Mathematical Logic 2
Course summary

Lecture 1. General notion of logic, regular logic, examples. The compactness and Löwenheim-Skolem properties. Lindström's Theorem. (Ebbinghaus-Flum-Thomas, last chapter — XII or XIII depending on the edition).

Class. Counterexamples to possible strengthenings of Lindström's Theorem. Finite occurence property. Compactness, countable compactness and Löwenheim-Skolem (or their failure) in various logics. Variants of Lindström's Theorem (most of this material is taken from Barwise, Feferman (ed.), *Model-Theoretic Logics*, Chapters II and III).

Lecture 2. Quantifier elimination. Typical benefits. Model-theoretic criterion for q.e. Simple examples: dense linear orders, the random graph. Corollaries: completeness, decidability, $\aleph_0$-categoricity (Marker, Section 3.1; for the random graph, Hodges, Section 6.4).

Class. Simple examples for q.e. and lack of it (Marker, exercises to Chapter 3). The 0-1 law for first order logic via the random graph (Hodges, Section 6.4).

Lecture 3. Another sufficient condition for q.e., proof of q.e. for DAG (theory of nontrivial divisible torsion-free abelian groups) and ACF (theory of algebraically closed fields). Corollaries: decidability of DAG and ACF, completeness for DAG and ACF_p, Chevalley's Theorem (the projection of a boolean combination of algebraic subsets of an acf is also a boolean combination of algebraic sets). Categoricity of DAG and ACF_p in uncountable cardinalities. (Marker, Sections 3.1 and 3.2)

Class. The Ax-Grothendieck Theorem via completeness of ACF_p (Marker, Section 2.2). Sketch of q.e. for (the right formulation of) arithmetic of $+$ (Presburger Arithmetic); model completeness (Marker, Section 3.1 and exercises to Chapter 3).

Lecture 4. Formally real fields, ordered fields, real closed fields. Tarski-Seidenberg Theorem: q.e. for RCF (theory of real closed ordered fields); still missing: uniqueness of real closure via Sturm's Theorem. Corollaries: completeness, decidability, o-minimality. Hilbert's 17th problem. (Marker, Section 3.3 and Appendix B, partly Bochnak, Coste, Roy, *Real Algebraic Geometry*, Chapter 1)

Class. Theories with q.e. are $\forall_2$ axiomatizable (in Hodges, Section 7.3, but with different proof). Semialgebraic sets, o-minimality (Marker, Section 3.3). Failure of a version of Hilbert's 17th for polynomials (Bochnak-Coste-Roy, Section 6.3).

Lecture 5. Proof of uniqueness of real closure, Sturm's Theorem (Marker, Appendix B). Concluding comments about o-minimality and q.e. Types, realizing and omitting types, statement of the Omitting Types Theorem (Marker, Sections 4.1 and 4.2 or Hodges, Sections 5.2 and 6.2).

Class. Real closed fields, semialgebraic sets, o-minimality (continuation). Types over $\mathbb{Q}$ (Marker, Section 4.1).

Lecture 6. Proof of the Omitting Types Theorem. Atomic and saturated models, existence criteria in the countable case. Characterization of $\aleph_0$-categoricity (Marker, Sections 4.2-4.4 or Hodges, Sections 6.2 and 6.3 - Hodges has more equivalent characterizations of $\aleph_0$-categoricity).

Class. Types, relation to automorphisms, omitting types, complete theories with finitely many countable models (never exactly 2, but 3 or more possible) (Marker, Chapter 4, also exercises to Chapter 2).

Lecture 7. Statements of: Vaught's Conjecture, Morley's theorem on the number of countable models (proof in Marker, Section 4.4.), Morley's Theorem on Uncountable Categoricity, the Baldwin-Lachlan Theorem (proof in Marker, Section 6.1, building on Sections 4.2-4.3, 5.1-5.2). An application of omitting types: countable completeness (thus countable compactness) for FO with the "exists uncountably many" quantifier (in Barwise-Fererman, Chapter IV: different proof, but similar in outline).

Class. Prime models, saturated models (Marker, Sections 4.2-4.3 and exercises to Chapter 4). Remarks on Baldwin-Lachlan. Strongly minimal theories: general notion of algebraic closure, the exchange principle (Marker, Section 6.1).

Warning. Starting with Lecture 8, I will not be relying on any source text too much. A large part of the material we will cover is in Kaye, and I will try to include references for various topics whenever possible, but they should be taken with a grain of salt.

Lecture 8. Foundational theories. ZFC set theory, PA^- (theory of non-negative parts of discretely ordered rings), PA (Peano Arithmetic). Interpretations, comparing foundational theories. ZFC interprets PA. The formula classes Δ_0, $\Delta_0(exp)$, Σ_n, Π_n, theories $I\Delta_0$, $I\Delta_0 + exp$, $I\Sigma_n$. Numerals. Σ_1 completeness of PA^-. Coding sequences in $I\Delta_0 + exp$. Interpreting some set theory in fragments of PA. Nasty, brutish and short introduction to computability/decidability (if you want a milder one, contact me or simply come to office hours). Relationship between computability/decidability and definability in arithmetic. (Much of this is in Kaye, cf. fragments of Sections 0.2, 2.1, 2.2, 3.1, 4.1, 5.3, 7.1; the stuff we skipped — the coding needed to define exponentiation — is Sections 5.1 and 5.2.)

Class. Strongly minimal theories: algebraic closure, bases, dimension, categoricity in uncountable cardinals (Marker, Section 6.1). Representability of decidable sets and computable functions in PA^-.

Lecture 9. Gödel numbering. Gödel's Diagonal Lemma. Tarski's Theorem on the Undefinability of Truth, Gödel's Incompleteness Theorems (This is roughly covered in Section 3.2 of Kaye, but the treatment is rather superficial: for example, Gödel's 2nd is only discussed in the exercises. For a more in-depth treatment, you might want to look e.g. at Smoryński's chapter *The incompleteness theorems* in Barwise (ed.), *Handbook of Mathematical Logic.*

Class. Order-type of models of PA (Kaye, Section 6.2). Rosser's Theorem and some consequences.

Lecture 10. Sketch of proof of provable Σ_1-completeness. Universal formulas Sat_{Σ_n} (for a painstakingly detailed account, consult Chapter 9 of Kaye).

Class. Essential undecidability of PA^-. The arithmetical hierarchy (Kaye, beginning of Section 7.1, end of Section 9.3).

Lecture 11. Coded sets, Tennenbaum's Theorem (Kaye, Sections 11.1 and 11.3)

Class. Consistency strengh (I don't have a good book reference to suggest: for a nice related discussion, you can have a look at: http://plato.stanford.edu/entries/independence-large-cardinals). Löb's Theorem (e.g. Section 4.1 of Smoryński's chapter in the Hanbook). The collection principle (Kaye, Section 7.1).

Lecture 12. The collection principle, definable elements and non-finite axiomatizability of PA (Kaye, Section 10.1). The Infinite Ramsey Theorem.

Class. Example of a false ω-consistent theory. Π_2-conservativity of $B\Sigma_1$ over $I\Delta_0$ (a more general result is in Kaye, Section 10.2, specifically Corollary 10.9). End-extendability of countable models of PA (Marker, Theorem 4.2.5; for a more general result, see Kaye, Section 8.2).

Lecture 13 and Class. The Paris-Harrington and Kanamori-McAloon Theorems (Marker, Section 5.4 or Kaye, Section 14.3)