the conclusion. A coinductive interpretation is indicated with double lines.

The greatest fixpoint $\to$ of F may be obtained by transfinitely iterating F starting with $T \times T$. More precisely, define an ordinal-indexed sequence $(\to^\alpha)_\alpha$ by:

- $\to^0 = T \times T$,
- $\to^{\alpha+1} = F(\to^\alpha)$,
- $\to^\lambda = \bigcap_{\alpha<\lambda} \to^\alpha$ for a limit ordinal λ.

Then there exists an ordinal ζ such that $\to\ = \ \to^\zeta$. Note also that $\to^\alpha \subseteq \to^\beta$ for $\alpha \geq \beta$ (we often use this fact implicitly). See e.g. [9, Chapter 8]. The relation $\to^\alpha$ is called the α-approximant of $\to$. Note that the α-approximants depend on a particular definition of $\to$ (i.e. on the function F), not solely on the relation $\to$ itself.

It is instructive to note that the coinductive rules for $\to$ may also be interpreted as giving rules for the $\alpha+1$-approximants, for any ordinal α.

$$\frac{}{x \to^{\alpha+1} x}\ (1) \qquad \frac{t \to^\alpha t'}{A(t) \to^{\alpha+1} A(t')}\ (2) \qquad \frac{s \to^\alpha s' \quad t \to^\alpha t'}{B(s,t) \to^{\alpha+1} B(s',t')}\ (3) \qquad \frac{t \to^\alpha t'}{A(t) \to^{\alpha+1} B(t',t')}\ (4)$$

In this paper we are interested in proving by coinduction statements of the form[5]

$$\psi(R_1,\ldots,R_m) = \forall x_1 \ldots x_n . \varphi(\vec{x}) \to R_1(g_1(\vec{x}),\ldots,g_k(\vec{x})) \wedge \ldots \wedge R_m(g_1(\vec{x}),\ldots,g_k(\vec{x})).$$

Statements with an existential quantifier may be reduced to statements of this form by skolemizing, as explained in Example 1.

To prove $\psi(R_1,\ldots,R_m)$ it suffices to show by transfinite induction that $\psi(R_1^\alpha,\ldots,R_m^\alpha)$ holds for each ordinal $\alpha \leq \zeta$, where R_i^α is the α-approximant of R_i. The reader may easily check that because of the special form of ψ and the fact that R_i^0 is the full relation, the base case $\alpha = 0$ and the cases of α a limit ordinal are trivial. Hence it remains to show the inductive step for α a successor ordinal. It turns out that a coinductive proof of ψ may be interpreted as a proof of this inductive step for a successor ordinal, with the ordinals left implicit and the phrase "coinductive hypothesis" used instead of "inductive hypothesis".

▶ **Example 2.** On terms from T (see Example 1) we define the operation of substitution by guarded corecursion.

$$\begin{array}{rclcrcl} y[t/x] & = & y \quad \text{if } x \neq y & \qquad & (A(s))[t/x] & = & A(s[t/x]) \\ x[t/x] & = & t & & (B(s_1,s_2))[t/x] & = & B(s_1[t/x], s_2[t/x]) \end{array}$$

We show by coinduction: if $s \to s'$ and $t \to t'$ then $s[t/x] \to s'[t'/x]$, where $\to$ is the relation from Example 1. Formally, the statement we show by transfinite induction on $\alpha \leq \zeta$ is: for $s,s',t,t' \in T$, if $s \to s'$ and $t \to t'$ then $s[t/x] \to^\alpha s'[t'/x]$. For illustrative purposes, we indicate the α-approximants with appropriate ordinal superscripts, but it is customary to omit these superscripts.

[5] See footnote 1.

Let us proceed with the proof. The proof is by coinduction with case analysis on $s \to s'$. If $s = s' = y$ with $y \neq x$, then $s[t/x] = y = s'[t'/x]$. If $s = s' = x$ then $s[t/x] = t \to^{\alpha+1} t' = s'[t'/x]$ (note that $\to = \to^{\zeta} \subseteq \to^{\alpha+1}$). If $s = A(s_1)$, $s' = A(s_1')$ and $s_1 \to s_1'$, then $s_1[t/x] \to^{\alpha} s_1'[t'/x]$ by the coinductive hypothesis. Thus $s[t/x] = A(s_1[t/x]) \to^{\alpha+1} A(s_1'[t'/x]) = s'[t'/x]$ by rule (2). If $s = B(s_1, s_2)$, $s' = B(s_1', s_2')$ then the proof is analogous. If $s = A(s_1)$, $s' = B(s_1', s_1')$ and $s_1 \to s_1'$, then the proof is also similar. Indeed, by the coinductive hypothesis we have $s_1[t/x] \to^{\alpha} s_1'[t'/x]$, so $s[t/x] = A(s_1[t/x]) \to^{\alpha+1} B(s_1'[t'/x], s_1'[t'/x]) = s'[t'/x]$ by rule (4).

The reduction of coinduction to transfinite induction outlined here gives a simple criterion to check the correctness of coinductive proofs, using established principles. However, it is perhaps not the best way to understand coinduction intuitively. The author's intuition is that, in the context of the present paper, coinduction formalises the "and so on" arguments quite common when informally explaining proofs of properties of infinite discrete structures.[6] Such intuitions are necessarily vague and can only be shaped through experience.

One thing that remains to be explained is what guarded corecursion is, and why the equations given above define the substitution operation uniquely. However, the author hopes this part is fairly standard and well-understood. Intuitively, guardedness means that each corecursive invocation has to be fed directly as an argument to a constructor, and the result of this cannot be manipulated further.

In practice, when doing proofs by coinduction the following simple but a bit informal criteria need to be kept in mind.

- When we conclude from the coinductive hypothesis that a certain relation $R(t_1, \ldots, t_n)$ holds, this really means that only its approximant $R^{\alpha}(t_1, \ldots, t_n)$ holds. Usually, we need to infer that the next approximant $R^{\alpha+1}(s_1, \ldots, s_n)$ holds (for some other elements $s_1, \ldots, s_n$) by using $R^{\alpha}(t_1, \ldots, t_n)$ as a premise of an appropriate rule. But we cannot, e.g., inspect (do case reasoning on) $R^{\alpha}(t_1, \ldots, t_n)$, use it in any lemmas, or otherwise treat it as $R(t_1, \ldots, t_n)$.
- An element e obtained from an existential quantifier in the coinductive hypothesis is not really the element itself, but a corecursive invocation of the implicit Skolem function. Usually, we need to put it inside some constructor c, e.g. producing $c(e)$, and then exhibit $c(e)$ in the proof of an existential statement. Applying at least one constructor to e is necessary to ensure guardedness of the implicit Skolem function. But we cannot, e.g., inspect e, apply some previously defined functions to it, or otherwise treat it as if it was really given to us.
- In the proofs of existential statements, the implicit Skolem function cannot depend on the ordinal α. However, this is the case as long as we do not violate the first point, because if the ordinals are never mentioned and we do not inspect the approximants obtained from the coinductive hypothesis, then there is no way in which we could possibly introduce a dependency on α.

The above explanation of coinduction is generalised and elaborated in much more detail in [8]. Also [29] may be helpful as it gives many examples of coinductive proofs written in a style similar to the one used here. The book [33] is an elementary introduction to coinduction and bisimulation (but the proofs there are written in a different style than here). A good

[6] How does one show that a Böhm tree M of a finite lambda-term does not contain β-redexes? If $M = \bot$ then it is obvious. Otherwise $M = \lambda x_1 \ldots x_n.yM_1 \ldots M_m$ does not contain β-redexes, except perhaps in $M_1, \ldots, M_m$. And so on, we continue the argument for $M_1, \ldots, M_m$.

way of learning coinduction is by doing non-trivial coinductive proofs. Some people may initially find a proof assistant helpful for this purpose. The chapters [3, 5] explain coinduction in Coq from a practical viewpoint. A reader interested in foundational matters should also consult [19, 32] which deal with the coalgebraic approach to coinduction.

In the rest of this paper we shall freely use coinduction in the style explained above, giving routine coinductive proofs in as much (or as little) detail as it is customary with inductive proofs of analogous difficulty. After all, our aim is to prove results in infinitary rewriting, not to give a mathematically trivial coinduction tutorial. A reader not familiar with coinduction should treat the apparent difficulty of some proofs as an opportunity to learn doing non-trivial proofs by coinduction.

3 Infinitary term rewiting systems

▶ **Definition 3.** A *signature* is a set of symbols with associated arities. By $T(\Sigma)$ we denote the set of finite terms over a signature Σ. By $T^\infty(\Sigma)$ we denote the set of finite and infinite terms over Σ. We denote the set of variables by V. Formally, a term $t \in T^\infty(\Sigma)$ is a partial function from $\mathbb{N}^*$ to $\Sigma \cup V$, satisfying appropriate conditions, see e.g. [21]. A *position* is an element of $\mathbb{N}^*$. A position p is *below* q if q is a prefix of p (not necessarily proper prefix – we allow $p = q$). The subterm at a given position is defined in the standard way. See e.g. [21] for details. The set of terms $T^\infty(\Sigma)$ could also be defined coinductively, giving essentially the same thing.

A *rewrite rule* is a pair $\langle l, r\rangle \in T(\Sigma) \times T^\infty(\Sigma)$ such that l is not a variable and all variables of r are present in l. Note that we require l to be finite. An *infinitary term rewriting system* (iTRS) is a pair $\mathcal{S} = \langle \Sigma, S\rangle$ where Σ is a signature and S a set of rewrite rules. We often confuse $\mathcal{S}$ with S. A *substitution* is a function from V to $T^\infty(\Sigma)$. A substitution σ is extended to a function $\sigma^* : T^\infty(\Sigma) \to T^\infty(\Sigma)$ coinductively.

$$\begin{array}{rcl} \sigma^*(x) & = & \sigma(x) \\ \sigma^*(f(t_1, \ldots, t_n)) & = & f(\sigma^*(t_1), \ldots, \sigma^*(t_n)) \end{array}$$

We often confuse σ^* with σ.

In what follows by a term we mean a member of $T^\infty(\Sigma)$, unless otherwise qualified. We use $=$ to denote identity of terms. Unless otherwise stated, we use $t, s, r, \ldots$ for terms, and $x, y, \ldots$ for variables, and $f, g, \ldots$ for symbols in Σ, and $\sigma, \sigma', \ldots$ for substitutions.

Let S be an iTRS. A term t is an S*-redex* by a rule $\langle l, r\rangle \in S$ with substitution σ, and s is its S*-reduct*, if $\sigma(l) = t$ and $\sigma(r) = s$. We define the relation $\overline{S} \subseteq T^\infty(\Sigma) \times T^\infty(\Sigma)$ by: $\langle t, s\rangle \in \overline{S}$ iff t is an S-redex and s its S-reduct. The *compatible closure* $\to_S$ of S is defined inductively by the following rules.

$$\frac{\langle t, s\rangle \in \overline{S}}{t \to_S s} \qquad \frac{t \to_S s}{f(t_1, \ldots, t_{k-1}, t, t_{k+1}, \ldots, t_n) \to_S f(t_1, \ldots, t_{k-1}, s, t_{k+1}, \ldots, t_n)}$$

By $\to_S^*$ we denote the transitive-reflexive closure of $\to_S$, and by $\to_S^{\equiv}$ the reflexive closure of $\to_S$. The *parallel closure* $\Rightarrow_S$ of S is defined coinductively.

$$\frac{\langle t, s\rangle \in \overline{S}}{\overline{\overline{t \Rightarrow_S s}}} \qquad \frac{}{\overline{\overline{x \Rightarrow_S x}}} \qquad \frac{t_i \Rightarrow_S t_i' \text{ for } i = 1, \ldots, n}{\overline{\overline{f(t_1, \ldots, t_n) \Rightarrow_S f(t_1', \ldots, t_n')}}}$$

Given a set $\mathcal{U}$ of terms we define the relation $\sim_{\mathcal{U}}$ analogously to parallel closure except that in the premise of the first rule we use $t, s \in \mathcal{U}$. The *infinitary closure* $\to_S^\infty$ of S is defined coinductively by the following rules.

$$\frac{t \to_S^* x}{t \to_S^\infty x} \qquad \frac{t \to_S^* f(t_1, \ldots, t_n) \quad t_i \to_S^\infty t_i' \text{ for } i = 1, \ldots, n}{t \to_S^\infty f(t_1', \ldots, t_n')}$$

We define $\to_S^{2\infty}$ in an analogous way to $\to_S^\infty$ except that in the first premise of the second rule we use $t \to_S^\infty f(t_1, \ldots, t_n)$.

A term l is *linear* if no variable occurs in l more than once. A rule $\langle l, r\rangle \in S$ is left-linear if l is linear. An iTRS S is left-linear if every rule in S is left-linear. Two rules $\langle l_1, r_1\rangle$ and $\langle l_2, r_2\rangle$ (renamed to have no variables in common) *overlap* if l_1 unifies with a non-variable subterm of l_2, or vice versa. We say that $\langle l_1, r_1\rangle$ and $\langle l_2, r_2\rangle$ *overlap at the root*, or that they form a *root overlap*, when l_1 unifies with l_2. An iTRS S is *nearly orthogonal* if it is left-linear, there are no non-root overlaps and for all rules $\langle l_1, r_1\rangle, \langle l_2, r_2\rangle \in S$ overlapping at the root there is s such that $\sigma(r_1) \to_S^\infty s$ and $\sigma(r_2) \Rightarrow_S s$, where σ is the mgu of l_1 and l_2 (note that this also implies that there is s' with $\sigma(r_1) \Rightarrow_S s'$ and $\sigma(r_2) \to_S^\infty s'$).

A rule $\langle l, r\rangle \in S$ is *collapsing* if r is a variable. A term t is a *collapsing redex* if it is a redex by a collapsing rule. A term t is *collapse-stable* if there is no collapsing redex s with $t \to_S^\infty s$. A term t is *hypercollapsing* if there is no collapse-stable s with $t \to_S^\infty s$. In other words, t is hypercollapsing if for every s with $t \to_S^\infty s$ there is a collapsing redex u such that $s \to_S^\infty u$. A term t is *root-stable* if there is no redex s with $t \to_S^\infty s$. A term t is *root-active* if there is no root-stable s with $t \to_S^\infty s$. By $\mathcal{H}$ we denote the set of hypercollapsing terms, and by $\mathcal{R}$ the set of root-active terms. Let $\mathcal{U}$ be a set of terms. An iTRS S is *confluent modulo* $\sim_\mathcal{U}$ when the following condition holds: if $t \sim_\mathcal{U} s$, $t \to_S^\infty t'$ and $s \to_S^\infty s'$ then there exist t'', s'' such that $t'' \sim_\mathcal{U} s''$, $t' \to_S^\infty t''$ and $s' \to_S^\infty s''$. An iTRS S has the *unique normal forms property* when the following condition holds: if $t \to_S^\infty t'$, $t \to_S^\infty t''$ and t', t'' are in normal form, then $t' = t''$.

The relation $\to_S$ is often called the *contraction relation* of S, and $\to_S^*$ the *reduction relation*. A *root contraction* is a contraction $t \to_S s$ such that t is the contracted redex. A *collapsing contraction* is a contraction of a collapsing redex.

The standard notion of an infinitary reduction is that of a strongly convergent reduction. In an appendix we prove that for left-linear iTRSs our coinductive definition corresponds, in the sense of existence, to strongly convergent reductions. The proof of this fact is a straightforward adaptation of [15, Theorem 3]. As a side-effect, this also yields a proof of the Compression Lemma for left-linear iTRSs.

▶ **Example 4.** Let $t_1 = A(B(t_1))$ and $t_2 = A(C(t_2))$. The following is an example of a nearly orthogonal iTRS. By capital letters we denote function symbols.

$$Z \to t_1 \qquad Z \to t_2 \qquad B(x) \to C(x) \qquad C(x) \to B(x) \qquad B(x) \to x \qquad C(x) \to x$$

Let $s_1 = M(A, s_1)$ and $s_2 = M(B, s_2)$. Here is another example of a nearly orthogonal iTRS.

$$Z \to s_1 \qquad Z \to s_2 \qquad A \to C \qquad B \to C$$

The following iTRS is *not* nearly orthogonal.

$$Z \to t_1 \qquad Z \to t_2 \qquad B(x) \to A(x) \qquad C(x) \to A(x)$$

Neither is this one: $\quad Z \to t_1 \qquad Z \to t_2 \qquad B(x) \to C(x)$.

The standard counterexample shows that it is not sufficient to require joinability of root critical pairs: $\quad A \to B \qquad B \to A \qquad A \to C \qquad B \to D$.

The following simple lemma will often be used implicitly.

▶ **Lemma 5.** *Let S an iTRS and $\mathcal{U}$ a set of terms. Then the following conditions hold for all terms t, s, s':*
1. $t \to_S^\infty t$ *and* $t \sim_{\mathcal{U}} t$,
2. *if* $t \to_S^* s \to_S^\infty s'$ *then* $t \to_S^\infty s'$,
3. *if* $t \to_S^* s$ *then* $t \to_S^\infty s$,
4. *if* $t \Rightarrow_S s$ *then* $t \to_S^\infty s$,
5. *if* $t \sim_{\mathcal{U}} s$ *then* $s \sim_{\mathcal{U}} t$,
6. *if* $t \to_S s$ *(respectively* $t \Rightarrow_S s$ *or* $t \to_S^\infty s$*) then* $\sigma(t) \to_S \sigma(s)$ *(respectively* $\sigma(t) \Rightarrow_S \sigma(s)$ *or* $\sigma(t) \to_S^\infty \sigma(s)$*),*
7. *if* $\sigma(x) \to_S^\infty \sigma'(x)$ *(respectively* $\sigma(x) \Rightarrow_S \sigma'(x)$*) for all variables x, then* $\sigma(t) \to_S^\infty \sigma'(t)$ *(respectively* $\sigma(t) \Rightarrow_S \sigma'(t)$*).*

Proof. The first point follows by coinduction. The second point follows by case analysis on $s \to_S^\infty s'$. The third point follows from the previous two. Points 4-6 follow by coinduction. The last point follows by coinduction with case analysis on t. Note that the last point does not hold with $\to_S^*$ instead of $\to_S^\infty$, because t may contain infinitely many variables. ◀

4 Confluence

Our aim is to prove the following theorem.

▶ **Theorem 34** (Confluence modulo $\sim_{\mathcal{R}}$ of nearly orthogonal iTRSs).
Let S be a nearly orthogonal iTRS. If $t \sim_{\mathcal{R}} s$, $t \to_S^\infty t'$ and $s \to_S^\infty s'$ then there exist t'', s'' such that $t' \to_S^\infty t''$, $s' \to_S^\infty s''$ and $t'' \sim_{\mathcal{R}} s''$.

Because $\sim_{\mathcal{R}}$ commutes with $\to_S^\infty$ (Lemma 18) and $\sim_{\mathcal{R}}$ is transitive (Lemma 20), it suffices to prove the theorem in the case $t = s$. The general strategy of the proof is illustrated in Figure 1. We show that for every term s' there exists a term w such that $s' \rightsquigarrow_s w$, i.e., s' reduces to s via a certain "standard" auxiliary "normalizing" reduction which disregards root-active subterms (see Definition 25 and Lemma 30). In contrast to infinitary N-reductions from [8], this "standard" reduction need not be unique and it is not really normalizing, but it is "regular" enough to show that it commutes with $\to_S^\infty$ (Lemma 33). The "normal" forms obtained through $\rightsquigarrow_s$ are not really in normal form, but they are closely related to Böhm trees. They may differ only in root-active subterms. Our overall proof strategy is partly similar to the strategy for the proof of the unique normal forms property of orthogonal iTRSs in [28].

Subdiagram (1) in Figure 1 is obtained by showing that $\to_S^\infty$ may be prepended to $\rightsquigarrow_s$ (Corollary 29), i.e., if $t \to_S^\infty s' \rightsquigarrow_s w$ then $t \rightsquigarrow_s w$. Subdiagram (2) follows from commutation of $\to_S^\infty$ and $\rightsquigarrow_s$ (Lemma 33). Subdiagrams (3) and (4) follow from the fact that $\rightsquigarrow_s$ decomposes into $\to^\infty$ and $\sim_{\mathcal{R}}$ (Lemma 31). Subdiagram (5) follows from the commutation of $\to_S^\infty$ and $\sim_{\mathcal{R}}$ (Lemma 18).

We also give a proof of confluence modulo $\sim_{\mathcal{H}}$. In this case a different and a bit more complicated method similar to [21] is necessary. Initially, we show some lemmas used in both proofs. In the rest of this section we fix a nearly orthogonal iTRS $\mathcal{S} = \langle \Sigma, S \rangle$. We write $\to$, $\Rightarrow$, $\to^\infty$, etc., for $\to_S$, $\Rightarrow_S$, $\to_S^\infty$, etc.

▶ **Lemma 6.** *Suppose l is finite and linear. If $t \to^\infty \sigma(l)$ then there is a substitution σ' such that $t \to^* \sigma'(l)$ and $\sigma'(x) \to^\infty \sigma(x)$ for all variables x.*

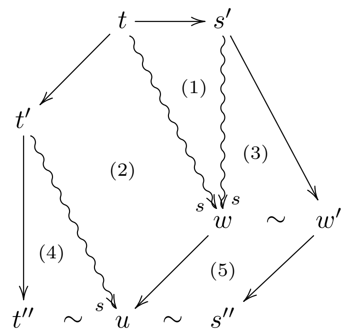

Figure 1 Confluence modulo $\sim_{\mathcal{R}}$ of nearly orthogonal iTRSs.

Proof. This follows from the definition of $\to^\infty$ and the fact that l is finite and linear. Indeed, we just need to go deep enough in the derivation tree of $t \to^\infty \sigma(l)$ to get below variable positions of l, concatenating the $\to^*$ prefixes along the way. ◀

Note that the finiteness of l is crucial in the above lemma. As a counterexample consider $l = A^\omega$, $t = B$ and an iTRS with a single rule $B \to A(B)$.

▶ **Lemma 7.** *If $t \to^\infty s \to u$ then $t \to^\infty u$.*

Proof. By coinduction. If $s = x$ then $u = x$ and thus $t \to^\infty u$. Otherwise $s = f(s_1, \ldots, s_n)$. First assume that s is the redex contracted in $s \to u$. Suppose the contraction is by a rule $\langle l, r\rangle \in S$ with substitution σ. By Lemma 6 there is σ' with $t \to^* \sigma'(l)$ and $\sigma'(x) \to^\infty \sigma(x)$ for all variables x. Then $t \to^* \sigma'(l) \to \sigma'(r) \to^\infty \sigma(r) = u$. Hence $t \to^\infty u$.

So assume that $s \to u$ is not a root contraction. Then $u = f(s_1, \ldots, s_{k-1}, s_k', s_{k+1}, \ldots, s_n)$ with $s_k \to s_k'$. Also $t \to^* f(t_1, \ldots, t_n)$ with $t_i \to^\infty s_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis $t_k \to^\infty s_k'$. Hence $t \to^\infty u$. ◀

▶ **Lemma 8.** *If $t \to^\infty s \to^\infty u$ then $t \to^\infty u$.*

Proof. By coinduction. If $u = x$ then $t \to^\infty s \to^* u$, so $t \to^\infty u$ by Lemma 7. Otherwise $u = f(u_1, \ldots, u_n)$, $s \to^* f(s_1, \ldots, s_n)$ and $s_i \to^\infty u_i$ for $i = 1, \ldots, n$. By Lemma 7 we have $t \to^\infty f(s_1, \ldots, s_n)$. Thus $t \to^* f(t_1, \ldots, t_n)$ with $t_i \to^\infty s_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis $t_i \to^\infty u_i$. Therefore $t \to^\infty f(u_1, \ldots, u_n) = u$. ◀

▶ **Corollary 9.** *If $t \in \mathcal{H}$ (resp. $t \in \mathcal{R}$, t is collapse-stable, t is root-stable) and $t \to^\infty t'$ then $t' \in \mathcal{H}$ (resp. $t' \in \mathcal{R}$, t' is collapse-stable, t' is root-stable).*

▶ **Corollary 10.** *If $t \Rightarrow^* s$ then $t \to^\infty s$.*

▶ **Lemma 11.** *If $t \to^{2\infty} s$ then $t \to^\infty s$.*

Proof. By coinduction, using Lemma 8. ◀

Note that the proofs of the above lemmas depend only on the left-linearity of S. The next lemma is crucial in our proof. It fails for weakly orthogonal iTRSs. As a counterexample consider the weakly orthogonal iTRS from the introduction and the term $P(S(P^\omega))$.

▶ **Lemma 12.** *If $\langle l, r\rangle \in S$ and $\sigma(l) \to t$ by a non-root contraction, then t is a redex by $\langle l, r\rangle$ with a substitution σ' such that $\sigma(x) \to^= \sigma'(x)$ for every variable x.*

Proof. The contraction $\sigma(l) \to t$ must occur below a variable position of l, because there are no non-root overlaps. Because S is left-linear, t is still a redex by $\langle l, r\rangle$, with a substitution σ' satisfying the requirements of the lemma. ◀

▶ **Lemma 13.** *If $f(t_1, \ldots, t_n)$ is a redex by a rule $\langle f(l_1, \ldots, l_n), r\rangle \in S$ with substitution σ, and $t_i \to^\infty t_i'$ (respectively $t_i \Rightarrow t_i'$) for $i = 1, \ldots, n$, then there is σ' such that $\sigma'(l_i) = t_i'$ for $i = 1, \ldots, n$, and $\sigma(x) \to^\infty \sigma'(x)$ (respectively $\sigma(x) \Rightarrow \sigma'(x)$) for every variable x.*

Proof. This follows from left-linearity and the fact that there are no non-root overlaps: all contractions in $t_i \to^\infty t_i'$ (respectively $t_i \Rightarrow t_i'$) must occur below variable positions of l_i. Formally, one applies the definition of $t_i \to^\infty t_i'$ (respectively $t_i \Rightarrow t_i'$) repeatedly until one reaches variable positions of l_i, using Lemma 12 to show that the contractions in the $\to^*$ prefixes occur below variable positions. ◀

▶ **Lemma 14.** *If $t \Rightarrow t_1$ and $t \Rightarrow t_2$ then there is s with $t_1 \Rightarrow s$ and $t_2 \to^\infty s$.*

Proof. By coinduction. If t is a redex and t_1, t_2 are both its reducts, then there is s with $t_2 \to^\infty s$ and $t_1 \Rightarrow s$, because S is nearly orthogonal. Suppose $t = f(u_1, \ldots, u_n)$ is a redex by a rule $\langle l, r\rangle \in S$ with substitution σ, but $t_2 = f(w_1, \ldots, w_n)$ with $u_i \Rightarrow w_i$ for $i = 1, \ldots, n$. By Lemma 13 there is σ' with $\sigma'(l) = t_2$ and $\sigma(x) \Rightarrow \sigma'(x)$ for every variable x. Then $t_1 = \sigma(r) \Rightarrow \sigma'(r)$ and $t_2 = \sigma'(l) \to \sigma'(r)$, so we may take $s = \sigma'(r)$. The remaining cases, when neither $t \Rightarrow t_1$ nor $t \Rightarrow t_2$ contracts at the root, are trivial or follow directly from the coinductive hypothesis. ◀

▶ **Lemma 15** (Infinitary Parallel Moves Lemma)**.** *If $t \to^\infty t_1$ and $t \Rightarrow t_2$ then there is s with $t_1 \Rightarrow s$ and $t_2 \to^\infty s$.*

Proof. By coinduction we show that if $t \to^\infty t_1$ and $t \Rightarrow t_2$ then there is s with $t_1 \Rightarrow s$ and $t_2 \to^{2\infty} s$. This suffices by Lemma 11. If $t_1 = x$ then $t \to^* t_1$ and the claim follows from Lemma 14 and Lemma 8. Otherwise $t \to^* u = f(u_1, \ldots, u_n)$, $t_1 = f(w_1, \ldots, w_n)$ and $u_i \to^\infty w_i$ for $i = 1, \ldots, n$. By Lemma 14 and Lemma 8 there is t_2' with $t_2 \to^\infty t_2'$ and $u \Rightarrow t_2'$. If $u \Rightarrow t_2'$ is a root contraction by a rule $\langle l, r\rangle \in S$ with substitution σ, then by Lemma 13 there is σ' with $t_1 = \sigma'(l)$ and $\sigma(x) \to^\infty \sigma'(x)$ for all variables x. Then $t_1 = \sigma'(l) \to \sigma'(r)$ and $t_2 \to^\infty t_2' = \sigma(r) \to^\infty \sigma'(r)$, so $t_2 \to^\infty \sigma'(r)$ by Lemma 8. Thus we may take $s = \sigma'(r)$. If $u \Rightarrow t_2'$ does not contract at the root, then $t_2' = f(v_1, \ldots, v_n)$ with $u_i \Rightarrow v_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis we obtain $s_1, \ldots, s_n$ with $v_i \to^{2\infty} s_i$ and $w_i \Rightarrow s_i$ for $i = 1, \ldots, n$. Take $s = f(s_1, \ldots, s_n)$. Then $t_2 \to^{2\infty} s$, because $t_2 \to^\infty f(v_1, \ldots, v_n)$, and $t_1 = f(w_1, \ldots, w_n) \Rightarrow s$. ◀

In the following lemmas $\mathcal{U}$ stands for either $\mathcal{H}$ or $\mathcal{R}$. We say that a term t is *active* if $t \in \mathcal{U}$. We say that a term is *stable* if it is collapse-stable and $\mathcal{U} = \mathcal{H}$, or it is root-stable and $\mathcal{U} = \mathcal{R}$. An *active redex* is a collapsing redex if $\mathcal{U} = \mathcal{H}$, or just a redex if $\mathcal{U} = \mathcal{R}$. Note that:

- t is stable iff there is no active redex s with $t \to^\infty s$,
- t is active ($t \in \mathcal{U}$) iff there is no stable s with $t \to^\infty s$,
- t is active iff for every s with $t \to^\infty s$ there is an active redex s' with $s \to^\infty s'$.

▶ **Lemma 16.** *If $\langle l, r\rangle \in S$ and $s \in \mathcal{U}$ is a proper subterm of $\sigma(l)$, then s occurs in $\sigma(l)$ below a variable position of l.*

Proof. Since $s \in \mathcal{U}$, by Lemma 6 there is a redex u such that $s \to^* u$. Because s is a proper subterm of $\sigma(l)$, by Lemma 12 the term t, which is $\sigma(l)$ with the subterm s replaced with u, is a redex by $\langle l, r\rangle$. But then u (and thus also s) must occur below a variable position of l, because otherwise there would be a non-root overlap between $\langle l, r\rangle$ and the rule by which u is a redex. ◀

▶ **Lemma 17.** *If $t \to t'$ and $t \sim_{\mathcal{U}} s$ then there is s' with $s \to^{=} s'$ and $t' \sim_{\mathcal{U}} s'$.*

Proof. Induction on $t \to t'$. If $t, s \in \mathcal{U}$ then also $t' \in \mathcal{U}$, so $t' \sim_{\mathcal{U}} s$ and we may take $s' = s$. If $t = x$ then $t' = s' = x$. Otherwise $t = f(t_1, \ldots, t_n)$, $s = f(s_1, \ldots, s_n)$ and $t_i \sim_{\mathcal{U}} s_i$ for $i = 1, \ldots, n$. First assume that $t \to t'$ is a root contraction, i.e., there are $\langle l, r\rangle \in S$ and σ such that $\sigma(l) = t$ and $\sigma(r) = t'$. By Lemma 16 all proper active subterms of t are below variable positions of l. This implies that $s = \sigma'(l)$ with some σ' such that $\sigma(x) \sim_{\mathcal{U}} \sigma'(x)$ for every variable x. Then $s = \sigma'(l) \to \sigma'(r) \sim_{\mathcal{U}} \sigma(r) = t'$. Therefore we may take $s' = \sigma'(r)$. If $t \to t'$ is not a root contraction then the claim follows directly from the inductive hypothesis. ◀

▶ **Lemma 18.** *If $t \to^{\infty} t'$ and $t \sim_{\mathcal{U}} s$ then there is s' with $s \to^{\infty} s'$ and $t' \sim_{\mathcal{U}} s'$.*

Proof. By coinduction. If $t' = x$ then $t \to^{*} x$ and the claim follows from Lemma 17. Otherwise $t \to^{*} u = f(u_1, \ldots, u_n)$, $t' = f(t'_1, \ldots, t'_n)$ and $u_i \to^{\infty} t'_i$ for $i = 1, \ldots, n$. By Lemma 17 there is u' with $s \to^{*} u'$ and $u \sim_{\mathcal{U}} u'$. If $u, u' \in \mathcal{U}$ then $t' \in \mathcal{U}$ by Corollary 9, because $u \to^{\infty} t'$. Hence $t' \sim_{\mathcal{U}} u'$ and we may take $s' = u'$. Otherwise $u' = f(u'_1, \ldots, u'_n)$ with $u_i \sim_{\mathcal{U}} u'_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis we obtain s_i with $u'_i \to^{\infty} s_i$ and $t'_i \sim_{\mathcal{U}} s_i$, for $i = 1, \ldots, n$. Take $s' = f(s_1, \ldots, s_n)$. Then $t' = f(t'_1, \ldots, t'_n) \sim_{\mathcal{U}} s'$ and $s \to^{\infty} s'$. ◀

▶ **Lemma 19.** *If $t \in \mathcal{U}$ and $t \sim_{\mathcal{U}} s$ then $s \in \mathcal{U}$.*

Proof. Assume $s \to^{\infty} s'$. Then by Lemma 18 there is t' with $t \to^{\infty} t' \sim_{\mathcal{U}} s'$. Because $t \in \mathcal{U}$, there is an active redex t'' such that $t' \to^{\infty} t''$. By Lemma 18 there is s'' such that $s' \to^{\infty} s'' \sim_{\mathcal{U}} t''$. If $t'', s'' \in \mathcal{U}$ then there is another active redex u with $s' \to^{\infty} s'' \to^{\infty} u$, so $s' \to^{\infty} u$ by Lemma 8, and thus s' is not stable. Otherwise $t'' = f(t_1, \ldots, t_n)$, $s'' = f(s_1, \ldots, s_n)$ and $t_i \sim_{\mathcal{U}} s_i$ for $i = 1, \ldots, n$. Since t'' is an active redex, there is a rule $\langle l, r\rangle \in S$ and a substitution σ with $\sigma(l) = t''$. By Lemma 16 all proper active subterms of t'' are below variable positions of l. This implies that s'' is also an active redex by the rule $\langle l, r\rangle$. Hence s' is not stable. Since s' was arbitrary with $s \to^{\infty} s'$, we conclude that $s \in \mathcal{U}$. ◀

▶ **Lemma 20.** *If $t \sim_{\mathcal{U}} s \sim_{\mathcal{U}} u$ then $t \sim_{\mathcal{U}} u$.*

Proof. By coinduction, using Lemma 19 when $t, s \in \mathcal{U}$ or $s, u \in \mathcal{U}$. ◀

▶ **Corollary 21.** *The relation $\sim_{\mathcal{U}}$ is an equivalence relation.*

We write $t \to_{\mathrm{ncr}} s$ if $t \to s$ and this is not a collapsing contraction at the root. So $t \to^{*}_{\mathrm{ncr}} s$ if $t \to^{*} s$ and there are no collapsing root contractions in the reduction.

▶ **Lemma 22.** *If $f(t_1, \ldots, t_n) \to^{*}_{\mathrm{ncr}} s$ and $t_i \to^{\infty} t'_i$ for $i = 1, \ldots, n$ then $s = g(s_1, \ldots, s_m)$ and there are $s'_1, \ldots, s'_m$ with $f(t'_1, \ldots, t'_n) \Rightarrow^{*} g(s'_1, \ldots, s'_m)$ and $s_j \to^{\infty} s'_j$ for $j = 1, \ldots, m$.*

Proof. Let $t = f(t_1, \ldots, t_n)$. It suffices to consider the case $t \to_{\mathrm{ncr}} s$. The general case then follows by induction.

If t is the redex contracted in $t \to_{\mathrm{ncr}} s$ then it is contracted by some non-collapsing rule $\langle l, r\rangle \in S$ with substitution σ. Then $r = g(r_1, \ldots, r_m)$. By Lemma 13 there is σ' such that $f(t'_1, \ldots, t'_i) = \sigma'(l)$ and $\sigma(x) \to^{\infty} \sigma'(x)$ for every variable x. Thus $\sigma(r_j) \to^{\infty} \sigma(r'_j)$ for $j = 1, \ldots, m$. Since $s = \sigma(r) = g(\sigma(r_1), \ldots, \sigma(r_m)) \to^{\infty} g(\sigma'(r_1), \ldots, \sigma'(r_m))$ and $f(t'_1, \ldots, t'_i) = \sigma'(l) \to \sigma'(r) = g(\sigma'(r_1), \ldots, \sigma'(r_m))$, we may take $s'_j = \sigma'(r_j)$ for $j = 1, \ldots, m$.

So assume the contraction $t \to_{\mathrm{ncr}} s$ does not occur at the root. Then $s = f(s_1, \ldots, s_n)$ with $t_i \to^= s_i$ for $i = 1, \ldots, n$. By Lemma 15 there are $s_1', \ldots, s_n'$ with $t_i' \Rightarrow s_i'$ and $s_i \to^\infty s_i'$ for $i = 1, \ldots, n$. Then $f(t_1', \ldots, t_n') \Rightarrow f(s_1', \ldots, s_n')$, so we may take $g = f$ and $m = n$. ◀

▶ **Lemma 23.** *If for every s with $t \to^* s$ there is an active redex u with $s \to^\infty u$, then $t \in \mathcal{U}$.*

Proof. Assume t satisfies the antecedent and $t \to^\infty t'$. Then $t \to^* f(t_1, \ldots, t_n)$, $t' = f(t_1', \ldots, t_n')$ and $t_i \to^\infty t_i'$ for $i = 1, \ldots, n$. By assumption and Lemma 6 there is an active redex $s = g(s_1, \ldots, s_m)$ such that $f(t_1, \ldots, t_n) \to^*_{\mathrm{ncr}} s$. By Lemma 22 there are $s_1', \ldots, s_m'$ such that $t' = f(t_1', \ldots, t_n') \Rightarrow^* g(s_1', \ldots, s_m')$ and $s_j \to^\infty s_j'$ for $j = 1, \ldots, m$. Let $s' = g(s_1', \ldots, s_m')$. By Lemma 13 we conclude that s' is a redex by the same rule as s, i.e., s' is an active redex. By Corollary 10 we have $t' \to^\infty s'$. Hence t' is not stable. Since t' was arbitrary with $t \to^\infty t'$, we conclude that $t \in \mathcal{U}$. ◀

▶ **Lemma 24.** *If $t \to^\infty t'$ and $t' \in \mathcal{U}$ then $t \in \mathcal{U}$.*

Proof. Suppose $t \to^* s$. By Lemma 15 there is s' with $s \to^\infty s'$ and $t' \Rightarrow^* s'$. We have $t' \to^\infty s'$ by Corollary 10. Since also $t' \in \mathcal{U}$, there is an active redex u with $s \to^\infty s' \to^\infty u$. Then $s \to^\infty u$ by Lemma 8. By Lemma 23 this implies $t \in \mathcal{U}$. ◀

4.1 Confluence modulo $\sim_{\mathcal{R}}$

We now proceed to show that nearly orthogonal iTRSs are confluent modulo $\sim_{\mathcal{R}}$. None of the lemmas in this subsection are needed in the proof of confluence modulo $\sim_{\mathcal{H}}$. The method of the present section does not work if $\mathcal{H}$ is used instead of $\mathcal{R}$, because then the proof of Lemma 33 does not go through.

▶ **Definition 25.** The relation $\rightsquigarrow_s$ is defined coinductively.

$$\frac{t \to^* x}{t \rightsquigarrow_s x} \qquad \frac{t, s \in \mathcal{R}}{t \rightsquigarrow_s s}$$

$$\frac{t \to^* f(t_1, \ldots, t_n) \quad t_i \rightsquigarrow_s t_i' \text{ for } i = 1, \ldots, n \quad f(t_1, \ldots, t_n) \text{ is root-stable}}{t \rightsquigarrow_s f(t_1', \ldots, t_n')}$$

The relation $\rightsquigarrow_a$ is defined coinductively in the same way as $\rightsquigarrow_s$ except that in the first premise of the last rule we use $t \to^\infty f(t_1, \ldots, t_n)$ instead of $t \to^* f(t_1, \ldots, t_n)$.

The relation $\rightsquigarrow_s$ denotes a "standard" reduction to "normal" form. The "normal" forms are not really in normal form, but they are closely related to Böhm trees. In fact, it is not difficult to show by coinduction that if $t \rightsquigarrow_s s \to^\infty s'$ then $s \sim_{\mathcal{R}} s'$. Bahr and Ketema [1, 2, 24] define similar reductions to Böhm-like trees, but they do not seem to use them to obtain new proofs of infinitary confluence. The author has not studied the mentioned papers in enough depth to give a detailed comparison.

The relation $\rightsquigarrow_a$, which turns out to be the same as $\rightsquigarrow_s$ (Lemma 28), is a technical notion needed to help in some proofs.

▶ **Lemma 26.** *If $t \to^\infty s \rightsquigarrow_a u$ then $t \rightsquigarrow_a u$.*

Proof. If $s, u \in \mathcal{R}$ then also $t \in \mathcal{R}$ by Lemma 24, so $t \rightsquigarrow_a u$. If $u = x$ then $s \to^* x$, and thus $t \to^\infty x$ by Lemma 7, so $t \rightsquigarrow_a u$. Otherwise $u = f(u_1, \ldots, u_n)$, $s \to^\infty f(s_1, \ldots, s_n)$, $s_i \rightsquigarrow_a u_i$ for $i = 1, \ldots, n$, and $f(s_1, \ldots, s_n)$ is root-stable. By Lemma 8 we have $t \to^\infty f(s_1, \ldots, s_n)$. Thus $t \rightsquigarrow_a f(u_1, \ldots, u_n) = u$. ◀

▶ **Lemma 27.** *If $s = f(s_1, \ldots, s_n)$ is root-stable and $t_i \to^\infty s_i$ for $i = 1, \ldots, n$, then $t = f(t_1, \ldots, t_n)$ is also root-stable.*

Proof. Suppose t is not root-stable. Hence by Lemma 6 there is a redex u such that $t \to^*_{\mathrm{ncr}} u = g(u_1, \ldots, u_m)$. By Lemma 22 there is $u' = g(u'_1, \ldots, u'_m)$ such that $u_j \to^\infty u'_j$ for $j = 1, \ldots, m$ and $s \Rightarrow^* u'$. By Lemma 13 we conclude that u' is still a redex. Since $s \to^\infty u'$ by Corollary 10, we conclude that s is not root-stable. Contradiction. ◀

▶ **Lemma 28.** *$t \rightsquigarrow_s s$ iff $t \rightsquigarrow_a s$.*

Proof. The implication from left to right follows by straightforward coinduction. We show the other direction by coinduction. If $s = x$ then $t \to^* x$, so $t \rightsquigarrow_s s$. If $t, s \in \mathcal{R}$ then $t \rightsquigarrow_s s$. Otherwise $t \to^\infty f(t_1, \ldots, t_n)$, $s = f(s_1, \ldots, s_n)$, $f(t_1, \ldots, t_n)$ is root-stable, and $t_i \rightsquigarrow_a s_i$ for $i = 1, \ldots, n$. Then $t \to^* f(t'_1, \ldots, t'_n)$ with $t'_i \to^\infty t_i$ for $i = 1, \ldots, n$. By Lemma 27 we conclude that $f(t'_1, \ldots, t'_n)$ is root-stable. By Lemma 26 we have $t'_i \rightsquigarrow_a s_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis $t'_i \rightsquigarrow_s s_i$ for $i = 1, \ldots, n$. Thus $t \rightsquigarrow_s f(s_1, \ldots, s_n) = s$. ◀

▶ **Corollary 29.** *If $t \to^\infty s \rightsquigarrow_s u$ then $t \rightsquigarrow_s u$.*

Proof. Follows from Lemma 26 and Lemma 28. ◀

At this point a reader might conjecture that the following may be easily shown:

($\star$) if $t \rightsquigarrow_s t_1$ and $t \rightsquigarrow_s t_2$ then $t_1 \sim_{\mathcal{R}} r_2$.

However, this is not the case, because a priori t might reduce to two essentially different root-stable terms. Thus it is not clear how to prove ($\star$) coinductively. Using Lemma 14 it is not difficult to show that if $t \to^* s_1$, $t \to^* s_2$ and s_1, s_2 are root-stable then s_1 and s_2 have the same root symbol. But they may still differ below the root.

Note that confluence modulo $\sim_{\mathcal{R}}$ would easily follow from ($\star$), Lemma 30 and Lemma 31. There are two methods which could probably be used to show ($\star$), though the author doubts whether any of them would lead to a much simpler confluence proof than via Lemma 33. The first method would be to adapt the proof of [28, Theorem 15]. The fact that the terms obtained through $\rightsquigarrow_s$ need not be in normal form might complicate this slightly. The second method would be to prove some standardisation result and proceed similarly to [8], using finitary standard reduction to a root-stable term in the definition of $\rightsquigarrow_s$ instead of ordinary finitary reduction. Then the proof of Corollary 29 would become more difficult, because there would be less freedom in the finitary reduction to a root-stable term in $\rightsquigarrow_s$.

▶ **Lemma 30.** *For every term t there is s with $t \rightsquigarrow_s s$.*

Proof. By coinduction. If $t \in \mathcal{R}$ then $t \rightsquigarrow_s t$. Otherwise $t \to^* t'$ for some root-stable t', by Lemma 23. If $t' = x$ then $t \rightsquigarrow_s x$. Otherwise $t' = f(t_1, \ldots, t_n)$. By the coinductive hypothesis we obtain $s_1, \ldots, s_n$ with $t_i \rightsquigarrow_s s_i$ for $i = 1, \ldots, n$. Thus $t \rightsquigarrow_s f(s_1, \ldots, s_n)$. ◀

▶ **Lemma 31.** *If $t \rightsquigarrow_s s$ then there is u with $t \to^\infty u \sim_{\mathcal{R}} s$.*

Proof. By coinduction. If $t, s \in \mathcal{R}$ then $t \sim_{\mathcal{R}} s$ and we may take $u = t$. If $s = x$ then $t \to^* x$, so $t \to^\infty s$ and we may take $u = s$. Otherwise $s = f(s_1, \ldots, s_n)$, $t \to^* f(t_1, \ldots, t_n)$ and $t_i \rightsquigarrow_s s_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis we obtain u_i with $t_i \to^\infty u_i \sim_{\mathcal{R}} s_i$, for $i = 1, \ldots, n$. Take $u = f(u_1, \ldots, u_n)$. Then $t \to^\infty u \sim_{\mathcal{R}} s$. ◀

▶ **Lemma 32.** *If $t \Rightarrow t'$ and $t \rightsquigarrow_s t''$ then there is s with $t' \rightsquigarrow_s s$ and $t'' \Rightarrow s$.*

Proof. By Lemma 28 it suffices to show that if $t \Rightarrow t'$ and $t \rightsquigarrow_s t''$ then there is s with $t' \rightsquigarrow_a s$ and $t'' \Rightarrow s$. We proceed by coinduction. If $t'' = x$ then the claim follows from Lemma 15. If $t, t'' \in \mathcal{R}$ then by Corollary 9 we have $t' \in \mathcal{R}$, so $t' \rightsquigarrow_s t''$ and we may take $s = t''$. Otherwise $t \to^* f(t_1, \ldots, t_n)$, $t'' = f(t''_1, \ldots, t''_n)$, $f(t_1, \ldots, t_n)$ is root-stable and $t_i \rightsquigarrow_s t''_i$ for $i = 1, \ldots, n$. By Lemma 15 there is u with $t' \to^\infty u$ and $f(t_1, \ldots, t_n) \Rightarrow u$. Because $f(t_1, \ldots, t_n)$ is root-stable, u is also root-stable and $u = f(u_1, \ldots, u_n)$ with $t_i \Rightarrow u_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis we obtain $s_1, \ldots, s_n$ with $u_i \rightsquigarrow_a s_i$ and $t''_i \Rightarrow s_i$. Take $s = f(s_1, \ldots, s_n)$. Then $t' \rightsquigarrow_a s$ and $t'' \Rightarrow s$. ◀

The proof of the following lemma fails if $\mathcal{H}$ is used instead of $\mathcal{R}$. This is because a collapse-stable term may contract at the root, in contrast to a root-stable term.

▶ **Lemma 33.** *If $t \rightsquigarrow_s t'$ and $t \to^\infty t''$ then there is s with $t' \to^\infty s$ and $t'' \rightsquigarrow_s s$.*

Proof. By Lemma 11 and Lemma 28 it suffices to show that if $t \rightsquigarrow_s t'$ and $t \to^\infty t''$ then there is s with $t' \to^{2\infty} s$ and $t'' \rightsquigarrow_a s$. We proceed by coinduction. If $t' = x$ then the claim follows from Lemma 15. If $t, t' \in \mathcal{R}$ then also $t'' \in \mathcal{R}$ by Corollary 9, so $t'' \rightsquigarrow_a t'$ and we may take $s = t'$. Otherwise $t \to^* f(t_1, \ldots, t_n)$, $t' = f(t'_1, \ldots, t'_n)$, $f(t_1, \ldots, t_n)$ is root-stable and $t_i \rightsquigarrow_s t'_i$ for $i = 1, \ldots, n$. By Lemma 15 there is u with $t'' \Rightarrow^* u$ and $f(t_1, \ldots, t_n) \to^\infty u$. Hence $f(t_1, \ldots, t_n) \to^* u' = g(u'_1, \ldots, u'_m)$, $u = g(u_1, \ldots, u_m)$ and $u'_j \to^\infty u_j$ for $j = 1, \ldots, m$. Because $f(t_1, \ldots, t_n)$ is root-stable, none of the contractions in $f(t_1, \ldots, t_n) \to^* u'$ may occur at the root. Thus $m = n$, $g = f$ and $t_i \to^* u'_i$ for $i = 1, \ldots, n$. By Lemma 32 there are $w_1, \ldots, w_n$ with $u'_i \rightsquigarrow_s w_i$ and $t'_i \Rightarrow^* w_i$. By the coinductive hypothesis we obtain $s_1, \ldots, s_n$ with $u_i \rightsquigarrow_a s_i$ and $w_i \to^{2\infty} s_i$ for $i = 1, \ldots, n$. Note that $u = f(u_1, \ldots, u_n)$ is root-stable by Corollary 9, because $f(t_1, \ldots, t_n)$ is root-stable and $f(t_1, \ldots, t_n) \to^\infty u$. Since $t'' \Rightarrow^* u$, by Corollary 10 we have $t'' \to^\infty u$, and thus $t'' \rightsquigarrow_a f(s_1, \ldots, s_n)$. Because $t' = f(t'_1, \ldots, t'_n) \Rightarrow^* f(w_1, \ldots, w_n)$, by Corollary 10 we have $t' \to^\infty f(w_1, \ldots, w_n)$, and thus $t' \to^{2\infty} f(s_1, \ldots, s_n)$. So we may take $s = f(s_1, \ldots, s_n)$. ◀

▶ **Theorem 34** (Confluence modulo $\sim_\mathcal{R}$ of nearly orthogonal iTRSs)**.** *Let S be a nearly orthogonal iTRS. If $t \sim_\mathcal{R} s$, $t \to^\infty_S t'$ and $s \to^\infty_S s'$ then there exist t'', s'' such that $t' \to^\infty_S t''$, $s' \to^\infty_S s''$ and $t'' \sim_\mathcal{R} s''$.*

Proof. See Figure 1 and the discussion just before it. ◀

▶ **Corollary 35.** *Any nearly orthogonal iTRS has the unique normal forms property.*

4.2 Confluence modulo $\sim_\mathcal{H}$

We only mention the following results, delegating the proofs to an appendix.

▶ **Theorem 36** (Confluence modulo $\sim_\mathcal{H}$ of nearly orthogonal iTRSs)**.** *Let S be a nearly orthogonal iTRS. If $t \sim_\mathcal{H} s$, $t \to^\infty_S t'$ and $s \to^\infty_S s'$ then there exist t'', s'' such that $t' \to^\infty_S t''$, $s' \to^\infty_S s''$ and $t'' \sim_\mathcal{H} s''$.*

▶ **Corollary 37.** *Any nearly orthogonal iTRS with no collapsing rules is confluent.*

Acknowledgements The author would like to thank the anonymous referees for helpful comments which led to substantial improvements in the presentation of this paper. The author also thanks Paula Severi for fruitful discussions regarding infinitary lambda calculus and infinitary rewriting, and Bartek Klin for teaching him coinduction.

References

1 Patrick Bahr. Partial order infinitary term rewriting and Böhm trees. In Christopher Lynch, editor, *Proceedings of the 21st International Conference on Rewriting Techniques and Applications, RTA 2010, July 11-13, 2010, Edinburgh, Scottland, UK*, volume 6 of *LIPIcs*, pages 67–84. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010.

2 Patrick Bahr. Partial order infinitary term rewriting. *Logical Methods in Computer Science*, 10(2), 2014.

3 Yves Bertot and Pierre Castéran. *Interactive Theorem Proving and Program Development: Coq'Art: the Calculus of Inductive Constructions*, chapter 13. Springer, 2004.

4 Marc Bezem, Keiko Nakata, and Tarmo Uustalu. On streams that are finitely red. *Logical Methods in Computer Science*, 8:1–20, 2012.

5 Adam Chlipala. *Certified Programming with Dependent Types*, chapter 5. The MIT Press, 2013.

6 Thierry Coquand. Infinite objects in type theory. In Henk Barendregt and Tobias Nipkow, editors, *Types for Proofs and Programs, International Workshop TYPES'93, Nijmegen, The Netherlands, May 24-28, 1993, Selected Papers*, volume 806 of *Lecture Notes in Computer Science*, pages 62–78. Springer, 1993.

7 Łukasz Czajka. A coinductive confluence proof for infinitary lambda-calculus. In Gilles Dowek, editor, *Rewriting and Typed Lambda Calculi - Joint International Conference, RTA-TLCA 2014, Held as Part of the Vienna Summer of Logic, VSL 2014, Vienna, Austria, July 14-17, 2014. Proceedings*, volume 8560 of *Lecture Notes in Computer Science*, pages 164–178. Springer, 2014.

8 Łukasz Czajka. Coinductive techniques in infinitary lambda-calculus. *CoRR*, abs/1501.04354, 2015.

9 Brian A. Davey and Hilary A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 2nd edition, 2002.

10 Jörg Endrullis, Clemens Grabmayer, Dimitri Hendriks, Jan Willem Klop, and Vincent van Oostrom. Unique normal forms in infinitary weakly orthogonal rewriting. In C. Lynch, editor, *Proceedings of the Conference on Rewriting Techniques and Applications (RTA 2010)*, volume 6 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 85–102. Dagstuhl, 2010.

11 Jörg Endrullis, Clemens Grabmayer, Dimitri Hendriks, Jan Willem Klop, and Vincent van Oostrom. Infinitary term rewriting for weakly orthogonal systems: properties and examples. *Logical Methods in Computer Science*, 10:1–33, 2014.

12 Jörg Endrullis, Helle Hvid Hansen, Dimitri Hendriks, Andrew Polonsky, and Alexandra Silva. A coinductive treatment of infinitary rewriting. Unpublished, available at `http://arxiv.org/abs/1306.6224`, 2013.

13 Jörg Endrullis, Dimitri Hendriks, and Jan Willem Klop. Highlights in infinitary rewriting and lambda calculus. *Theoretical Computer Science*, 464:48–71, 2012.

14 Jörg Endrullis, Dimitri Hendriks, Jan Willem Klop, and Andrew Polonsky. Discriminating lambda-terms using clocked Böhm trees. *Logical Methods in Computer Science*, 10(2), 2014.

15 Jörg Endrullis and Andrew Polonsky. Infinitary rewriting coinductively. In Nils Anders Danielsson and Bengt Nordström, editors, *TYPES*, volume 19 of *LIPIcs*, pages 16–27. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2011.

16 Eduardo Giménez. Codifying guarded definitions with recursive schemes. In Peter Dybjer, Bengt Nordström, and Jan M. Smith, editors, *TYPES*, volume 996 of *Lecture Notes in Computer Science*, pages 39–59. Springer, 1994.

17 Bernhard Gramlich. Confluence without termination via parallel critical pairs. In Hélène Kirchner, editor, *Trees in Algebra and Programming - CAAP'96, 21st International Col-*

loquium, Linköping, Sweden, April, 22-24, 1996, Proceedings, volume 1059 of *Lecture Notes in Computer Science*, pages 211–225. Springer, 1996.

18 Gérard P. Huet. Confluent reductions: Abstract properties and applications to term rewriting systems. *J. ACM*, 27(4):797–821, 1980.

19 Bart Jacobs and Jan J.M.M. Rutten. An introduction to (co)algebras and (co)induction. In *Advanced Topics in Bisimulation and Coinduction*, pages 38–99. Cambridge University Press, 2011.

20 Felix Joachimski. Confluence of the coinductive [lambda]-calculus. *Theoretical Computer Science*, 311(1-3):105–119, 2004.

21 Richard Kennaway and Fer-Jan de Vries. Infinitary rewriting. In Terese, editor, *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*, chapter 12, pages 668–711. Cambridge University Press, 2003.

22 Richard Kennaway, Jan Willem Klop, M. Ronan Sleep, and Fer-Jan de Vries. Transfinite reductions in orthogonal term rewriting systems. *Information and Computation*, 119(1):18–38, 1995.

23 Richard Kennaway, Vincent van Oostrom, and Fer-Jan de Vries. Meaningless terms in rewriting. *Journal of Functional and Logic Programming*, 1:1–35, 1999.

24 Jeroen Ketema. Böhm-like trees for term rewriting systems. In Vincent van Oostrom, editor, *Rewriting Techniques and Applications, 15th International Conference, RTA 2004, Aachen, Germany, June 3-5, 2004, Proceedings*, volume 3091 of *Lecture Notes in Computer Science*, pages 233–248. Springer, 2004.

25 Jeroen Ketema. Comparing Böhm-like trees. In Ralf Treinen, editor, *Rewriting Techniques and Applications, 20th International Conference, RTA 2009, Brasília, Brazil, June 29 - July 1, 2009, Proceedings*, volume 5595 of *Lecture Notes in Computer Science*, pages 239–254. Springer, 2009.

26 Jeroen Ketema. Reinterpreting compression in infinitary rewriting. In Ashish Tiwari, editor, *23rd International Conference on Rewriting Techniques and Applications (RTA'12) , RTA 2012, May 28 - June 2, 2012, Nagoya, Japan*, volume 15 of *LIPIcs*, pages 209–224. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2012.

27 Jeroen Ketema and Jakob Grue Simonsen. Infinitary combinatory reduction systems. *Information and Computation*, 209(6):893–926, 2011.

28 Jan Willem Klop and Roel C. de Vrijer. Infinitary normalization. In Sergei N. Artëmov, Howard Barringer, Artur S. d'Avila Garcez, Luís C. Lamb, and John Woods, editors, *We Will Show Them! Essays in Honour of Dov Gabbay, Volume Two*, pages 169–192. College Publications, 2005.

29 Dexter Kozen and Alexandra Silva. Practical coinduction. Draft, 2014.

30 Xavier Leroy and Hervé Grall. Coinductive big-step operational semantics. *Information and Computation*, 207(2):284–304, 2009.

31 Keiko Nakata and Tarmo Uustalu. Resumptions, weak bisimilarity and big-step semantics for while with interactive I/O: An exercise in mixed induction-coinduction. In L. Aceto and P. Sobociński, editors, *Seventh Workshop on Structural Operational Semantics (SOS '10)*, pages 57–75, 2010.

32 Jan J.M.M. Rutten. Universal coalgebra: a theory of systems. *Theoretical Computer Science*, 249(1):3–80, 2000.

33 Davide Sangiorgi. *Introduction to Bisimulation and Coinduction.* Cambridge University Press, 2012.

34 Yoshihito Toyama. Commutativity of term rewriting systems. In K. Fuchi and L. Kott, editors, *Programming of Future Generation Computers II*, pages 393–407. North-Holland, 1988.

35 Vincent van Oostrom. Development closed critical pairs. In Gilles Dowek, Jan Heering, Karl Meinke, and Bernhard Möller, editors, *HOA*, volume 1074 of *Lecture Notes in Computer Science*, pages 185–200. Springer, 1995.
36 Vincent van Oostrom. Developing developments. *Theoretical Computer Science*, 175(1):159–181, 1997.

A Proof of Theorem 36

It turns out that because nearly orthogonal iTRSs allow no non-root overlaps, all root overlaps with a collapsing rule must have a special form.

▶ **Lemma 38.** *Let $\langle l_1, x\rangle, \langle l_2, r_2\rangle \in S$, where l_1 and l_2 have no variables in common, and let σ be the mgu of l_1 and l_2. Then $\sigma(r_2) \Rightarrow \sigma(x)$.*

Proof. Because S is nearly orthogonal, there is s such that $\sigma(r_2) \Rightarrow s$ and $\sigma(x) \to^\infty s$. It suffices to show that $s = \sigma(x)$. If $\sigma(x)$ is a variable then this is obvious. Otherwise, because σ is an mgu of two linear terms with no variables in common and x occurs in l_1, we may assume that $\sigma(x)$ is a proper subterm of l_2 (we may assume the subterm is proper because l_1 is not a variable). But then $\sigma(x)$ cannot contain any redexes, because they would constitute a non-root overlap with the rule $\langle l_2, r_2\rangle$. Hence $\sigma(x) = s$. ◀

▶ **Definition 39.** A *hypercollapsing sequence* for a term t is an infinite sequence $(t_n)_{n\in\mathbb{N}}$ of terms satisfying:

- $t \to^\infty t_0$, and
- for each $n \in \mathbb{N}$ there is a collapsing rule $\langle l, x\rangle \in S$ and a substitution σ such that $t_n = \sigma(l) \to \sigma(x) \to^\infty t_{n+1}$.

The following lemma was shown for orthogonal iTRSs in [21, Lemma 12.8.4], by essentially the same proof.

▶ **Lemma 40.** *If there exists a hypercollapsing sequence for t then $t \in \mathcal{H}$.*

Proof. Assume that $(t_n)_{n\in\mathbb{N}}$ is a hypercollapsing sequence for t. It suffices to show that if $t \to s$ then there is a hypercollapsing sequence for s. Then it will follow from Lemma 23 that $t \in \mathcal{H}$.

Assume $t \to s$. We describe the construction of a hypercollapsing sequence $(s_n)_{n\in\mathbb{N}}$ for s. Assume the elements $s_0, \ldots, s_{n-1}$ of the sequence have been defined, and u, v are such that $u \Rightarrow v$, $u \to^\infty t_n$. In the base case $n = 0$ we take $u = t$ and $v = s$. By Lemma 15 there is v' with $v \to^\infty v'$ and $t_n \Rightarrow v'$. By the definition of a hypercollapsing sequence there are $\langle l, x\rangle \in S$ and σ such that $t_n = \sigma(l) \to \sigma(x) \to^\infty t_{n+1}$. If $t_n \Rightarrow v'$ by a root contraction, then $v' \to^\infty \sigma(x)$ by Lemma 38. Hence $v \to^\infty v' \to^\infty \sigma(x) \to^\infty t_{n+1}$, so $v \to^\infty t_{n+1}$ by Lemma 8. Then take $s_m = t_{m+1}$ for $m \geq n$ and finish the construction. So assume $t_n \Rightarrow v'$ is not a root contraction. Then $t_n = f(w_1, \ldots, w_k)$, $v' = f(w_1', \ldots, w_k')$ and $w_i \Rightarrow w_i'$ for $i = 1, \ldots, k$. By Lemma 13 there is σ' such that $v' = \sigma'(l)$ and $\sigma(x) \Rightarrow \sigma'(x)$. Hence $v \to^\infty v' \to \sigma'(x)$. Take $s_n = v'$ and continue the construction with $u := \sigma(x)$ and $v := \sigma'(x)$.

It follows by construction that $(s_n)_{n\in\mathbb{N}}$ is a hypercollapsing sequence for s. ◀

▶ **Definition 41.** The relation $\rightsquigarrow$ is defined coinductively.

$$\frac{t \to^* x}{t \rightsquigarrow x} \qquad \frac{t \to^* f(t_1, \ldots, t_n) \quad t_i \rightsquigarrow t_i' \text{ for } i = 1, \ldots, n}{t \rightsquigarrow f(t_1', \ldots, t_n')} \qquad \frac{t, s \in \mathcal{H}}{t \rightsquigarrow s}$$

The relation $\rightsquigarrow^\infty$ is defined coinductively in the same way as $\rightsquigarrow$ except that in the first premise of the second rule we use $t \to^\infty f(t_1, \ldots, t_n)$ instead of $t \to^* f(t_1, \ldots, t_n)$.

The intuitive interpretation of $\rightsquigarrow$ is quite different from the intuitive interpretation of $\rightsquigarrow_s$ in Section 4.1. If $t \rightsquigarrow s$ then s need not be "normal" in any sense. The crucial difference is that in the second rule we do not require $f(t_1,\dots,t_n)$ to be collapse-stable. Essentially, $t \rightsquigarrow s$ means that t infinitarily reduces to s, up to equivalence of hypercollapsing subterms. This intuition is validated by the following lemma.

▶ **Lemma 42.** *The following conditions are equivalent:*
1. $t \to^\infty u \sim_{\mathcal{H}} s$ *for some term* u,
2. $t \rightsquigarrow s$,
3. $t \rightsquigarrow^\infty s$.

Proof.

$(1 \Rightarrow 2)$ By coinduction, analysing $u \sim_{\mathcal{H}} s$. If $u, s \in \mathcal{H}$ then $t \in \mathcal{H}$ by Lemma 24, so $t \rightsquigarrow s$. If $u = s = x$ then $t \to^* x$, so $t \rightsquigarrow s$. If $u = f(u_1,\dots,u_n)$, $s = f(s_1,\dots,s_n)$ and $u_i \sim_{\mathcal{H}} s_i$ for $i = 1,\dots,n$, then $t \to^* f(t_1,\dots,t_n)$ with $t_i \to^\infty u_i$. By the coinductive hypothesis $t_i \rightsquigarrow s_i$ for $i = 1,\dots,n$. Thus $t \rightsquigarrow f(s_1,\dots,s_n) = s$.

$(2 \Rightarrow 3)$ Straightforward coinduction.

$(3 \Rightarrow 1)$ We show by coinduction that if $t \rightsquigarrow^\infty s$ then there is u with $t \to^{2\infty} u \sim_{\mathcal{H}} s$. This suffices by Lemma 11. If $s = x$ then $t \to^* x \sim_{\mathcal{H}} x$, so we may take $u = x$. If $t, s \in \mathcal{H}$ then $t \sim_{\mathcal{H}} s$ and we may take $u = t$. Otherwise $t \to^\infty f(t_1,\dots,t_n)$, $s = f(t'_1,\dots,t'_n)$ and $t_i \rightsquigarrow^\infty t'_i$ for $i = 1,\dots,n$. By the coinductive hypothesis we obtain $u_1,\dots,u_n$ with $t_i \to^{2\infty} u_i \sim_{\mathcal{H}} t'_i$. Take $u = f(u_1,\dots,u_n)$. Then $t \to^{2\infty} u \sim_{\mathcal{H}} s$. ◀

▶ **Lemma 43.** *If* $s \sim_{\mathcal{H}} t \Rightarrow t'$ *then there is* s' *with* $s \Rightarrow s' \sim_{\mathcal{H}} t'$.

Proof. By coinduction. If $t, s \in \mathcal{H}$ then $t' \in \mathcal{H}$ by Corollary 9, so $t' \sim_{\mathcal{H}} s$ and we may take $s' = s$. If $t = x$ then $t' = x$ and we may take $s' = s$. Otherwise $s = f(s_1,\dots,s_n)$, $t = f(t_1,\dots,t_n)$ and $s_i \sim_{\mathcal{H}} t_i$ for $i = 1,\dots,n$. If $t \Rightarrow t'$ is a root contraction then the claim follows from Lemma 17. If $t \Rightarrow t'$ does not contract at the root, then the claim follows directly from the coinductive hypothesis. ◀

▶ **Lemma 44.** *If* $t \Rightarrow t_1$ *and* $t \rightsquigarrow t_2$ *then there is* s *with* $t_1 \rightsquigarrow s$ *and* $t_2 \Rightarrow s$.

Proof. Follows from Lemma 42, Lemma 15 and Lemma 43. ◀

The construction of a hypercollapsing sequence in the proof of the following lemma is similar to the construction in [21, Lemma 12.8.14].

▶ **Lemma 45.** *If* $t \notin \mathcal{H}$, $t \to^\infty t'$ *and* $t \rightsquigarrow u$ *then one of the following holds:*
1. $t' \to^* x$ *and* $u \to^* x$ *for some variable* x, *or*
2. *there are* $s = f(s_1,\dots,s_n)$, $u' = f(u_1,\dots,u_n)$ *and* $w = f(w_1,\dots,w_n)$ *such that* $t \to^* s$, $t' \to^\infty w$, $u \to^\infty u'$, $s_i \to^\infty w_i$ *and* $s_i \rightsquigarrow u_i$ *for* $i = 1,\dots,n$.

Proof. By Lemma 40 it suffices to show that if neither 1 nor 2 holds then a hypercollapsing sequence $(v_k)_{k\in\mathbb{N}}$ for t may be constructed.

If $t' = x$ then by Lemma 44 we have $u \to^* x$, so 1 holds. If t' is not a variable then we have $t' = f(t'_1,\dots,t'_n)$. Because $t \to^\infty t'$, there are $t_1,\dots,t_n$ with $t \to^* t_0 = f(t_1,\dots,t_n)$ and $t_i \to^\infty t'_i$ for $i = 1,\dots,n$. By Lemma 44 and Corollary 10 there is u' with $u \to^\infty u'$ and $t_0 \rightsquigarrow u'$. If $u' = x$ then $u \to^* x$ and $t_0 \to^* x$, and thus $t' \to^* x$, by Lemma 15 and Corollary 10, so point 1 is true. Hence assume $u' = g(u_1,\dots,u_m)$. Because $t \notin \mathcal{H}$, also $t_0 \notin \mathcal{H}$ by Lemma 24. Thus $t_0 \to^* s = g(s_1,\dots,s_m)$ with $s_j \rightsquigarrow u_j$ for $j = 1,\dots,m$.

If $t_0 \to^*_{\mathrm{ncr}} s$ then by Lemma 22 and Corollary 10 there is $w = g(w_1,\dots,w_m)$ with $t' \to^\infty w$ and $s_j \to^\infty w_j$ for $j = 1,\dots,m$. Since also $t \to^* s$, $u \to^\infty u'$ and $s_j \to^\infty u_j$ for $j = 1,\dots,m$, then point 2 is true.

So suppose there is a collapsing root contraction in the reduction $t_0 \to^* s$, i.e., $t_0 \to^* \sigma(l) \to \sigma(x) \to^* s$ for some collapsing rule $\langle l, x \rangle \in S$ and some substitution σ. Since $t_0 \to^* \sigma(x)$ and $t_0 \to^\infty t'$, by Lemma 15 and Corollary 10 there is t'' with $t' \to^\infty t''$ and $\sigma(x) \to^\infty t''$. Note that also $\sigma(x) \notin \mathcal{H}$ and $\sigma(x) \rightsquigarrow u'$, by Lemma 24 because $t \to^* \sigma(x)$. Note that if the points 1-2 hold for $\sigma(x), t'', u'$ then they also hold for t, t', u, by Lemma 8. So we may take $v_k = \sigma(l)$ as the next element of the hypercollapsing sequence, and continue the construction with $t := \sigma(x)$, $t' := t''$ and $u := u'$.

Ultimately, we will either conclude that 1 or 2 holds, or we will construct a hypercollapsing sequence $(v_k)_{k \in \mathbb{N}}$ for t. ◀

▶ **Lemma 46.** *If $t \to^\infty t_1$ and $t \rightsquigarrow t_2$ then there is s with $t_1 \rightsquigarrow^\infty s$ and $t_2 \to^{2\infty} s$.*

Proof. By coinduction. If $t \in \mathcal{H}$ then $t_1, t_2 \in \mathcal{H}$ by Corollary 9, Lemma 42 and Lemma 19, so $t_1 \rightsquigarrow^\infty t_2$ and we may take $s = t_2$. So assume $t \notin \mathcal{H}$. Then by Lemma 45 either $t_1, t_2 \to^* x$ for some variable x, and then we may take $s = x$, or there are $v = f(v_1, \ldots, v_n)$, $u = f(u_1, \ldots, u_n)$ and $w = f(w_1, \ldots, w_n)$ such that $t_1 \to^\infty w$, $t_2 \to^\infty u$, and $v_i \to^\infty w_i$ and $v_i \rightsquigarrow u_i$ for $i = 1, \ldots, n$. By the coinductive hypothesis we obtain $s_1, \ldots, s_n$ with $w_i \rightsquigarrow^\infty s_i$ and $u_i \to^{2\infty} s_i$ for $i = 1, \ldots, n$. Take $s = f(s_1, \ldots, s_n)$. Then $t_1 \rightsquigarrow^\infty s$ and $t_2 \to^{2\infty} s$. ◀

▶ **Theorem 36** (Confluence modulo $\sim_{\mathcal{H}}$ of nearly orthogonal iTRSs).
Let S be a nearly orthogonal iTRS. If $t \sim_{\mathcal{H}} s$, $t \to_S^\infty t'$ and $s \to_S^\infty s'$ then there exist t'', s'' such that $t' \to_S^\infty t''$, $s' \to_S^\infty s''$ and $t'' \sim_{\mathcal{H}} s''$.

Proof. Assume $t \sim_{\mathcal{H}} s$, $t \to^\infty t'$ and $s \to^\infty s'$. By Lemma 18 there is u with $s \to^\infty u \sim_{\mathcal{H}} t'$. Hence $s \rightsquigarrow t'$ by Lemma 42. By Lemma 46, Lemma 42 and Lemma 11 there are t'', s'' with $t' \to^\infty t''$ and $s' \to^\infty s'' \sim_{\mathcal{H}} t''$. ◀

B Strongly convergent reductions

In this section we prove that for left-linear iTRSs the existence of coinductive infinitary reductions is equivalent to the existence of strongly convergent reductions. As a corollary, this also yields ω-compression of strongly convergent reductions. The equivalence proof is virtually the same as in [15]. The notion of strongly convergent reductions is the standard notion of infinitary reductions used in non-coinductive treatments of infinitary rewriting. See e.g. [21] for details. In the rest of this section we fix a left-linear iTRS $\mathcal{S} = \langle \Sigma, S \rangle$.

▶ **Definition 47.** On the set of terms we define a metric d by

$$d(t, s) = \inf\{2^{-n} \mid t^{\restriction n} = s^{\restriction n}\}$$

where $r^{\restriction n}$ for $r \in T^\infty(\Sigma)$ is defined as the term obtained by replacing all subterms of r at depth n by a fresh constant $\bot$. This defines a metric topology on the set of terms. Let α be an ordinal. A map $\phi : \{\beta \leq \alpha\} \to T^\infty(\Sigma)$ together with contraction steps $\sigma_\beta : \phi(\beta) \to_S \phi(\beta + 1)$ for $\beta < \alpha$ is a *strongly convergent S-reduction sequence of length α from $\phi(0)$ to $\phi(\alpha)$* if the following conditions hold:

1. if $\gamma \leq \alpha$ is a limit ordinal then $f(\gamma)$ is the limit in the metric topology on infinitary terms of the ordinal-indexed sequence $(\phi(\beta))_{\beta < \gamma}$,
2. if $\gamma \leq \alpha$ is a limit ordinal then for every $d \in \mathbb{N}$ there exists $\beta < \gamma$ such that for all β' with $\beta \leq \beta' < \gamma$ the redex contracted in the step $\sigma_{\beta'}$ occurs at depth greater than d.

We write $s \xrightarrow{Q,\alpha}_S t$ if Q is a strongly convergent S-reduction sequence of length α from s to t.

▶ **Theorem 48.**

1. *If $s \to_S^\infty t$ then there exists a strongly convergent R-reduction sequence from s to t of length at most ω.*
2. *If there exists a strongly convergent S-reduction sequence from s to t then $s \to_S^\infty t$.*

Proof. The proof is a straightforward adaptation of the proof of Theorem 3 in [15].

Suppose that $s \to_S^\infty t$. By traversing the infinite derivation tree of $s \to_S^\infty t$ and accumulating the finite prefixes by concatenation, we obtain a reduction sequence of length at most ω which satisfies the depth requirement by construction.

For the other direction, by induction on α we show that if $s \xrightarrow{Q,\alpha}_S t$ then $s \to_S^{2\infty} t$, which suffices for $s \to_S^\infty t$ by Lemma 11 (recall that the proofs of lemmas 6-11 depended only on the left-linearity of S). There are three cases.

- $\alpha = 0$. If $s \xrightarrow{Q,0}_S t$ then $s = t$, so $s \to_S^{2\infty} t$.
- $\alpha = \beta + 1$. If $s \xrightarrow{S,\beta+1}_S t$ then $s \xrightarrow{Q',\beta}_S s' \to_S t$. Hence $s \to_S^{2\infty} s'$ by the inductive hypothesis. Then $s \to_S^\infty s' \to_S t$ by Lemma 11. So $s \to_S^\infty t$ by Lemma 7.
- α is a limit ordinal. By coinduction we show that if $s \xrightarrow{Q,\alpha}_S t$ then $s \to_S^{2\infty} t$. By the depth condition there is $\beta < \alpha$ such that for every $\gamma \geq \beta$ the redex contracted in S at γ occurs at depth greater than zero. Let t_β be the term at index β in Q. Then by the inductive hypothesis we have $s \to_S^{2\infty} t_\beta$, and thus $s \to_S^\infty t_\beta$ by Lemma 11. There are two cases.
  - $t_\beta = x$. This is impossible because then there can be no contraction of t_β at depth greater than zero.
  - $t_\beta = f(t_1, \ldots, t_n)$. Then $t = f(u_1, \ldots, u_n)$ and the tail of the reduction S past β may be split into n parts: $t_i \xrightarrow{Q_i,\delta_i}_S u_i$ with $\delta_i \leq \alpha$ for $i = 1, \ldots, n$. Then $t_i \to_S^{2\infty} u_i$ by the inductive and/or the coinductive hypothesis. Since $s \to_S^\infty f(t_1, \ldots, t_n)$ we obtain $s \to_S^{2\infty} f(u_1, \ldots, u_n) = t$.

◀

▶ **Corollary 49** (ω-compression)**.** *If there exists a strongly convergent S-reduction sequence from s to t then there exists such a sequence of length at most ω.*