Adaption makes it easy to integrate functions with unknown singularities

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We study numerical integration $I(f) = \int_{0}^{T} f(x) \, dx$ for functions $f$ with singularities. Nonadaptive methods are inefficient in this case, and we show that the problem can be efficiently solved by adaptive quadratures at cost similar to that for functions with no singularities.

Consider first a class $\mathcal{F}_r$ of functions whose derivatives of order up to $r$ are continuous and uniformly bounded for any but one singular point. We propose adaptive quadratures $Q_n^*$, each using at most $n$ function values, whose worst case errors $\sup_{f \in \mathcal{F}_r} |I(f) - Q_n^*(f)| = \Theta(n^{-r})$. On the other hand, the worst case error of nonadaptive methods is $\Omega(n^{-1})$.

These worst case results do not extend to the case of functions with two or more singularities; however, adaption shows its power even for such functions in the asymptotic setting. That is, let $F_r^\infty$ be the class of $r$-smooth functions with arbitrary (but finite) number of singularities. Then a generalization of $Q_n^*$ yields adaptive quadratures $Q_n^{**}$ such that $|I(f) - Q_n^{**}(f)| = O(n^{-r})$ for any $f \in F_r^\infty$. In addition, we show that for any sequence of nonadaptive methods there are ‘many’ functions in $F_r^\infty$ for which the errors converge no faster than $n^{-1}$.

Results of numerical experiments are also presented.