On the optimal convergence rate of universal and non-universal algorithms for multivariate integration and approximation

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We discuss the optimal rate of convergence of algorithms for multivariate integration and approximation of d-variate functions. We consider functions which belong to reproducing kernel Hilbert spaces $H(K_d)$. Here K_d is an arbitrary kernel whose all partial derivatives up to order r satisfy a Hölder-type condition with exponent 2β. We study algorithms which use n function values and analyze their rate of convergence as n tends to infinity. Without loss of generality it is enough to consider linear algorithms. We focus on *universal* algorithms, whose weights and sample points are only dependent on d, r and β but not on the specific kernel K_d, and *non-universal* algorithms whose weights and sample points may depend additionally on the specific kernel K_d.

We prove that the optimal (largest possible) rate of convergence of universal algorithms is $(r+\beta)/d$. This holds for both, multivariate integration and approximation. Furthermore, we prove that the optimal rate of convergence of non-universal algorithms is $1/2+(r+\beta)/d$ for multivariate integration and $a+(r+\beta)/d$ with $a \in [1/(4+4(r+\beta)/d), 1/2]$ for multivariate approximation. Hence, the optimal rate of convergence of universal algorithms suffers from the curse of dimensionality, i.e., it goes linearly with d^{-1} to zero and can be arbitrarily small if d is large relative to $r+\beta$, whereas the optimal rate of convergence of non-universal algorithms does *not* suffer from the curse of dimensionality since it is always at least $1/2$ for multivariate integration, and $1/4$ for multivariate approximation. This is the price we have to pay for using universal algorithms. On the other hand, if $r+\beta$ is large relative to d then the optimal rates of convergence for universal and non-universal algorithms are approximately the same.

We also consider the case when we have the additional knowledge that the kernel K_d has product structure, $K_{d,r,\beta} = \otimes_{j=1}^{d} K_{r_j,\beta_j}$. Here K_{r_j,β_j} are some univariate kernels which are r_j times continuously differentiable and whose derivatives up to order r_j satisfy a Hölder-type condition with exponent $2\beta_j$. In this case, the optimal order of convergence of universal algorithms is $q := \min_{j=1,2,\dots,d}(r_j+\beta_j)$ for both, multivariate integration and approximation. For non-universal algorithms the optimal rate of convergence is $1/2+q$ for multivariate integration and $a+q$ with $a \in [1/(4+4q), 1/2]$ for multivariate approximation. The optimal rate of convergence of universal algorithms for product kernels now depends on d only through the minimum of local regularities $r_j+\beta_j$. If we assume, for example, that $r_j \geq 1$ or $\beta_j \geq \beta > 0$ for all j, then the optimal rate is at least $\min(1,\beta)$, and the curse of dimensionality is *not* present for product kernels. This shows that the product form of reproducing kernels is essential and breaks the curse of dimensionality even by universal algorithms.