How unprovable is Rabin's decidability theorem?

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August 2, 2016

Abstract

We study the strength of set-theoretic axioms needed to prove Rabin's theorem on the decidability of the MSO theory of the infinite binary tree. We first show that over the second-order arithmetic theory ACA_0, the complementation theorem for nondeterministic tree automata is equivalent to a statement expressing the determinacy of all Gale-Stewart games given by $\mathrm{Bool}(\mathbf{\Sigma^0_2})$ sets. It follows that the complementation theorem is provable from Π^1_3- but not Δ^1_3-comprehension.

We then use results due to MedSalem-Tanaka, Möllerfeld and Heinatsch-Möllerfeld to prove that

- the complementation theorem for nondeterministic tree automata,
- the decidability of the Π^1_3 fragment of MSO on the infinite binary tree,
- the positional determinacy of parity games,
- the determinacy of $\mathrm{Bool}(\mathbf{\Sigma^0_2})$ Gale-Stewart games, and
- the Π^1_3-reflection principle for Π^1_2-comprehension

are all equivalent over Π^1_2-comprehension. It follows in particular that Rabin's decidability theorem is not provable from Δ^1_3-comprehension.

1 Introduction

Rabin's decidability theorem [22] says that the monadic second order (MSO) theory of the infinite binary tree $\{0,1\}^*$ with the left and right successor relations is decidable. In the words of the book "The Classical Decision Problem" [2], Rabin's result is "one of the most important decidability theorems for mathematical theories and has numerous applications in several areas of mathematics and computer science" (for a discussion, see e.g. [2, Chapter 7]).

Unlike other prominent decidability results, such as the ones for Presburger arithmetic, real-closed fields or even the MSO theory of $(\mathbb{N}, \leq)$, Rabin's theorem appears likely to involve significant logical strength, in the sense of being unprovable without axioms asserting the existence of very abstract and complicated sets. This is suggested by the fact that MSO on $\{0,1\}^*$ is able to express

*Partially supported by Polish National Science Centre grant no. 2013/09/B/ST1/04390.

†Partially supported by Polish National Science Centre grant no. 2014/13/B/ST6/03595

the determinacy of certain infinite games with Borel winning conditions, and such determinacy is used in typical modern proofs of Rabin's theorem, dating back to [6]. Determinacy principles are notorious for requiring very large logical strength.

The framework of *reverse mathematics* (see [25]) offers a natural way of measuring the logical strength of a theorem. The idea is that many mathematical theorems can be formalized in the language of *second-order arithmetic*, a foundational axiomatic theory used already by Hilbert and Bernays. The most important axiom of second-order arithmetic is the comprehension scheme, stating the existence of any set of natural numbers defined by first- and second-order quantification over $\mathbb{N}$. Reverse mathematics proceeds by analyzing various mathematical statements and proving their equivalence, over a suitable weak base theory, to some rather limited form of comprehension. Most mathematical theorems analyzed in this fashion have turned out to require no more than the theory ACA_0 allowing only *arithmetical* comprehension, that is, the existence of sets defined without any second-order quantifiers. Theorems requiring strictly more than Π^1_1-comprehension, or the existence of sets defined in terms of one second-order quantifier, are quite exceptional. One such exception is the famous graph minor theorem, cf. [5]. A less known example is provided in [18], where a theorem in general topology is proved equivalent to Π^1_2-comprehension.

Determinacy theorems are a more extreme exception. Π^1_1-comprehension is barely enough to prove the determinacy of games in which the winning condition is the intersection of an open set and a closed set. Π^1_2-comprehension proves that F_σ games are determined [26], but can no longer do so for games given by arbitrary boolean combinations of F_σ sets (essentially[1] [14]). For arbitrary boolean combinations of $F_{\sigma\delta}$ sets, proving determinacy requires going beyond second-order arithmetic, an immensely strong theory in most other respects [17].

The aim of this paper is to analyze, reverse mathematics–style, the logical strength of Rabin's theorem. We first focus our attention on the complementation theorem for nondeterministic automata on infinite trees, which is the key ingredient in typical proofs of Rabin's theorem. We prove that the complementation theorem is equivalent over ACA_0 to the determinacy of Gale-Stewart games given by arbitrary boolean combinations of $\mathbf{\Sigma^0_2}$ (i.e., F_σ) sets. Using earlier work on determinacy, we conclude that the complementation theorem is provable in Π^1_3-, but not Π^1_2- or even Δ^1_3-comprehension.

We then consider Rabin's decidability result itself. Using the work of [14], we prove that over Π^1_2-comprehension, already the statement "the Π^1_3 fragment of the MSO theory of $\{0,1\}^*$ is decidable" is equivalent to the determinacy of arbitrary $\mathrm{Bool}(\mathbf{\Sigma^0_2})$ games, which makes it unprovable in Δ^1_3-comprehension. On the other hand, any version of the decidability theorem that can be stated in second-order arithmetic follows from Π^1_3-comprehension.

The final part of our work relies on techniques developed in Michael Möllerfeld's PhD thesis [16], which links Π^1_2-comprehension with an arithmetical version of the μ-calculus. Using a slight strengthening of Möllerfeld's results, we show that over Π^1_2-comprehension the following conditions are equivalent:

- the complementation theorem for nondeterministic tree automata,

[1] The original reasoning presented in [14] applies to $\mathbf{\Delta^0_3}$ sets, but it is easily adaptable to the case of boolean combinations of F_σ sets. We present this argument in Section 4.

- Rabin's decidability theorem,
- the determinacy of all $\mathrm{Bool}(\mathbf{\Sigma^0_2})$ games,
- the positional determinacy of all parity games and
- a purely logical *reflection principle*: "all Π^1_3 sentences provable using Π^1_2-comprehension are true".

Note that the unprovablity of this reflection principle in Π^1_2-comprehension follows from Gödel's second incompleteness theorem.

Section 2 presents the necessary background in reverse mathematics, automata theory and games. In Section 3, we give a first approximation of the logical strength of the complementation theorem for tree automata, proving that it implies $\mathrm{Bool}(\mathbf{\Sigma^0_2})$ determinacy and is implied by the positional determinacy of all parity games. We review some known results on the reverse mathematics of determinacy principles in Section 4. We then use those results to get a more exact characterization of the strength of the complementation theorem for automata (Section 5), and to analyze the strength of Rabin's decidability theorem (Section 6). Our final result linking complementation for automata, decidability of MSO, and determinacy statements with Π^1_3-reflection for Π^1_2-comprehension is discussed in Section 7.

2 Basic notions

2.1 Second-order arithmetic

Second-order arithmetic is a natural framework for studying the strength of axioms needed to prove theorems of countable mathematics, that is, the part of mathematics concerned with objects that can be represented using no more than countably many bits of information. The two-sorted language of second-order arithmetic, L_2, contains *first-order* variables $x, y, z, \ldots$, intended to range over natural numbers, and *second-order* variables $X, Y, Z, \ldots$, intended to range over sets of natural numbers. L_2 includes the usual arithmetic functions and relations $+, \cdot, \leq, 0, 1$ on the first-order sort, and the $\in$ relation which has one first-order and one second-order argument.

Full second-order arithmetic, Z_2, has axioms of three types: (i) axioms stating that the first-order sort is the non-negative part of a discretely ordered ring; (ii) comprehension axioms, or sentences of the form

$$\forall \bar{Y}\, \forall \bar{y}\, \exists X\, \forall x\, (x \in X \Leftrightarrow \varphi(x, \bar{Y}, \bar{y})),$$

where φ is an arbitrary formula of L_2 not containing the variable X; (iii) the induction axiom,

$$\forall X\ [0 \in X \wedge \forall x\, (x \in X \Rightarrow x+1 \in X) \Rightarrow \forall x\, (x \in X)]\,.$$

The language L_2 is surprisingly expressive, as the first-order sort can be used to encode arbitrary finite objects and the second-order sort can encode even such objects as complete separable metric spaces, continuous functions between them, and Borel sets within them (cf. [25, Chapters II.5, II.6, V.3]). Moreover, the theory Z_2 is very strong: almost all theorems from a typical undergraduate course that are expressible in L_2 can be proved in Z_2. In fact, the basic observation underlying the programme of reverse mathematics [25] is that many important theorems are equivalent to various fragments of Z_2, where the equivalence is proved in some specific weaker fragment, referred to as the *base theory*.

The most commonly used base theory is RCA_0, which guarantees only the existence of decidable sets. For relatively strong theorems such as the ones studied in this paper, a reasonable base theory is ACA_0, the fragment of Z_2 obtained by restricting the comprehension scheme to instances where the formula φ is *arithmetical*, that is, does not contain second-order quantifiers (second-order free variables are allowed). Stronger fragments can be obtained by allowing comprehension for φ with a fixed number of second-order quantifiers. A formula is Π^1_n if it has the form

$$\forall X_1\, \exists X_2 \ldots \mathsf{Q} X_n\, \psi$$

with ψ arithmetical; it is Σ^1_n if it is the negation of a Π^1_n formula; it is Δ^1_n if it is equivalent to both a Π^1_n and a Σ^1_n formula. The theory Π^1_n-CA_0 is obtained by restricting the comprehension scheme to instances where φ is Π^1_n. In the subtheory Δ^1_n-CA_0 of Π^1_n-CA_0, the comprehension scheme takes the form

$$\forall \bar{Y}\, \forall \bar{y}\, \left[\forall x\, (\varphi(x, \bar{Y}, \bar{y}) \Leftrightarrow \neg\psi(x, \bar{Y}, \bar{y})) \Rightarrow \exists X\, \forall x\, (x \in X \Leftrightarrow \varphi(x, \bar{Y}, \bar{y}))\right],$$

where both φ and ψ are Π^1_n.

The induction scheme Σ^1_n-IND consists of the sentences

$$\forall \bar{Y}\, \forall \bar{y}\, \left[\varphi(0, \bar{Y}, \bar{y}) \wedge \forall x\, (\varphi(x, \bar{Y}, \bar{y}) \Rightarrow \varphi(x+1, \bar{Y}, \bar{y})) \Rightarrow \forall x\, \varphi(x, \bar{Y}, \bar{y})\right]$$

where φ is Π^1_n or Σ^1_n. By the induction and comprehension axioms, Π^1_n-CA_0 proves Σ^1_n-IND. On the other hand, Π^1_n-CA_0 does not prove Σ^1_{n+1}-IND, while $\bigcup_{n\in\omega} \Sigma^1_n$-$\mathsf{IND}$ does not prove Π^1_n-CA_0 even assuming Π^1_{n-1}-CA_0 (cf. [25]).

In this paper, the most prominent theory is Π^1_2-CA_0. We present some important principles provable in Π^1_2-CA_0. The first two principles are related to countable sequences of sets. Such a sequence $\langle X_i \rangle_{i\in\mathbb{N}}$ can be represented by a single set X if we let

$$x \in X_i \Leftrightarrow \langle i, x\rangle \in X,$$

where $\langle \cdot, \cdot \rangle$ is some standard pairing function. We can then write $X \in \langle X_i \rangle_{i\in\mathbb{N}}$ if there is some i such that $X = X_i$.

Definition 2.1. *The* Σ^1_2 axiom of choice, Σ^1_2-AC*, is the axiom scheme consisting of sentences of the form*

$$\forall \bar{Z}\, \forall \bar{z}\, [\forall x\, \exists Y\, \varphi(x, Y, \bar{Z}, \bar{z}) \Rightarrow \exists \langle Y_x \rangle_{x\in\mathbb{N}}\, \forall x\, \varphi(x, Y_x, \bar{Z}, \bar{z})],$$

where φ is Σ^1_2.

Theorem 2.2. *(see [25, Theorem VII.6.9])* Π^1_2-$\mathsf{CA}_0 \vdash \Sigma^1_2$-$\mathsf{AC}$.

A sequence $\langle X_i \rangle_{i\in\mathbb{N}}$ can be regarded as a countable model for the language L_2, with $\mathbb{N}$ as the first-order universe and $\{X_i\}_{i\in\mathbb{N}}$ as the second-order universe. Such a model is called *a countable coded model* and typically denoted M.

Definition 2.3. *A countable coded model M is a* β_2-model *if for all $\bar{x} \in \mathbb{N}, \bar{X} \in M$, and each Π^1_2 formula $\varphi(\bar{x}, \bar{X})$,*

$$M \models \varphi(\bar{x}, \bar{X}) \text{ iff } \varphi(\bar{x}, \bar{X}) \text{ is true.}$$

(A completely formal version of this definition involves truth definitions for Π^1_2 sentences and can be found for instance in [25, Definition VII.7.2]). Note that any true Π^1_3 sentence remains true in each β_2 model.

Theorem 2.4. *(see [25, Theorem VII.6.9 and VII.7.4])* Π^1_2-CA_0 *proves:*

$$\forall X\, \exists M\, (M \textit{ is a } \beta_2\textit{-model and } X \in M).$$

The final principle we have to discuss concerns fixpoints of iterations of certain operators on sets. A Δ^1_2 *operator* is given by a pair of Π^1_2 formulas $\varphi(x, X), \psi(x, X)$ (possibly with parameters) such that

$$\forall x\, \forall X\, (\varphi(x, X) \Leftrightarrow \neg\psi(x, X)).$$

We think of φ and ψ as defining an operator $\Gamma_{\varphi,\psi}$ on sets that maps a set X to $\{x \in \mathbb{N} : \varphi(x, X)\}$.

Definition 2.5. *The axiom scheme* Δ^1_2-ID *asserts the following for every* Δ^1_2 *operator given by formulas* φ, ψ*: there exists a prewellordering (reflexive transitive relation connected on its field)* $\preccurlyeq$ *with field* P *such that*

$$\forall x\, \forall z\, [x \preccurlyeq z \Leftrightarrow x \prec z \vee x \in \Gamma_{\varphi,\psi}(\{y : y \prec z\})]$$

and

$$\Gamma_{\varphi,\psi}(P) \subseteq P.$$

If $\preccurlyeq, P$ are as in Definition 2.5, then P can be regarded as a fixed point of the operator $X \mapsto \Gamma_{\varphi,\psi}(X) \cup X$ generated inductively in the usual way. If φ is a Σ^1_1 formula, then ψ can of course be simply $\neg\varphi$; the scheme consisting of the instances of Δ^1_2-ID for $\varphi \in \Sigma^1_1$ is known as Σ^1_1-ID.

Theorem 2.6. *[14, Theorem 5.1]* Π^1_2-$\mathsf{CA}_0 \vdash \Delta^1_2$-$\mathsf{MI}$.

Notational convention. As above, we will use the letter $\mathbb{N}$ to denote the natural numbers as formalized in second-order arithmetic, that is, the domain of the first-order sort. On the other hand, the symbol ω will stand for the concrete, or standard, natural numbers. For instance, given a theory T and a formula $\varphi(x)$, "T proves $\varphi(n)$ for all $n \in \omega$" will mean "$\mathrm{T} \vdash \varphi(0), \mathrm{T} \vdash \varphi(1), \ldots$", which does not have to imply $\mathrm{T} \vdash \forall x \in \mathbb{N}\, \varphi(x)$.

The letter n will be used exclusively to denote elements of ω. As formal variables of the first-order sort, we will typically use $x, y, z, \ldots$, but sometimes also i, j, k, ℓ or other lowercase letters different from n.

2.2 Automata and MSO

The study of monadic second order logic on the infinite binary tree relies heavily on the theory of automata on infinite words and infinite trees. Our presentation of automata theory and MSO is based on [27], with some modifications.

Given a finite set Σ (the so-called *alphabet*), a *word over alphabet* Σ is simply a mapping $f \colon \mathbb{N} \to \Sigma$. A *tree* (or more precisely, *labelled binary tree*) *over* Σ is a mapping $T \colon \{0,1\}^* \to \Sigma$, where $\{0,1\}^*$ stands for the set of all finite binary strings[2]. Note that words and trees over Σ have another natural representation as structures with universe $\mathbb{N}$ resp. $\{0,1\}^*$ and $\mathrm{card}(\Sigma)$ disjoint unary relations whose union is the universe.

Definition 2.7 (Nondeterministic Parity Word Automaton)**.** *A* nondeterministic parity word automaton *over a finite alphabet* Σ *is a tuple* $\mathcal{A} = \langle \Sigma, Q, q^I, \Delta, \mathrm{rk}\rangle$ *where* Q *is a finite* set of states*,* $q^I \in Q$ *is the* initial state*,* $\Delta \subseteq Q \times \Sigma \times Q$ *is the* transition relation*, and* $\mathrm{rk}\colon Q \to \mathbb{N}$ *is the* rank function*.*

We use the letter δ to denote individual *transitions* of $\mathcal{A}$, i.e. elements of Δ.

A *run* of a nondeterministic parity word automaton $\mathcal{A}$ on a word f over Σ is a labelling $\rho\colon \mathbb{N} \to \Delta$ which is consistent, that is for every position $i \in \mathbb{N}$ in f, if $\rho(i) = (q_i, a_i, q_{i+1})$, then

1. $f(i) = a_i$,
2. $\rho(i+1) = (q_{i+1}, a_{i+1}, q_{i+2})$,

and $q_0 = q^I$, that is the first state in the run is the initial state of the automaton. Intuitively, the transitions in a run have to be such that after moving from a position i to the successor position $i+1$, the run continues from the state q_{i+1} indicated by the transition $\rho(i)$.

A run ρ on a word f is *accepting* if the states $q_0, q_1, \ldots$ along this run satisfy *the parity acceptance condition*:

$$\liminf_{i \in \mathbb{N}} \mathrm{rk}(q_i) \text{ is even.}$$

A word f is *accepted* by the automaton $\mathcal{A}$ if there exists an accepting run of $\mathcal{A}$ on f.

Definition 2.8 (Nondeterministic Parity Tree Automaton)**.** *A* nondeterministic parity tree automaton *over a finite alphabet* Σ *is a tuple* $\mathcal{A} = \langle \Sigma, Q, q^I, \Delta, \mathrm{rk}\rangle$ *where* Q *is a finite* set of states*,* $q^I \in Q$ *is the* initial state*,* $\Delta \subseteq Q \times \Sigma \times Q \times Q$ *is the* transition relation*, and* $\mathrm{rk}\colon Q \to \mathbb{N}$ *is the* rank function*.*

If a transition $\Delta \ni \delta$ has the form (q, a, q_0, q_1), we may write $q_0 = \delta(a, q, 0)$ and $q_1 = \delta(a, q, 1)$. The idea is that if the automaton is in some vertex of the tree and reads the letter a while in state q, it may use δ to move simultaneously to the left son of the vertex in state q_0 and to the right son in state q_1.

A *run* of a nondeterministic parity tree automaton $\mathcal{A}$ on a tree T over Σ is a labelling $\rho\colon \{0,1\}^* \to \Delta$ which is consistent, that is for every vertex $v \in \{0,1\}^*$ if $\rho(v) = (q_v, a_v, q_{v0}, q_{v1})$, then

1. $T(v) = a_v$,
2. $\rho(vi) = (q_{vi}, a_{vi}, q_{vi0}, q_{vi1})$ for $i = 0, 1$,

and $q_\epsilon = q^I$ where ϵ denotes the empty string. Intuitively, the transitions in a run have to be chosen so that after moving from a vertex v in the tree to its sons $v0, v1$, the run continues from the states q_{v0}, q_{v1} indicated by the transition $\rho(v)$.

[2] In automata theory, a more typical choice of symbols would be w and t instead of, respectively, f and T. However, in this paper we decided to use capital letters to emphasize that these are objects of the second-order sort.

A run ρ on a tree T is *accepting* if for every branch $\pi \in \{0,1\}^{\mathbb{N}}$, the states along this branch, $q_{\pi\restriction_0}, q_{\pi\restriction_1}, \ldots$ satisfy *the parity acceptance condition*:

$$\liminf_{i\in\mathbb{N}} \mathrm{rk}(q_{\pi\restriction_i}) \text{ is even.}$$

A tree T is *accepted* by the automaton $\mathcal{A}$ if there exists an accepting run of $\mathcal{A}$ on T.

Definition 2.9 (Deterministic automaton)**.** *We call a nondeterministic parity word automaton $\mathcal{A} = \langle \Sigma, Q, q^I, \Delta, \mathrm{rk}\rangle$ a* deterministic word automaton *if the transition relation Δ is a graph of a function $Q \times \Sigma \to Q$. Similarly, we call a nondeterministic parity tree automaton $\mathcal{A} = \langle \Sigma, Q, q^I, \Delta, \mathrm{rk}\rangle$ a* deterministic tree automaton *if the transition relation δ is a graph of a function $Q \times \Sigma \to Q \times Q$.*

Note that a deterministic word (resp. tree) automaton has exactly one run on each word (resp. tree).

We consider monadic second order logic MSO over the structure $(\{0,1\}^*, S_0, S_1)$, where S_0 and S_1 are the left and right successor relations, respectively ($S_0(v,w)$ holds iff $w = v0$ and $S_1(v,w)$ holds iff $w = v1$). The language of MSO over $(\{0,1\}^*, S_0, S_1)$ contains first-order variables $x, y, \ldots$ ranging over elements of $\{0,1\}^*$ and second-order variables $X, Y, \ldots$ ranging over subsets of $\{0,1\}^*$. Atomic formulas have the form $x = y$, $S_0(x,y)$, $S_1(x,y)$ and $x \in X$. The language of MSO has the usual boolean connectives and the quantifiers $\exists x, \exists X$. In the language with just the two successors, as opposed to the arithmetical language with $+$ and $\cdot$, there is no way to define a pairing function in MSO, so the restriction to *unary* second-order quantifiers is very important.

2.3 Games

We will be concerned with games in which winning conditions take the form of boolean combinations of Σ^0_2 properties. To talk about arbitrary boolean combinations of Σ^0_2 statements, we formalize a version of the so-called *difference hierarchy* over Σ^0_2.

Let f be a distinguished second-order variable which is assumed to represent (the graph of) a function from $\mathbb{N}$ to $\mathbb{N}$. A $(\Sigma^0_2)_x$ *formula* $\varphi(f)$ is given by a number x and a Π^0_2 formula $\psi(y,f)$, possibly with other parameters, such that for all f, $\psi(x,f)$ always holds, and for all $z<y<x$, if $\psi(z,f)$, then $\psi(y,f)$. We say that $\varphi(f)$ *holds* if

$$\psi(0,f) \vee \psi(2,f) \vee \ldots \vee \psi(2\lfloor x/2\rfloor, f);$$

or formally, if the smallest $y \leq x$ such that $y = x \vee \psi(y,f)$ is even.

Note that in ACA_0 a $(\Sigma^0_2)_1$ formula is the same thing as a Π^0_2 formula, in the sense that for every $(\Sigma^0_2)_1$ formula $\varphi(f)$ there is a Π^0_2 formula $\xi(f)$ such that $\mathsf{ACA}_0 \vdash \forall f\,(\varphi(f) \Leftrightarrow \xi(f))$, and vice versa. Similarly, a $(\Sigma^0_2)_{x+1}$ formula is the same thing as the disjunction of a Π^0_2 formula and a negated $(\Sigma^0_2)_x$ formula. So, our class $(\Sigma^0_2)_x$ is dual to the usual x-th level of the difference hierarchy over Σ^0_2. It is a matter of routine if tedious verification that every concrete boolean combination of Σ^0_2 properties can be expressed by a $(\Sigma^0_2)_n$ formula for some $n \in \omega$.

A $(\mathbf{\Sigma^0_2})_x$ *Gale-Stewart game* (briefly, a $(\mathbf{\Sigma^0_2})_x$ *game*) is given by a $(\Sigma^0_2)_x$ formula $\varphi(f)$, again, possibly with other parameters (in accordance with the conventions from descriptive set theory, the boldface font serves precisely to indicate the possible presence of parameters). The game is played by two players, 0 and 1, who alternately choose natural numbers $f(0), f(1), \ldots$, building an infinite sequence $f \in \mathbb{N}^{\mathbb{N}}$. Player 0 wins the game if $\varphi(f)$ holds, and player 1 wins otherwise. The notions of a *strategy* and *winning strategy* for each player are defined as usual. The game given by $\varphi(f)$ is *determined* if one of the players has a winning strategy. For precise definitions, see e.g. [9].

Definition 2.10. $(\mathbf{\Sigma^0_2})_x$-Det *is the* Π^1_3 *statement "all* $(\mathbf{\Sigma^0_2})_x$ *games are determined".*

$(\mathbf{\Sigma^0_2})_x$-Det* *is the same statement restricted to games in the Cantor space* $\{0,1\}^{\mathbb{N}}$ *instead of the Baire space* $\mathbb{N}^{\mathbb{N}}$*, that is to games where the players are required to choose only numbers from* $\{0,1\}$ *in each move.*

Obviously, $(\mathbf{\Sigma^0_2})_x$-Det implies $(\mathbf{\Sigma^0_2})_x$-Det*. On the other hand, it easily follows from [19, Lemma 4.2] that, provably in ACA_0, $(\mathbf{\Sigma^0_2})_{x+1}$-Det* implies $(\mathbf{\Sigma^0_2})_x$-Det. So, we have:

Proposition 2.11. $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ *and* $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}^*]$ *are equivalent in* ACA_0.

A *parity game of index* $(0, x)$ is a tuple $G = \langle V_\exists, V_\forall, v_I, E, \mathrm{rk}\rangle$, where: $V_\exists$ and $V_\forall$ are disjoint sets with union $V := V_\exists \cup V_\forall$; v_I is an element of V; $E \subseteq V^2$ is such that for every $v \in V$ there is some w with $(v, w) \in E$; and the *rank function* rk is a function from V to $\{0, \ldots, x\}$. The game is played on the *arena* V by the two players $\exists$ and $\forall$. The game starts in the *initial position* v_I; and if it reaches position $v \in V_p$, $p \in \{\exists, \forall\}$, then player p moves to some w such that $(v, w) \in E$. Formally, a *play* of G is a sequence $\langle v_i\rangle_{i\in\mathbb{N}}$ such that $v_0 = v_I$ and $(v_i, v_{i+1}) \in E$ for all i. Player $\exists$ wins the play exactly if

$$\liminf_{i\in\mathbb{N}} \mathrm{rk}(v_i) \text{ is even.}$$

$(\mathbf{\Sigma^0_2})_x$-Det implies that all parity games of index $(0, x)$ are determined. However, we also need a stronger notion of determinacy. A *positional strategy* (also known as a *memoryless*, or *forgetful*, strategy) for $\exists$ (resp. $\forall$) is a function σ from $V_\exists$ (resp. $V_\forall$) into V such that for all v in the domain, $(v, \sigma(v)) \in E$. A positional strategy σ is *winning* if the player using σ wins every play consistent with σ. The game G is *positionally determined* if one of the two players has a positional winning strategy. Basic notions related to parity games come originally from [4].

Definition 2.12 (Tree-like parity games)**.** *A parity game* G *is* tree-like *if* $V = \{0,1\}^{<\mathbb{N}}$, $V_\exists = \bigcup_{i\in\mathbb{N}} \mathbb{N}^{2i}$*, and* $(v, w) \in E$ *iff* $w = v0$ *or* $w = v1$.

Our notion of $(\Sigma^0_2)_x$ formula was chosen to make the proof of the following lemma rather straightforward.

Lemma 2.13. ACA_0 *proves that for every* x, $(\mathbf{\Sigma^0_2})_x$-Det* *holds if and only if all tree-like parity games of index* $(0, x)$ *are positionally determined.*

Proof. Clearly, a tree-like parity game is determined iff it is positionally determined. Moreover, such a game of index $(0, x)$ can be thought of as a $(\mathbf{\Sigma^0_2})_x$ Gale-Stewart game in the Cantor space given by x and the formula

$$\psi(y, f) := \exists z \leq y \, \forall i \, \exists j > i \, (\mathrm{rk}(f \restriction_j) = z).$$

Note that ψ is Π^0_2 in ACA_0. This proves the left-to-right direction.

In the other direction, consider a $(\mathbf{\Sigma^0_2})_x$ Gale-Stewart game given by the Π^0_2 formula $\psi(y, f)$. By a normal form result for Π^0_2 analogous to the normal form for Σ^0_1 presented as [25, Theorem II.2.7], we may assume that $\psi(y, f)$ has the shape $\forall z \, \exists u \, \delta(y, z, u, f \restriction_u)$ with all quantifiers in δ bounded by u. Define the tree-like parity game of index $(0, x)$ by letting $\mathrm{rk}(f \restriction_i)$ equal the smallest such $y < x$ that for some j,

$$\forall z < j \, \exists u \leq i \, \delta(y, z, u, f \restriction_u)$$

holds, but the number of $k < i$ with $\mathrm{rk}(f \restriction_i) = y$ is strictly smaller than j. If there is no such y, let $\mathrm{rk}(f \restriction_i)$ equal x. It is easy to verify in ACA_0 that for every $f \in \{0, 1\}^{\mathbb{N}}$, $\liminf_i \mathrm{rk}(f \restriction_i)$ is exactly the smallest y such that $y = x \vee \psi(y, f)$. Thus, determinacy of the parity game implies determinacy of the game given by x and ψ. $\square$

3 Complementation: first take

In this section, we begin to study the logical strength of the complementation theorem for nondeterministic tree automata. We prove that the theorem implies $\forall x [(\mathbf{\Sigma^0_2})_x\text{-Det}]$ and is in turn implied by the positional determinacy of all parity games.

The proofs of our implications work in ACA_0 and, unlike the proofs in later sections of this paper, do not rely on any earlier results related to the logical strength of determinacy principles.

Theorem 3.1. ACA_0 *proves the implications* (1) $\Rightarrow$ (2) *and* (2) $\Rightarrow$ (3) *for:*

(1) *all parity games are positionally determined,*

(2) *for every nondeterministic tree automaton $\mathcal{A}$ there exists a nondeterministic tree automaton $\mathcal{B}$ such that for any tree T, $\mathcal{B}$ accepts T iff $\mathcal{A}$ does not accept T,*

(3) $\forall x [(\mathbf{\Sigma^0_2})_x\text{-Det}]$, *or equivalently, all treelike parity games are positionally determined.*

Remark. It would be possible to give a somewhat technical definition of a class of games whose positional determinacy can be proved equivalent to the complementation theorem by the arguments used to prove Theorem 3.1. However, we refrain from doing that since we are later able to improve Theorem 3.1 to a nicer equivalence result, namely Theorem 5.1.

Proof of Theorem 3.1. (1) $\Rightarrow$ (2). We formalize a standard proof of the complementation theorem for tree automata, similar e.g. to the one in [27]. Let $\mathcal{A} = \{\Sigma_{\mathcal{A}}, Q_{\mathcal{A}}, q^I_{\mathcal{A}}, \Delta_{\mathcal{A}}, \mathrm{rk}_{\mathcal{A}}\}$ be a parity tree automaton. We may assume w.l.o.g. that for each $a \in \Sigma_{\mathcal{A}}$, $q \in Q_{\mathcal{A}}$, there is at least one transition in $\Delta_{\mathcal{A}}$ of the form $(q, a, \cdot, \cdot)$. This is because $\mathcal{A}$ can be easily modified so as to satisfy this condition without changing the class of accepted trees.

Given a labelled binary tree T, consider the following game $G_{\mathcal{A},T}$ between two players, Automaton and Pathfinder. The set of Automaton's positions is $\{0,1\}^* \times Q_{\mathcal{A}}$, with $(\emptyset, q^I_{\mathcal{A}})$ the starting position. The set of Pathfinder's positions is $\{0,1\}^* \times \Delta_{\mathcal{A}}$. Given a position (w, q), Automaton can choose a transition from q, that is, move to the position (w, δ), where $\Delta_{\mathcal{A}} \ni \delta = (q, T(w), q_0, q_1)$. Pathfinder responds by deciding which direction to take from w, that is, by moving either to position $(w0, q_0)$ or to $(w1, q_1)$. A play of the game induces a sequence of states $\langle q_i \rangle_{i \in \mathbb{N}} \in (Q_{\mathcal{A}})^{\mathbb{N}}$, and Automaton wins if and only if

$$\liminf_{i \in \mathbb{N}} \mathrm{rk}_{\mathcal{A}}(q_i) \text{ is even.}$$

Because of the assumption that $\mathcal{A}$ has at least one transition from every letter and state, $G_{\mathcal{A},T}$ can easily be formalized as a parity game.

$G_{\mathcal{A},T}$ is defined in such a way that $\mathcal{A}$ accepts T if and only if Automaton has a positional winning strategy. Thus, by positional determinacy, $\mathcal{A}$ does not accept T if and only if Pathfinder has a positional winning strategy in $G_{\mathcal{A},T}$.

Let $\mathcal{S}$ be the (finite) set of all maps from $\Delta_{\mathcal{A}}$ into $\{0,1\}$. Note that a positional strategy for Pathfinder can be represented as a labelled binary tree S such that $S(w) \in \mathcal{S}$ for all $w \in \{0,1\}^*$. The strategy is winning if for any choice of transitions $\langle \delta_w \rangle_{w \in \{0,1\}^*}$ consistent with the labelling of T, the ranks of states of $\mathcal{A}$ appearing on the path determined by S and $\langle \delta_w \rangle_{w \in \{0,1\}^*}$ do not satisfy the parity condition.

We will construct an automaton $\mathcal{B}$ which accepts a tree T if and only Pathfinder has a positional winning strategy in $G_{\mathcal{A},T}$. The construction of $\mathcal{B}$ proceeds in four standard steps, of which three are elementary and one invokes McNaughton's determinization theorem for word automata ([13]), discussed separately below.

In the first step of the construction we build a deterministic word automaton $\mathcal{A}_1$ which accepts all infinite words $\langle (s_i, a_i, \delta_i, \pi_i) \rangle_{i \in \mathbb{N}} \in (\mathcal{S} \times \Sigma_{\mathcal{A}} \times \Delta_{\mathcal{A}} \times \{0,1\})^{\mathbb{N}}$ such that if

$$\forall i \in \mathbb{N}\, (s_i(\delta_i) = \pi_i) \tag{$\star$}$$

and if we define

$$\begin{aligned} q_0 &= q^I_{\mathcal{A}} \\ q_{i+1} &= \delta_i(a_i, q_i, \pi_i), \end{aligned}$$

then either at some point q_{i+1} cannot be defined (i.e. δ_i is not a transition from state q_i and letter a_i) or

$$\liminf_{i \in \mathbb{N}} \mathrm{rk}_{\mathcal{A}}(q_i) \text{ is odd.}$$

The set of states of $\mathcal{A}_1$ is $Q_{\mathcal{A}} \cup \{\top\}$, where $\mathrm{rk}_{\mathcal{A}_1}(q) = \mathrm{rk}_{\mathcal{A}}(q) + 1$ for $q \in Q_{\mathcal{A}_1}$ and $\top$ is an additional accepting state (i.e. with rank 0). The transitions are defined in the natural way, except that whenever the rule $(\star)$ is violated or q_{i+1} cannot be defined, we redirect the computation to state $\top$.

In the second step we consider the following property of infinite words $\langle (s_i, a_i, \pi_i) \rangle_{i \in \mathbb{N}} \in (\mathcal{S} \times \Sigma_{\mathcal{A}} \times \{0,1\})^{\mathbb{N}}$: for every sequence $\langle \delta_i \rangle_{i \in \mathbb{N}} \in (\Delta_{\mathcal{A}})^{\mathbb{N}}$, the word $\langle (s_i, a_i, \delta_i, \pi_i)\rangle_{i\in\mathbb{N}}$ is accepted by $\mathcal{A}_1$. Note that thanks to the fact that $\mathcal{A}_1$ is deterministic, the complement of this property is recognized by a nondeterministic word automaton. By determinization of word automata ([13], see below) we can find a deterministic parity word automaton $\mathcal{A}_2$ recognizing this property.

In the third step we define a tree automaton $\mathcal{A}_3$ over the alphabet $\mathcal{S} \times \mathcal{A}$ such that a tree (S, T) is accepted if for every infinite path $\pi \in \{0,1\}^{\mathbb{N}}$, if $\langle (s_i, a_i)\rangle_{i\in\mathbb{N}}$ is the sequence of labels appearing on this path, then the infinite word $\langle (s_i, a_i, \pi_i)\rangle_{i\in\mathbb{N}}$ is accepted by the automaton $\mathcal{A}_2$. So, $\mathcal{A}_3$ accepts (S, T) iff S encodes a positional winning strategy for Pathfinder in $G_{\mathcal{A},T}$. The states of $\mathcal{A}_3$ are the same as those of $\mathcal{A}_2$, and defining the transition function is unproblematic thanks to the fact that $\mathcal{A}_2$ is deterministic.

Finally, in the fourth step we define the nondeterministic tree automaton $\mathcal{B}$ as accepting a given tree T over $\Sigma_{\mathcal{A}}$ if there exists a labelling $S\colon \{0,1\}^* \to \mathcal{S}$ such that the tree (S, T) is accepted by $\mathcal{A}_3$. During its computation on T, $\mathcal{B}$ uses nondeterminism to "guess" the labels $S(w)$ for $w \in \{0,1\}^*$. By construction $\mathcal{B}$ accepts T iff Pathfinder has a positional winning strategy in $G_{\mathcal{A},T}$.

Determinization of word automata. To complete the proof of (1) ⇒ (2), we need to make sure that $\mathsf{ACA}_0 + (1)$ is able to prove that McNaughton's result [13] that every nondeterministic word automaton is equivalent to a deterministic automaton. In fact, this is provable in ACA_0 and does not require the full power of ACA_0.

The determinization of a nondeterministic word automaton with a parity condition proceeds in two steps. The first is to replace the original automaton by a nondeterministic Büchi automaton (i.e. with index $(0, 1)$). This is straightforward: assume that the original automaton, say $\mathcal{A}$, has index $(0, x)$, so that $\mathcal{A}$ accepts a word if it has a computation in which the lim inf of ranks of states is some odd number $y \leq x$. The Büchi automaton $\mathcal{B}$ behaves like $\mathcal{A}$, except that it additionally uses nondeterminism to do two things. Firstly, in the very first transition it guesses the value of y. Secondly, at some point in the computation it guesses that states with rank $> y$ will no longer appear; from that point onwards, it assigns rank 0 to states with rank y in $\mathcal{A}$, rank 1 to states with rank $< y$ in $\mathcal{A}$, and aborts the computation if $\mathcal{A}$ wants to enter a state with rank $> y$.

It remains to build a deterministic parity word automation simulating a nondeterministic Büchi word automaton $\mathcal{B}$. This can be carried out by means of the *Safra construction*, originally due to [23]. In the Safra construction, the states of the new deterministic automaton $\mathcal{D}$ are certain (bounded-size) trees with vertices labelled by letters from a fixed finite alphabet. The combinatorial details of the construction are a bit involved, but the logical strength engaged is quite modest. The construction of $\mathcal{D}$ from $\mathcal{B}$ is completely elementary, the proof that acceptance of $\mathcal{B}$ implies acceptance of $\mathcal{D}$ requires nothing beyond defining number sequences by recursion with an arithmetical condition in the recursion step, and the other direction additionally makes use of König's Lemma in the form known as *weak* König's Lemma, WKL; all this is well-known to be readily formalizable in ACA_0. (For details of the Safra construction, see e.g. [20, 27][3].)

[3]The proof presented in [27] yields a deterministic automaton with the so-called Rabin acceptance condition, which can be simulated by a parity condition without difficulty but at exponential cost; the modified construction of [20] leads directly to a deterministic parity automaton.

Remark. In fact, the determinization of word automata does not require Weak König's Lemma: by [12], it can be proved in the fragment of ACA_0 consisting of RCA_0 extended by the induction scheme for Σ^0_2 formulas. Without Σ^0_2 induction, the basic notions of automata theory on infinite structures make little sense (in particular the lim inf of ranks appearing in a computation might not exist) and the basic results of the theory tend to fail; see [12] for details.

(2) $\Rightarrow$ (3). By Proposition 2.11 and Lemma 2.13, it is enough to show that for every $x \in \mathbb{N}$, all treelike parity games of index $(0, x)$ are positionally determined. Given fixed x, we can represent treelike parity games with index $(0, x)$ by labelled binary trees over a suitable alphabet Σ_x. The label of a node v in such a tree should encode the rank of of v in the game, so the alphabet Σ_x has to contain at least $x + 1$ symbols.

Fix $x \in \mathbb{N}$. A tree over Σ_x representing a treelike game G of index $(0, x)$ has the property $W^{\exists}_{0,x}$ (resp. $W^{\forall}_{0,x}$) if $\exists$ (resp. $\forall$) has a positional winning strategy in G. Note that both $W^{\exists}_{0,x}$ and $W^{\forall}_{0,x}$ can be expressed by MSO sentences with 4 blocks of quantifiers. For instance, the sentence expressing $W^{\exists}_{0,x}$ is the prenex normal form of

$$\exists S\, \forall P \left[\alpha(S, P) \vee \bigvee_{0 \leq y \leq \lfloor x/2 \rfloor} \beta_{2y}(S, P) \right],$$

where S represents a positional strategy for player $\exists$, P represents a potential play of the game, α is a purely existential first-order formula stating that the play P is inconsistent with the strategy S, and each β_{2y} is an $\forall\exists \wedge \exists\forall$ first-order formula stating that $2y$ is the lim inf of ranks appearing in the play P. The sentence expressing $W^{\forall}_{0,x}$ is defined analogously.

Now assume (2). It is routine to verify in ACA_0 that any quantifier-free expressible property of labelled trees is recognized by a nondeterministic tree automaton. The usual argument by induction on formula complexity (see e.g. [27, Theorem 6.7, cf. Theorem 3.1]), using (2) in the step for $\neg$ and the nondeterminism of automata in the step for a block of second-order $\exists$'s, proves that every property of labelled trees expressible by an MSO sentence with at most 5 quantifier blocks can be recognized by a tree automaton. Hence, for any fixed x there is an automaton $\mathcal{A}_x$ recognizing $\neg W^{\exists}_{0,x} \wedge \neg W^{\forall}_{0,x}$. It remains to show that this implies (3).

A labelled tree T is *regular* if there is some bound $d \in \mathbb{N}$ with the following property: for every vertex $v \in \{0,1\}^*$ there exists $w \in \{0,1\}^{<d}$ such that the subtree of T under w is the same as the subtree under v, that is, for all $u \in \{0,1\}^*$, $T(vu) = T(wu)$. We complete the argument by proving two lemmas about regular trees which together immediately imply that $\mathcal{A}_x$ cannot accept any tree, and therefore (3) holds. The first lemma expresses a completely standard fact, whereas the second is interesting only in a context where positional determinacy of treelike parity games is not known in advance.

Lemma 3.2. ACA_0 *proves that any tree automaton which accepts some tree also accepts a regular tree.*

Proof. Let $\mathcal{A}$ be a tree automaton of index $(0, x)$. Consider the following parity game $G_{\mathcal{A}}$. The game arena is split into $V_{\exists} = Q_{\mathcal{A}}$ and $V_{\forall} = \Delta_{\mathcal{A}}$ with $q^I_{\mathcal{A}}$ as the initial position. When the game is in position $q \in Q_{\mathcal{A}}$, player $\exists$ moves to some $\delta \in \Delta_{\mathcal{A}}$ of the form (q, a, q', q'') for some $a \in \Sigma_{\mathcal{A}}, q', q'' \in Q_{\mathcal{A}}$. Now the game is in position δ, and player $\forall$ decides whether to go left or right, that is, moves to either q' or q''. Player $\exists$ wins if $\liminf$ of ranks of states visited during the play is even (formally this is achieved by setting the ranks of all $\delta \in \Delta_{\mathcal{A}}$ to x).

The positional determinacy of parity games on finite arenas is known to be provable even in S^2_2, a very weak subtheory of ACA_0 [1]. Moreover, if $\mathcal{A}$ accepts some tree T, then player $\exists$ clearly has a winning strategy in $\mathcal{A}$, and thus also a positional winning strategy, say σ. Now construct a labelled tree T_σ in the following way: to determine the label $T(w)$ for $w \in \{0,1\}^*$, simulate a play of $G_{\mathcal{A}}$ in which $\exists$ plays according to σ, $\forall$ chooses directions so as to reach w, and for each $i \leq \mathrm{lh}(w)$ the label $T(w{\restriction}_i)$ is chosen to correspond to the transition δ chosen by $\exists$. T_σ is accepted by $\mathcal{A}$ because σ is a winning strategy. Additionally, T_σ is a regular tree because $G_{\mathcal{A}}$ has a finite arena and σ is positional. $\square$

Lemma 3.3. ACA_0 *proves that for each x, the treelike parity game represented by a regular tree T over Σ_x is determined.*

Proof of Lemma 3.3. Let T be a regular tree over Σ_x encoding a treelike parity game G. Define U to be the set consisting of those $v \in \{0,1\}^*$ for which there is no w length-lexicographically smaller than v such that the lengths of v and w are congruent mod 2 and $T(vu) = T(wu)$ for all u (the additional congruence condition is needed to distinguish between moves of different players). The formula defining U is arithmetical, so U is indeed a set. Moreover, since T is regular, the set U is finite.

Consider the following parity game H with arena U: the assignment of positions to players and the ranks of positions are inherited from G, and a move from v to w is possible in H if and only if the subtree of T under w is identical to the subtree under a son vb of v. The game H exists by arithmetical comprehension. By positional determinacy of parity games on finite arenas, H is positionally determined provably in ACA_0. Let σ_H be a positional winning strategy for one of the players, say $\exists$.

We can use σ_H to define a positional strategy σ_G in the game G represented by T: at position v, player $\exists$ finds the unique vertex $w \in U$ such that the lengths of w and v are congruent mod 2 and T under v is the same as T under w, and then moves from v analogously to the move σ_H would make at w. The strategy σ_G exists by arithmetical comprehension, and since any play of G consistent with σ_G corresponds to a play of H consistent with σ_H, σ_G is in fact a winning strategy for $\exists$ in G. $\square$

The proofs of the Lemmas conclude the proof of Theorem 3.1. $\square$

4 Determinacy vs comprehension

We now turn our attention to the question which set-theoretic existence axioms are needed to prove the determinacy statements considered in Theorem 3.1. For the statement $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$, sharp upper and lower bounds on the requisite amount of comprehension follow from earlier work on determinacy statements in second-order arithmetic, most importantly [16, 7, 14].

We begin with the upper bound. The following result is a direct corollary of the proof of Theorem 4.3 in [14]:

Theorem 4.1 ([14])**.** $\mathsf{ACA}_0 + \Delta^1_2\text{-}\mathsf{ID} + \Sigma^1_3\text{-}\mathsf{IND} \vdash \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$.

By Theorem 2.6, it follows that $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ is provable from $\Pi^1_2\text{-}\mathsf{CA}_0 + \Sigma^1_3\text{-}\mathsf{IND}$, and hence also from $\Pi^1_3\text{-}\mathsf{CA}_0$. In fact, it is known that $\Pi^1_3\text{-}\mathsf{CA}_0$ already suffices to formalize Davis' proof of $\mathbf{\Sigma^0_3}$ determinacy.

On the other hand, $\Pi^1_2\text{-}\mathsf{CA}_0$ by itself can only prove a smaller amount of determinacy:

Theorem 4.2 ([7], relying heavily on [16])**.** *Over* ACA_0*, the theory*

$$\{(\mathbf{\Sigma^0_2})_n\text{-Det} : n \in \omega\}$$

implies all the Π^1_1 *consequences of* $\Pi^1_2\text{-}\mathsf{CA}_0$*.*

In fact, the methods of [16] can be used to prove a strengthening of Theorem 4.2. We state the stronger version, accompanied by a proof sketch, as Theorem 7.1 below.

Remark. In [3, Section VII A], the authors state that they "suspect" that Δ^1_2 comprehension in the language of S_0, S_1 suffices to axiomatize the MSO theory of $(\{0,1\}^*, S_0, S_1)$.

However, Theorem 4.2 shows that this cannot be the case. The MSO theory of $(\{0,1\}^*, S_0, S_1)$ contains, in particular, sentences expressing the determinacy of $(\mathbf{\Sigma^0_2})_n$ games for each $n \in \omega$. By Theorem 4.2 and known relationships between $\Delta^1_2\text{-}\mathsf{CA}_0$ and $\Pi^1_2\text{-}\mathsf{CA}_0$, not all these sentences can be provable in $\Delta^1_2\text{-}\mathsf{CA}_0$ (in fact, all of them are unprovable). The reduct of any (possibly nonstandard) model of $\Delta^1_2\text{-}\mathsf{CA}_0$ to the language of S_0, S_1 satisfies Δ^1_2 comprehension in that language — however, by the reasoning above, the reduct may have a different MSO theory than the true theory of $(\{0,1\}^*, S_0, S_1)$.

Theorem 4.2 and a standard argument imply the following corollary, which is proved exactly as the formally slightly weaker Theorem 5.6 in [14]:

Corollary 4.3 (essentially in [14])**.** $\Delta^1_3\text{-}\mathsf{CA}_0 \nvdash \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$.

Proof sketch. $\Delta^1_3\text{-}\mathsf{CA}_0$ is Π^1_4-conservative over $\Pi^1_2\text{-}\mathsf{CA}_0$, so it suffices to show that $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ is unprovable in $\Pi^1_2\text{-}\mathsf{CA}_0$.

By Theorem 2.4, $\Pi^1_2\text{-}\mathsf{CA}_0$ proves the existence of a β_2-model. Since every true Π^1_3 sentence remains true in a β_2-model, for every Π^1_3 sentence φ we have:

$$\Pi^1_2\text{-}\mathsf{CA}_0 \vdash \varphi \Rightarrow \mathrm{Con}(\varphi),$$

where Con is a formalized consistency statement (cf. [25, Corollary VII.7.8.2]).

Now apply this to $\mathsf{ACA}_0 \wedge \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ as φ (ACA_0 is well-known to be axiomatizable by a single Π^1_2 sentence). If φ were provable in $\Pi^1_2\text{-}\mathsf{CA}_0$, then also $\mathrm{Con}(\varphi)$ would be provable. But $\mathrm{Con}(\varphi)$ is Π^0_1, so by Theorem 4.2 it would follow from $\mathsf{ACA}_0 + (\mathbf{\Sigma^0_2})_n\text{-Det}$ for some $n \in \omega$, and hence from φ itself. The theory $\mathsf{ACA}_0 + \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ would then contradict Gödel's second incompleteness theorem. $\square$

Therefore, $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$, and by Theorem 3.1 also the complementation theorem for tree automata, is provable from Π^1_3- but not Δ^1_3-comprehension. On the other hand, the argument used to prove Theorem 4.2 in [14] gives:

Theorem 4.4 ([14]). $\mathsf{ACA}_0 + \Delta^1_2$-$\mathsf{ID}$*, and thus also* Π^1_2-CA_0*, proves* $(\mathbf{\Sigma^0_2})_n$-Det *for all* $n \in \omega$*, as well as*

$$\forall x\,((\mathbf{\Sigma^0_2})_x\text{-Det} \Rightarrow (\mathbf{\Sigma^0_2})_{x+1}\text{-Det}).$$

Corollary 4.5 ([14]). $\mathsf{ACA}_0 + \Delta^1_2$-$\mathsf{ID}$*, and thus also* Π^1_2-CA_0*, proves* $(\mathbf{\Sigma^0_2})_n$-Det* *for all* $n \in \omega$*, as well as*

$$\forall x\,((\mathbf{\Sigma^0_2})_x\text{-Det}^* \Rightarrow (\mathbf{\Sigma^0_2})_{x+1}\text{-Det}^*).$$

Proof. This follows immediately from Theorem 4.4, $(\mathbf{\Sigma^0_2})_x$-Det $\Rightarrow (\mathbf{\Sigma^0_2})_x$-Det*, and $(\mathbf{\Sigma^0_2})_{x+1}$-Det$^* \Rightarrow (\mathbf{\Sigma^0_2})_x$-Det. □

It is worth pointing out that implications between existence of fixpoints and determinacy run also in the other direction. In this context we only mention the following result, which is more than we need in Section 5.

Theorem 4.6 ([15]). *Over* ACA_0*,* $\mathbf{\Sigma^0_2}$-Det *implies* Σ^1_1-ID*.*

We now turn to the positional determinacy of parity games. Of course, by Proposition 2.11 and Lemma 2.13, $\forall x[(\mathbf{\Sigma^0_2})_x$-Det$]$ is equivalent to a special case of the positional determinacy of parity games, so the lower bounds on provability of $\forall x[(\mathbf{\Sigma^0_2})_x$-Det$]$ remain valid here: the positional determinacy of all parity games is not provable in Δ^1_3-CA_0. The lemma below shows that also the upper bound of Π^1_2-$\mathsf{CA}_0 + \Sigma^1_3$-$\mathsf{IND}$, and a fortiori Π^1_3-CA_0, remains valid.

Lemma 4.7. *For each* $n \in \omega$*,* Π^1_2-CA_0 *proves that parity games of index* $(0, n)$ *are positionally determined. Moreover,* Π^1_2-CA_0 *proves that for every* x*, if parity games of index* $(0, x)$ *are positionally determined, then so are parity games of index* $(0, x + 1)$*.*

Remark. Our proof of Lemma 4.7 is again the formalization of a standard argument in the spirit of the one presented in [27]. We strongly believe that the MedSalem-Tanaka argument showing Theorem 4.4 can be adapted to give a proof of the step from $(0, x)$ to $(0, x + 1)$ in $\mathsf{ACA}_0 + \Delta^1_2$-$\mathsf{ID}$, but we have not attempted to do it.

Proof. Since parity games of index $(0, 0)$ are trivially positionally determined, it is enough to prove the second part of the statement. So, assume that all parity games of index $(0, x)$ are positionally determined and let $G = \langle V_\exists, V_\forall, v_I, E, \mathrm{rk}\rangle$ be a game of index $(0, x+1)$. For each $v \in V$, we will write G_v to denote a game defined just like G but with starting position v instead of v_I. Π^1_2-comprehension guarantees the existence of the set

$$W_\forall = \{v : \text{ player } \forall \text{ has a positional winning strategy in } G_v\}.$$

Let U be $V \setminus W_\forall$. Our aim is to prove that player $\exists$ has a positional winning strategy in G_v for each $v \in U$. Of course, this will in particular imply the positional determinacy of G. We first prove a useful claim.

Claim. Player $\forall$ has a single positional strategy $\sigma_\forall$ which is winning in each G_v for $v \in W_\forall$. Moreover, each play of G_v, $v \in W_\forall$, consistent with $\sigma_\forall$ never leaves $W_\forall$.

To see that the claim is true, note that by Theorem 2.2 we can apply Σ^1_2-AC to conclude that there exists a sequence of positional strategies $\langle \sigma_v \rangle_{v \in W_\forall}$ such that σ_v is winning for p in G_v. Π^1_1-comprehension implies the existence of a sequence $\langle U_v \rangle_{v \in W_\forall}$ such that for each v,

$$U_v = \{w \in W_\forall : \sigma_w \text{ is winning for } \forall \text{ in } G_v\}.$$

Observe that U_v is nonempty for each $v \in W_\forall$. Now define $\sigma_\forall(v)$, $v \in W_\forall \cap V_\forall$, to be $\sigma_w(v)$ for $w = \min(U_v)$; for $v \notin W_\forall$, define $\sigma_\forall(v)$ arbitrarily. Given a play $\langle v_i \rangle_{i \in \mathbb{N}}$ consistent with $\sigma_\forall$ and such that $v_0 \in W_\forall$, we can prove by induction on i that $v_i \in W_\forall$ and that the sequence $\langle \min(U_{v_i}) \rangle_{i \in \mathbb{N}}$ is nonincreasing. Thus, from some round i onwards the play is actually consistent with some fixed strategy σ_w which is winning for $\forall$ in G_{v_i}. This proves the Claim.

Note that for $v \in U \cap V_\forall$, all edges from v must lead to vertices in U. The Claim also implies that for $v \in U \cap V_\exists$, at least one edge from v must lead to a vertex in U.

Let N be the set $U \cap \mathrm{rk}^{-1}[\{0\}]$. Consider the following formula $\varphi(x, X)$:

$$\begin{aligned} x \in U \wedge [x \in N \vee x \in X \\ &\vee (x \in V_\exists \text{ and there exists an edge from } x \text{ into } X) \\ &\qquad \vee (x \in V_\forall \text{ and all edges from } x \text{ lead into } X).] \end{aligned}$$

Theorem 2.6 implies the existence of an inductively generated fixed point P of the operation $X \mapsto \{x \in \mathbb{N} : \varphi(x, X)\} \cup X$ and the corresponding prewellordering $\preccurlyeq$. Since X appears in φ only positively, it is easily checked that P is in fact the least fixed point of $X \mapsto \{x \in \mathbb{N} : \varphi(x, X)\}$ and $x \in P \Leftrightarrow \varphi(x, \{y : y \prec x\})$. Note that $\exists$ has a positional strategy π on $P \setminus N$ such that π guarantees reaching a vertex in N in finitely many steps and staying inside P until that happens. The strategy is defined by: $\pi(v) =$ the $\leq$-smallest w such that $(v, w) \in E$ and $w \prec v$ (this is well-defined on $P \setminus N$ unless $N = \emptyset$, in which case $P = \emptyset$ as well).

Now let R be $U \setminus P$. Note that for $v \in R \cap V_\exists$, all edges from v must lead to vertices in $W_\forall \cup R$, while for $v \in R \cap V_\forall$, at least one edge from v must lead to a vertex in R. So, it makes sense to consider the parity game $G{\restriction_R}$ obtained by restricting G to the induced subgraph on R. All vertices in R have ranks from $\{1, \ldots, x+1\}$, so by the inductive assumption each game $(G{\restriction_R})_v$ for $v \in R$ is positionally determined. It is not hard to check, using the Claim, that a positional winning strategy for $\forall$ in $(G{\restriction_R})_v$ would also give a positional winning strategy for $\forall$ in G_v, so in fact it is player $\exists$ who wins each $(G{\restriction_R})_v$. By an obvious analogue of our Claim for $(G{\restriction_R})_v$, this means that $\exists$ has a single positional strategy ρ which wins $(G{\restriction_R})_v$ for each $v \in R$.

Define a positional strategy $\sigma_\exists$ for $\exists$ as follows:

$$\sigma_\exists(v) = \begin{cases} \pi(v) & v \in P \setminus N, \\ \rho(v) & v \in R, \\ \text{arbitrary outside } W_\forall & v \in N, \\ \text{arbitrary} & v \in W_\forall. \end{cases}$$

It is easy to verify that for each $v \in U$, $\sigma_\exists$ is winning for $\exists$ in G_v. $\square$

5 Complementation: full result

It turns out that implications from determinacy to set existence axioms (for instance in the form of Theorem 4.6), along with "positionalization of winning strategies" techniques developed in automata theory, make it possible to reverse implication (2) $\Rightarrow$ (3) of Theorem 3.1. In other words, we can prove:

Theorem 5.1. *The following are equivalent over* ACA_0*:*

(1) $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$,

(2) *for every nondeterministic tree automaton* $\mathcal{A}$ *there exists a nondeterministic tree automaton* $\mathcal{B}$ *such that for any tree* T*,* $\mathcal{B}$ *accepts* T *iff* $\mathcal{A}$ *does not accept* T*.*

Our proof of Theorem 5.1 is based on an analysis of the arguments of Klarlund and Kozen in [10, 11]. We assume that the reader is familiar with those two papers and has them at hand.

Proof of Theorem 5.1. Clearly, it is enough to verify that in the Automaton-Pathfinder games described in the proof of Theorem 3.1, a winning strategy for Pathfinder can be improved to a positional winning strategy provably in $\mathsf{ACA}_0 + \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$. We achieve this by analyzing the technology of pointer trees and Rabin measures developed in [10, 11].

As in those papers, we consider automata equipped not with a parity condition but the more general *Streett condition*: given a finite set $\{(R_\chi, I_\chi) : \chi \in C\}$ of pairs of subsets of Q (w.l.o.g. $0 \in X$ and $I_0 = \emptyset$), the automaton accepts if for every path π and every χ such that the states along π are in R_χ infinitely often, the states along π are also in I_χ infinitely often. Pathfinder's winning condition in $G_{\mathcal{A},T}$ is then the dual *Rabin condition*: there is some $\chi \in C$ such that states in R_χ appear infinitely often but states in I_χ do not. Pathfinder's strategy σ is winning if the graph $G_{\mathcal{A},T}\restriction_\sigma$ of possible plays in $G_{\mathcal{A},T}$ consistent with σ satisfies the same Rabin condition, in the sense that the condition holds on each infinite path through the graph.

A *pointer tree* (cf. [11, Section 4][4]) is a tree $T \subseteq \mathbb{N}^*$ (with the initial prefixes of sequence as its ancestors, as usual) together with a relation W such that for every $t \in T$, W_t is a well-ordering of the set of sons of t. The *Kleene-Brouwer ordering* on (T, W), $\preccurlyeq_{T,W}$, is the linear ordering on T such that $t \preccurlyeq_{T,W} s$ if either t is a descendant of s or t is lexicographically smaller than s, *where comparisons between siblings are as given by* W (this should be contrasted with the standard Kleene-Brouwer ordering on T, where comparisons among siblings are made according to the usual ordering of $\mathbb{N}$).

All the basic results on pointer trees proved in [11, Section 4] go through in ACA_0 without problems. The crucial fact that $\preccurlyeq_{T,W}$ is a well-ordering if T is well-founded [11, Lemma 2] is proved just like the analogous property for the standard Kleene-Brouwer ordering on T [25, Lemma V.1.3], with the usual ordering of $\mathbb{N}$ replaced by the well-orderings given by W.

A *coloured pointer tree* is defined as in [11, Definition 2]: every non-leaf $t \in T$ is coloured by some element $\xi(t) \in X$, the root gets colour 0, and no colour

[4] We have made slight changes to the definition of pointer trees from [11] in order to simplify formalization in second-order arithmetic. These changes do not affect any of the main results.

repeats along a path. The definition of a *Rabin measure* [11, Definition 3] is also unchanged: for a graph $G = (V, E)$ with the Rabin condition $\mathcal{C} = \{(R_\chi, I_\chi) : \chi \in X\}$ on it, a Rabin measure on $(G, \mathcal{C})$ is a coloured pointer tree (T, W, ξ) together with a mapping $\mu\colon V \to T$ such that (i) $v \notin I_\chi$ for all χ appearing as colours on the path from the root to $\mu(v)$, and (ii) for all $(u, v) \in E$, either $\mu(u) \succ_{T,W} \mu(v)$ or there is a common ancestor t of $\mu(u)$ and $\mu(v)$ such that $v \in R_{\xi(t)}$ (the disjunction of (i) and (ii) is briefly written as $u \rhd_\mu v$).

We say that G satisfies the condition $\mathcal{C}$ if for every infinite path $\langle v_i \rangle_{i\in\mathbb{N}}$ in G, there exists $\chi \in X$ such that $v_i \in R_\chi$ for infinitely many i but $v_i \in I_\chi$ only for finitely many i. The main result of [11] is:

Theorem 5.2. *[11, Theorem 1] G satisfies the Rabin condition $\mathcal{C}$ iff $(G, \mathcal{C})$ admits a Rabin measure.*

Proof of Theorem 5.2 in $\mathsf{ACA}_0 + \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$. The proof of ($\Leftarrow$) is straightforward and easily formalizes in ACA_0. We discuss the proof of ($\Rightarrow$), which is more subtle and requires stronger axioms.

For a set of colours C (in the intended applications $C \neq X$ and C may be infinite), a *colour set assignment* is, as in [11, Section 6.1], a map assigning to each $v \in V$ a nonempty subset of C — in second order arithmetic this is formalized as a relation CS such that CS_v is a nonempty subset of C for each v. The notions of an *enabled set* and *permissible colour set assignment* are defined as in [11].

[11, Lemma 3] states that for a permissible CS there is some $v \in V$ and $\chi \in C$ which belongs to CS_w for all descendants w of v. This lemma goes through in ACA_0: the main ingredient of the proof is an application of the Baire Category Theorem for a complete separable metric space; the theorem is well-known to be provable in the subtheory RCA_0 of ACA_0, cf. [25, Chapter II.5].

[11, Lemma 4] states that for a permissible CS there is a colouring $c\colon V \to C$ which is (i) consistent with CS, (ii) eventually constant on every path, and (iii) for some well-order J admits a partition $\langle V_j \rangle_{j\in J}$ of V into monochromatic subsets, with the additional property that for a vertex v and w a descendant of v, the block to which w belongs is no higher in the ordering J than the block to which v belongs.

The proof of this lemma is the first place in which an axiom going beyond ACA_0 seems required. In [11], the lemma is proved by a transfinite iteration of Lemma 3. In our setting, we can formalize this by means of the inductively generated fixed point of the operator $Z \mapsto \Gamma_\varphi(Z) \cup Z$, where $\varphi(x, \chi, Z)$ is the following arithmetical formula (we think of Z as a set of pairs $\langle \text{vertex}, \text{colour} \rangle$, and $(Z)_0$ is the projection onto the first coordinate):

(1) "$x \notin (Z)_0$, and

(2) there is some $y \notin (Z)_0$ for which there is some $\tilde{\chi}$ such that $\tilde{\chi} \in CS_w$ for all w which are reachable from y via a path that stays outside of $(Z)_0$, and

(3) for the $\leq$-smallest such y the $\leq$-smallest such $\tilde{\chi}$ is χ, and

(4) x is reachable from the smallest such y via a path that stays outside of $(Z)_0$."

Intuitively, at each step of constructing the fixed point, we look at G restricted to the vertices not yet assigned a colour, make a canonical choice of an element y witnessing Lemma 3 for that restricted graph, and assign a suitable colour to y and its descendants.

Σ^1_1-ID, available in $\mathsf{ACA}_0 + \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ by Theorem 4.6, gives us not only a fixed point P of the iteration of $\Gamma_\varphi(Z) \cup Z$, but also a prewellordering $\preccurlyeq$ representing the stages of the construction of P. At each stage there is a single canonically chosen element y entering P along with all of its remaining descendants; since $\preccurlyeq$ is a prewellordering, $\preccurlyeq$ restricted to the (arithmetically definable) set of the y's is a well-ordering which we can take as our J. For $y \in J$, the block V_y consists of all x's entering P as descendants of y in accordance with conjunct (4) of φ. It is not hard to check in ACA_0 that the colouring

$$x \mapsto \text{the unique } \chi \text{ such that } \langle x, \chi \rangle \in P,$$

together with the partition $\langle V_y \rangle_{y \in J}$, satisfies the requirements of Lemma 4.

We now come to the final part of the proof of ($\Rightarrow$) [11, Section 6.3]. The coloured pointer tree (T, W, ξ) used to define a Rabin measure on $(G, \mathcal{C})$ is constructed by an algorithm which at any point of its execution has access to an already constructed node $t \in T$, an "active" colour $\xi \in X$, and a *finite* set Y_t of "already used" colours. In addition, at every t there is a set $V_t \subseteq V$ (denoted by $W(t)$ in the paper) consisting of those v that still need to be "dealt with" at t or its descendants.

Originally, $\chi_{\langle\rangle}$ is the colour 0, $Y_{\langle\rangle} = \emptyset$, and $v_{\langle\rangle} = V$. Given t, χ_t, Y_t, V_t, we apply Lemma 4 to the induced graph on $V_t \setminus R_{\chi_t}$ and a colour set assignment defined by allowing, for v in $V_t \setminus R_{\chi_t}$, either all the colours $\chi \in X \setminus (Y_t \cup \{\chi_t\})$ such that $v \notin I_\chi$ or, if there are no such χ, a unique dummy colour $\perp_v$. The application of Lemma 4 gives rise to a well-ordered partition of $V_t \setminus R_{\chi_t}$ into monochromatic subsets. The sons of t in T correspond to the blocks of this partition, and for a given block the new Y is $Y \cup \{\chi\}$, the new χ is the colour of the block (unless that is a dummy colour, in which case the son will be a leaf of T), and the new V is simply the block itself.

We claim that this entire construction can be carried out by means of a single fixed point of an arithmetical operator. The crucial observation is that once a vertex t enters T and the corresponding χ_t, Y_t, V_t are known, the application of Lemma 4 to $V_t \setminus R_{\chi_t}$ may begin immediately — there is no need to wait for all siblings of t to appear or to know what the order type of the siblings will be. This lets us avoid nesting of fixed points.

The arithmetical formula φ which we use to define the fixed points has variables t, x, χ, Y, Z, where Y is *first-order* (it represents a finite set). Therefore, Z is viewed as a set of quadruples. The intended meaning of a tuple $\langle t, x, \chi, Y \rangle$ entering the fixed point is: t is in T, $x \in V_t$, $\chi_t = \chi$, $Y_t = Y$. The formula φ states: "either $t = \langle\rangle$, $\chi = 0$ and $Y = \emptyset$, or there exist $s, y, \tilde{\chi}, \tilde{Y}$ such that:

(1) $\tilde{\chi} \in X \setminus \tilde{Y} \wedge Y = \tilde{Y} \cup \{\tilde{\chi}\}$, and

(2) $\chi \notin Y$, and

(3) $t = s^\frown\langle y \rangle$, and

(4) $\langle s, y, \tilde{\chi}, \tilde{Y} \rangle \in Z$, and

(5) $y \notin R_{\tilde{\chi}}$, and

(6) there is no tuple in Z whose first component is a proper descendant of t and second component is y, and

(7) x satisfies (4)-(6), that is, $\langle s, x, \tilde{\chi}, \tilde{Y} \rangle \in Z$, $x \notin R_{\tilde{\chi}}$, and there is no tuple in Z whose first component is a proper descendant of t and second component is x, and

(8) among all w satisfying (4)-(6), y is the $\leq$-smallest for which there is a colour $\hat{\chi} \notin Y$ such that $w \notin I_{\hat{\chi}}$ and for all vertices $z \in G$ reachable from w via a path that consists of vertices satisfying (4)-(6) it holds that $z \notin I_{\hat{\chi}}$, and

(9) the $\leq$-smallest such $\hat{\chi}$ for y is χ, and

(10) x is reachable from y via a path that consists of vertices satisfying (4)-(6)."

Intuitively, the restriction in (8) to vertices w satisfying (4)-(5) leaves only the elements of $V_s \setminus R_{\chi_s}$ (where χ_s is $\tilde{\chi}$), and the restriction to w satisfying (6) additionally rules out vertices that have already been dealt with at earlier siblings of t.

Once we have a fixed point P of $Z \mapsto \Gamma_\varphi(Z) \cup Z$ along with the prewellordering $\preccurlyeq$ describing the stages of its inductive generation, the objects T, W, ξ and $\langle V_t \rangle_{t \in T}$ have natural arithmetical definitions in terms of $P, \preccurlyeq$. Moreover, it is easy to verify in ACA_0 that T, W, ξ, $\langle V_t \rangle_{t \in T}$ satisfy the various desirable properties listed in [11, Section 6.3, Claims 1-3]. In particular, the height of T is bounded by the size of X, so T is well-founded, and for each $v \in V$ there is a unique longest path $t_0, \ldots, t_\ell$ in T such that $v \in V_{t_i}$ for all $i \leq \ell$. As in [11], we can define $\mu(v)$ to be t_ℓ and check (in ACA_0) that T, W, ξ, μ give a Rabin measure on $(G, \mathcal{C})$. Thus, $\mathsf{ACA}_0 + \forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ proves Theorem 5.2. $\square$

We now briefly discuss the part of [10] where Theorem 5.2 is applied to the special case of positionalizing Pathfinder's winning strategies. Essentially, the relevant part of this paper requires much less axiomatic strength than [11] and goes through smoothly in ACA_0. This holds in particular for the proofs of [10, Lemmas 2-4]. Lemma 4 provides the crucial observation that a certain type of "surgery" on a graph with a Rabin measure leads to a graph that still has a Rabin measure.

This is applied to the graph $G_{\mathcal{A},T}\restriction_\sigma$ (denoted by $\mathcal{G}_\rho(\mathfrak{A}, \tau)$ in the paper) of possible plays in $G_{\mathcal{A},T}$ consistent with a winning strategy σ for Pathfinder. Since σ is winning, $G_{\mathcal{A},T}\restriction_\sigma$ satisfies a Rabin condition and thus by Theorem 5.2 admits a Rabin measure (T, W, ξ, μ). The Kleene-Brouwer ordering on (T, W) makes it possible to define, for each $(v, \delta) \in \{0,1\}^* \times \Delta_\mathcal{A}$ which requires Pathfinder to make a move in $G_{\mathcal{A},T}\restriction_\sigma$, a canonical game history leading to (v, δ). In the proof of [10, Lemma 9], some (arithmetically definable) surgery on $G_{\mathcal{A},T}\restriction_\sigma$ leads to a graph that still has a Rabin measure, but now contains only the canonical histories and thus corresponds to a positional strategy for Pathfinder. Another application of Theorem 5.2 shows that the new graph satisfies the Rabin condition, so the new strategy is also winning. $\square$

Remark. The fixed point construction needed in [11] can actually be carried out in Π^1_1-CA_0, which is equivalent to $(\mathbf{\Sigma^0_1})_2$-Det. Of course, we still need $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$ to know that there is a winning strategy to positionalize.

6 Decidability

Theorem 3.1, combined with the results discussed in Section 4, leads to an easy argument showing that Δ^1_3-CA_0 is not only unable to prove the complementation theorem for tree automata, but also unable to prove the decidability of the MSO theory of the infinite binary tree by any other method.

Theorem 6.1. ACA_0 *proves the implications* (1) $\Rightarrow$ (2_n)*, for all* $n \in \omega$*, and* $\mathsf{ACA}_0 + \Delta^1_2$-$\mathsf{ID}$ *(thus also* Π^1_2-CA_0*) proves the implications* $(2_n) \Rightarrow (1)$ *for all* $3 \leq n \in \omega$*, where:*

(1) $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$,

(2_n) *there is a Turing machine* $\mathfrak{t}$ *which halts on every input and accepts exactly the* Π^1_n *sentences of MSO true in* $(\{0,1\}^*, S_0, S_1)$.

Remark. In an "ideal version" of Theorem 6.1, the statements (2_n) would be replaced by a single statement (2): "there is a Turing machine $\mathfrak{t}$ which halts on every input and accepts exactly the sentences of MSO true in $(\{0,1\}^*, S_0, S_1)$". However, by Tarski's theorem on the undefinability of truth, the natural way of expressing "sentence φ is true in $(\{0,1\}^*, S_0, S_1)$" as a property of the variable φ cannot be formalized in the language of second-order arithmetic. On the other hand, it *is* possible to formulate a truth definition for Π^1_n sentences, in particular for Π^1_n sentences of MSO; as Theorem 6.1 shows, the case $n = 3$ is crucial here.

Proof. Both directions of the proof rely on ideas similar to those in the proof of (2) $\Rightarrow$ (3) in Theorem 3.1.

(1) $\Rightarrow$ (2_n). By Theorem 5.1, (1) implies the complementation theorem for nondeterministic tree automata. Moreover, the implication goes through in ACA_0.

Just like in the proof of (2) $\Rightarrow$ (3) in Theorem 3.1, complementation for tree automata makes it possible to formalize the usual algorithm constructing an automaton equivalent to an MSO formula φ on all labelled binary trees. Given a fixed $n \in \omega$, the correctness of this algorithm restricted to $\varphi \in \Pi^1_n$ is easily verified in ACA_0.

It remains to have a procedure to deciding whether a given tree automaton $\mathcal{A}$ accepts any tree at all. This is equivalent to the problem whether player $\exists$ wins the game $G_{\mathcal{A}}$ described in the proof of Lemma 3.2, and determining the winner of a parity game on a finite arena is a decidable problem provably well within ACA_0 [1].

$(2_3) \Rightarrow (1)$ As discussed in the proof of Theorem 3.1, for each $x \in \mathbb{N}$ there is a Π^1_3 MSO sentence ψ_x expressing the positional determinacy of treelike parity games of index $(0, x)$. By Lemma 2.13, ψ_x is true in $\{0,1\}^*$ exactly if $(\mathbf{\Sigma^0_2})_x\text{-Det}^*$ holds.

Let $\mathfrak{t}$ be given by (2_3). For each x, $\mathfrak{t}(\psi_x) = \mathsf{yes}$ exactly if $(\mathbf{\Sigma^0_2})_x\text{-Det}^*$ holds. Therefore, by Corollary 4.4, $\mathfrak{t}(\psi_0) = \mathsf{yes}$ and for every x, $\mathfrak{t}(\psi_x) = \mathsf{yes}$ implies $\mathfrak{t}(\psi_{x+1}) = \mathsf{yes}$.

By arithmetical comprehension, $X = \{x : \mathfrak{t}(\psi_x) = \mathsf{yes}\}$ exists as a set. Since X contains 0 and is closed under successor, we conclude that $X = \mathbb{N}$. Therefore each ψ_x is true in $\{0,1\}^*$ and so we have $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$. $\square$

7 Rabin's theorem as a reflection principle

Our aim now is to prove that over Π^1_2-CA_0, Rabin's decidability theorem, the complementation theorem for nondeterministic tree automata, and the determinacy statements appearing in Theorem 3.1 are all equivalent to a logical *reflection principle* stating that all Π^1_3 sentences provable in Π^1_2-CA_0 are true.

Our proofs in this section rely in an essential way on both the results and the techniques of Möllerfeld's thesis [16]. At present we see no way of avoiding reliance on these advanced techniques.

To achieve this, we first notice that slight changes to the argument of [16] yield a strengthening of Theorem 4.2.

Theorem 7.1. *Over* ACA_0*, the theory*

$$\{(\mathbf{\Sigma^0_2})_n\text{-Det} : n \in \omega\}$$

axiomatizes the Π^1_3 *consequences of* Π^1_2-CA_0*.*

Proof sketch. There is a well-known correspondence between sufficiently strong fragments of second-order arithmetic and weak systems of set theory formulated in the usual set-theoretic language $L_\in$ (cf. for instance [25, Chapter VII.3]). $L_\in$ is translated into L_2 by letting the $\in$-structure of sets be represented by well-founded trees in $\mathbb{N}^\mathbb{N}$. The inverse translation is the obvious one: $\exists x$ becomes $\exists x \in \omega^{\mathsf{set}}$, and $\exists X$ becomes $\exists x \subseteq \omega^{\mathsf{set}}$. (Here we use the symbol ω^{set} for the usual set-theoretic definition of the natural numbers as formalized in $L_\in$. This should not be confused with our use of ω in the earlier sections.)

Π^1_2-CA_0 corresponds to the set theory $\mathsf{Lim}(\preceq_1)$, which consists of the basic axioms known as BT^r (extensionality, pair, union, Δ_0-separation, and set foundation) and the axiom $\forall x\, \exists y\, [x \in y \wedge y \prec_1 \mathbb{V}]$, where $y \prec_1 \mathbb{V}$ means: "y is a transitive set, and for all $\bar{a} \in y$ and Σ_1 formulas φ (of $L_\in$), $\varphi(\bar{a})$ iff $y \models \varphi(\bar{a})$".

Now assume that Π^1_2-CA_0 proves the Π^1_3 sentence π. It follows that $\mathsf{Lim}(\preceq_1)$ implies the set-theoretic translation π^* of π. As written, π^* is a Π_3 sentence of $L_\in$:

$$\forall x \subseteq \omega^{\mathsf{set}} \exists y \subseteq \omega^{\mathsf{set}} \forall z \subseteq \omega^{\mathsf{set}} \delta(x, y, z),$$

with δ a bounded formula of $L_\in$. However, $\mathsf{Lim}(\preceq_1)$ proves the so-called Axiom β, which is a variant of the Mostowski collapse lemma: it states that any transitive relation r on a set a can be homomorphically mapped onto the set membership relation on some set b. (A statement of Axiom β, and a sketch of a proof in a subtheory of $\mathsf{Lim}(\preceq_1)$, can be found for instance in [21, Chapter 11.6].) Because of this, the Π^1_1 subformula of π^*, $\forall z \subseteq \omega^{\mathsf{set}} \delta(x, y, z)$, which is equivalent to the statement that a certain relation $r_{x,y}$ parametrized by x, y is well-founded, can provably in $\mathsf{Lim}(\preceq_1)$ be rewritten in a Σ_1 way: "there exists a Mostowski collapse of $r_{x,y}$". Furthermore, the existence of a Mostowski collapse of $r_{x,y}$ implies the well-foundedness of $r_{x,y}$ already in the weaker theory BT^r (thanks to the set foundation axiom). What this means is that there is a Π_2 sentence $\widetilde{\pi^*}$ such that

$$\mathsf{Lim}(\prec_1) \vdash \widetilde{\pi^*} \Leftrightarrow \pi^*$$
$$\mathsf{BT}^r \vdash \widetilde{\pi^*} \Rightarrow \pi^*.$$

By [16, Theorem 10.4], every Π_2 sentence of $L_\in$ provable in $\mathsf{Lim}(\prec_1)$ is also provable in a certain extension $\mathsf{BT}\sigma^r$ of BT^r. In particular, $\mathsf{BT}\sigma^r$ proves $\widetilde{\pi^*}$ and therefore also π^*. However, by [16, Theorem 8.15], every L_2 sentence whose translation is provable in a theory $\mathsf{Ref} \supseteq \mathsf{BT}\sigma^r$ is itself provable in an arithmetic version of the μ-calculus, which by [7] is in turn conservative over $\mathsf{ACA}_0 + \{(\mathbf{\Sigma^0_2})_n\text{-Det} : n \in \omega\}$. □

Even though Theorem 7.1 concerns relatively strong theories, the work in [7, 16] involved in its proof relies only on (sketches of) explicit primitive recursive constructions of proofs in various formal theories and, at one point, the cut elimination theorem for first-order logic. Therefore, we have the following corollary of the proof.

Corollary 7.2. *Primitive recursive arithmetic, and thus a fortiori* Π^1_2-CA_0*, proves that the theory*

$$\mathsf{ACA}_0 + \{(\mathbf{\Sigma^0_2})_x\text{-Det} : x \in \mathbb{N}\}$$

axiomatizes the Π^1_3 *consequences of* Π^1_2-CA_0*.*

The reflection principle $\mathrm{RFN}_{\Pi^1_3}(\Pi^1_2\text{-}\mathsf{CA}_0)$ is the formalized statement "every Π^1_3 sentence provable in Π^1_2-CA_0 is true", or

$$\forall\varphi\in\Pi^1_3\,[\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\varphi) \Rightarrow \mathrm{Tr}(\varphi)],$$

where Pr is a standard formalized provability predicate and Tr is a truth definition for Π^1_3 sentences. Note that $\mathrm{RFN}_{\Pi^1_3}(\Pi^1_2\text{-}\mathsf{CA}_0)$ implies $\mathrm{Con}(\Pi^1_2\text{-}\mathsf{CA}_0)$ and hence is unprovable in Π^1_2-CA_0.

Theorem 7.3. *The following are equivalent over* Π^1_2-CA_0*:*

(1) *all parity games are positionally determined,*

(2) $\forall x[(\mathbf{\Sigma^0_2})_x\text{-Det}]$,

(3) *for every nondeterministic tree automaton* $\mathcal{A}$ *there exists a nondeterministic tree automaton* $\mathcal{B}$ *such that for any tree* T*,* $\mathcal{B}$ *accepts* T *exactly if* $\mathcal{A}$ *does not accept* T*,*

(4) *there is a Turing machine* $\mathfrak{t}$ *which halts on every input and accepts exactly the* Π^1_3 *sentences of MSO true in* $(\{0,1\}^{<\mathbb{N}}, S_0, S_1)$*,*

(5) $\mathrm{RFN}_{\Pi^1_3}(\Pi^1_2\text{-}\mathsf{CA}_0)$.

Proof. Given Theorems 3.1 and 6.1, it is enough to prove (5) ⇒ (1) and (2) ⇒ (5). We reason in Π^1_2-CA_0.

The implication from (5) to (1) is essentially immediate. Let $\varphi(v)$ be a Π^1_3 formula with free variable v stating that all parity games of index $(0,x)$ are positionally determined. For each $x \in \mathbb{N}$, let φ_x be the Π^1_3 sentence obtained by substituting a canonical name (numeral) for x into $\varphi(v)$. By Lemma 4.7, we have $\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\varphi_0)$ and $\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\ulcorner\forall v\,(\varphi(v) \Rightarrow \varphi(v+1))\urcorner)$, so also also $\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\varphi_x)$ for each $x \in \mathbb{N}$. By (5), we obtain $\mathrm{Tr}(\varphi_x)$ for each x, so indeed all parity games are positonally determined.

Now assume (2) and consider a Π^1_3 sentence φ of the form

$$\forall X\,\exists Y\,\forall Z\,\psi(X,Y,Z),$$

with ψ arithmetical, such that $\mathrm{Pr}_{\Pi^1_2\text{-}\mathsf{CA}_0}(\varphi_0)$. By Corollary 7.2, there is some $x \in \mathbb{N}$ such that $\forall X\, \exists Y\, \forall Z\, \psi(X, Y, Z)$ is provable from $\mathsf{ACA}_0 + (\mathbf{\Sigma^0_2})_x$-Det.

Assume that φ is false. This means that there is some set S such that $\forall Y\, \exists Z\, \neg\psi(S, Y, Z)$. By Theorem 2.4, there is a countable coded β_2-model M such that $S \in M$. Since M is a β_2-model, we have

$$M \models \forall Y\, \exists Z\, \neg\psi(S, Y, Z).$$

However, since both ACA_0 and $(\mathbf{\Sigma^0_2})_x$-Det are true and have complexity no higher than Π^1_3, we also have

$$M \models \mathsf{ACA}_0 \wedge (\mathbf{\Sigma^0_2})_x\text{-Det}.$$

Now, take the proof π of $\forall X\, \exists Y\, \forall Z\, \psi(X, Y, Z)$ from $\mathsf{ACA}_0 \wedge (\mathbf{\Sigma^0_2})_x$-Det. Since M is countable, we can express the statement "the y-th formula in π is true in M" as an arithmetical formula $\tau(y)$. By induction on y, $\tau(y)$ holds for all y smaller than the length of π. But the last formula in π is $\forall X\, \exists Y\, \forall Z\, \psi(X, Y, Z)$, so

$$M \models \forall X\, \exists Y\, \forall Z\, \psi(X, Y, Z),$$

a contradiction. This proves (5). □

Corollary 7.4. *The statements (1)-(5) of Theorem 7.3 are all provable in* Π^1_2-$\mathsf{CA}_0 + \Sigma^1_3$-$\mathsf{IND}$ *(hence also in* Π^1_3-CA_0*) and unprovable in* Δ^1_3-CA_0*.*

Proof. From Theorem 7.3, Theorem 4.1, and Corollary 4.3. □

8 Further work

Positionalization in ACA_0 Our proof of Theorem 5.1 relies on the Klarlund-Kozen method of building positional strategies, which does not formalize in ACA_0. Is it possible to positionalize strategies (without assuming determinacy) in ACA_0, perhaps using the approach of [8]?

Determinacy As part of Theorem 7.3, we prove the equivalence of the determinacy principle $\forall x[(\mathbf{\Sigma^0_2})_x$-Det] and positional determinacy of all parity games. Our proof of this equivalence relies on the advanced techniques of [16], goes through an intermediate step involving the reflection principle $\mathrm{RFN}_{\Pi^1_3}(\Pi^1_2\text{-}\mathsf{CA}_0)$, and requires Π^1_2-CA_0 as a base theory.

Is there a simpler, more elementary proof of this equivalence with Π^1_2-CA_0 as the base theory? Is there a proof in ACA_0?

Weak MSO over the binary tree Can statements related to the decidability of the *weak* MSO theory of the infinite binary tree be characterized as determinacy principles, for instance as the determinacy of Boolean combinations of open games?

MSO over $\mathbb{R}$ In [24] it shown that the MSO theory of the real line with the natural ordering is undecidable. However, once the quantification is restricted to F_σ sets, the theory is decidable [22]. Is there a proof of this decidability result not relying on the full strength of Rabin's theorem?

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