Set Theory
Homework for April 6, 2016

15. Let $\{A_n : n \in \omega\}$ be a family of (exactly) countable sets. Prove that there exists a family $\{A'_n : n \in \omega\}$ of disjoint exactly countable sets such that $A'_n \subseteq A_n$ for each n and $\bigcup_n A_n = \bigcup_n A'_n$.

A closed set $F \subseteq \mathbb{R}^n$ is *perfect* if it is nonempty and has no isolated points.

16. Prove that every uncountable closed set $F \subseteq \mathbb{R}^n$ contains a perfect set.

17. Prove that every closed subset of $\mathbb{R}$ has cardinality either $\leqslant \aleph_0$ or $\mathfrak{c}$.

18. Prove the existence of a *Bernstein set*, i.e. a set $B \subseteq \mathbb{R}$ such that for every uncountable closed $F \subseteq \mathbb{R}$, both $F \cap B$ and $F \setminus B$ are nonempty.

19. Prove that no Bernstein set is Lebesgue-measurable.