APPOXIMATION & COMPLEXITY

HOMEWORKS

Due date is November 18, 2019

(1) Show that the following space of infinite sequences
$$c_0 := \{(x_1, x_2, \ldots) : \; x_i \in \mathbb{R}, \; \lim_{i\to\infty} x_i = 0\}$$
with the norm $\max_{i\geqslant 1} |x_i|$ is not strictly convex.

(2) Find the best approximation in $L_2([0,1])$ for the function $f(x) = x$ with respect to the space spanned by $v_1(x) = \mathrm{e}^x$ i $v_2(x) = \mathrm{e}^{2x}$.

(3) Suppose that none of the points $a_1, a_2, \ldots, a_n$ is in the interval $[a, b]$. Show that then
$$\operatorname{span}\left(\frac{1}{x-a_1}, \frac{1}{x-a_2}, \ldots, \frac{1}{x-a_n}\right)$$
is a Haar space in $C([a, b])$.

(4) For what values of $a < b$ the space spanned by the functions
(a) $\{1, \cos(x), \cos(2x), \ldots, \cos(nx)\}$
(b) $\{\sin(x), \sin(2x), \ldots, \sin(nx)\}$
is a Haar space in $C([a, b])$?

(5) Let D be the unit sphere,
$$D = \{\vec{x} \in \mathbb{R}^s : \; \|\vec{x}\|_2 = 1\} \subset \mathrm{R}^s.$$
For what values of s and n one can find Haar spaces of dimension n that are subspaces of $C(D)$?

(6) Find a polynomial p of degree $\leqslant 3$ that minimizes
$$\sup_{-1\leqslant x\leqslant 1} |\,|x| - p(x)\,|.$$

(7) Find a trigonometric polynomial of the form
$$v(t) = a_0 + a_1 \sin t + b_1 \cos t$$
that best approximates the function $\sin(t/2)$ in the uniform norm on the interval $[-\pi, \pi]$.

(8) In the set of polynomials p of degree $\leqslant n$ such that $p(0) = 1$ find p^* that has minimal uniform norm on $[1, 2]$.

(9) Let p be a polynomial of degee at most n such that $\|p\|_{C([-1,1])} \leqslant 1$. Show that then for any $|x| \geqslant 1$ we have $|p(x)| \leqslant |T_n(x)|$.

(10) Show that among all the polynomials of degree at most n such that $p'(1) = A$, the polynomial AT_n/n^2 has the minimal uniform norm in $[-1, 1]$.

Due date is December 9, 2019

(11) Let the operator $L : C([a, b]) \to C([a, b])$ be given as
$$(Lf)(x) = \sum_{i=1}^{n} f(x_i) g_i(x),$$

where $a \leqslant x_1 < \cdots < x_n \leqslant b$ and $g_i \in C([a,b])$. Show that L is positive if and only if all the functions g_i assume nonnegative values only.

(12) Let $L : C([a,b]) \to C([a,b])$ be a positive linear operator satisfying $Lw = w$ for all polynomials of degree at most 2. Show that then $Lf = f$ for all $f \in C([a,b])$.

(13) Show that if f is a polynomial of degree at most k then the same property possess all the Bernstein polynomials $B_n f$.

(14) Let $\mathcal{E}$ be the family of projections $L : C([a,b]) \to \mathcal{P}_{n+1}$, and $\hat{\mathcal{E}}$ be the family of projections $\hat{L} : C([-1,1]) \to \mathcal{P}_{n+1}$. Show that

$$\inf_{L_n \in \mathcal{E}} \|L_n\| = \inf_{\hat{L}_n \in \hat{\mathcal{E}}} \|\hat{L}_n\|.$$

(15) Let $L : C([-1,1]) \to \mathcal{P}_3$ be the interpolation operator corresponding to some points x_i, $i = 0,1,2$. Show that

$$\min_{-1 \leqslant x_0 < x_1 < x_2 \leqslant 1} \|L\| = \frac{5}{4},$$

and for the Chebyshev points we have $\|L\| = 5/3$. What is $\|L_2\|$ for the equispaced points $-1, 0, 1$?

(16) Show the following properties of the Chebyshev polynomial T_n.
(a) $(1 - x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0$
(b) $T_{2n}(x) = T_n(2x^2 - 1)$
(c) $T_n(T_m) = T_{nm}$

(17) Let $f(x) = x^3$. Find possibly minimal n such that $B_n f$ approximates f with error at most 10^{-8} with respect to the uniform norm on $[0,1]$?

(18) Show that if f i f' are continuous on $[0,1]$ then for any $\epsilon > 0$ there is a polynomial p such that $\|f - p\| \leqslant \epsilon$ and $\|f' - p'\| \leqslant \epsilon$, where the norm is uniform on $[0,1]$.

(19) Let $X = L_2([0,1])$, $f(x) = x^m$ and $v_k(x) = x^{p_k}$, $k = 1, 2, \ldots, n$, where

$$0 \leqslant m < p_1 < p_2 < \cdots < p_n.$$

Let $V_n = \operatorname{span}(v_1, v_2, \ldots, v_n)$. Show that

$$\operatorname{dist}(f, V_n)^2 = \frac{1}{2m+1} \prod_{k=1}^{n} \left(\frac{m - p_k}{m + p_k + 1} \right)^2.$$

Hint.

$$\det\left(\left[\frac{1}{a_k + b_k} \right]_{k,l=1}^{n} \right) = \frac{\prod_{k>l}(a_k - a_l)(b_k - b_l)}{\prod_{k,l}(a_k + b_l)}.$$

(20) Let $0 \leqslant m < p_1 < p_2 < \cdots$. Show that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(\frac{m - p_k}{m + p_k + 1} \right)^2 = 0$$

if and only if

$$\sum_{k=2}^{\infty} \frac{1}{p_k} = \infty.$$

(21) *(Münz theorem I)* Show that the space spanned by the functions $v_k(x) = x^{p_k}$, where $0 \leqslant p_1 < p_2 < \cdots$, is dense in $L_2([0,1])$ if and only if $\sum_{k=2}^{\infty} 1/p_k = \infty$. **Hint.** Use Problems 19 i 20 and the fact that algebraic polynomials are dense in $L_2([0,1])$.

(22) *(Münz theorem II)* Show that the space spanned by the functions $v_k(x) = x^{p_k}$, where $0 \leqslant p_1 < p_2 < \cdots$, is dense in $C([0,1])$ if and only if $p_1 = 0$ and $\sum_{k=2}^{\infty} 1/p_k = \infty$. **Hint.** Use Problem (21).

(23) Is the space spanned by $1, x^{p_1}, x^{p_2}, \ldots, x^{p_n}, \ldots$, where p_n are successive primes, dense in $C([0,1])$?

Due date is January 27, 2020

(24) Let $1 \leqslant \alpha \leqslant 2$. Find an example of a normed space G and a set $A \subset G$ such that $d(A) = \alpha\, r(A)$.

(25) Find an example of a normed space G and a set $A \subset G$ that does not have any center.

(26) Suppose $A \subset G$ is symmetric about some $g^* \in G$; i.e., if $g \in A$ then $2g^* - g \in A$. Show that then g^* is a center of A and $d(A) = 2\, r(A)$.

(27) Show that if for two linear information operators $N_1, N_2 : F \to \mathbb{R}$ we have $\ker N_1 = \ker N_2$ then for any solution operator $S : F \to G$ and for any class $\mathcal{F} \subset F$ of problem instances

$$\mathrm{rad}^{\mathrm{wor}}(N_1) = \mathrm{rad}^{\mathrm{wor}}(N_2).$$

(28) Consider the problem of uniform approximation of functions $f : [0,1] \to \mathbb{R}$ satisfying the Lipschitz condition with constant 1, based on information

$$N_n(f) = \big(f(t_1), f(t_2), \ldots, f(t_n)\big),$$

where $0 \leqslant t_1 < t_2 < \cdots < t_n \leqslant 1$. Show that the natural spline of degree 1 with knots t_i interpolating f at the same points t_i, $1 \leqslant i \leqslant n$, provides an optimal algorithm. What would be the answer if the uniform approximation were replaced by L^2-approximation?

(29) Consider the problem of weighted integration

$$S(f) = \int_0^{+\infty} f(x)\, \mathrm{e}^{-x}\, \mathrm{d}x,$$

for functions $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying the Lipschitz condition with constant 1 and such that $f(0) = 0$, based on information

$$N_n(f) = \big(f(t_1), f(t_2), \ldots, f(t_n)\big),$$

where $0 \leqslant t_1 < t_2 < \cdots < t_n < +\infty$. Find $\mathrm{rad}^{\mathrm{wor}}(N_n)$ and an optimal linear algorithm, if it exists.

(30) Is the composed trapezoid rule an optimal algorithm for the problem (29)?

(31) Let $\mathcal{F}$ be the class of functions $f \in C^1([0,1])$ such that $|f'(x)| \leqslant \psi(x)$, where ψ is nonnegative, nonincreasing, and continuous. Let $N(f) = (f(0), f(1))$. Find, if exist, central algorithm and optimal linear algorithm for the problems of:
(a) integration,

(b) uniform approximation.
What is the radius of information for both problems?

(32) Consider the problem of weighted integration as in the problem (29). Show that the nth optimal information is given as $N_n^*(f) = (f(t_1^*), \dots, f(t_n^*))$, where

$$t_i^* = -2 \ln \left(1 - \frac{i}{n+1}\right), \qquad 1 \leqslant i \leqslant n,$$

and the nth minimal radius

$$r^*(n) = \mathrm{rad}^{\mathrm{wor}}(N_n^*) = \frac{1}{n+1}.$$

(33) Let $F = C^1([0,1])$ and $N_n(f) = (f(0), f(1/n), f(2/n) \dots, f(1))$. How to construct a linear algorithm $\Phi_n : \mathbb{R}^n \to \mathcal{P}_{n+1}$ such that for all $f \in F$ it holds

$$\|f - \Phi_n(N_n(f))\|_C \leqslant \frac{\alpha}{n} \|f'\|_C,$$

for some α independent of n and f.