New bounds on the strength of some restrictions of Hindman's Theorem

Lorenzo Carlucci[1] carlucci@di.uniroma1.it, Leszek Aleksander Kołodziejczyk[2] lak@mimuw.edu.pl, Francesco Lepore[1] lepore90@gmail.com, and Konrad Zdanowski[3] k.zdanowski@uksw.edu.pl

[1] Department of Computer Science, University of Rome I,
[2] Institute of Mathematics, University of Warsaw,
[3] Faculty of Mathematics and Natural Sciences, Cardinal Stefan Wyszyński University in Warsaw

Abstract. We prove upper and lower bounds on the effective content and logical strength for a variety of natural restrictions of Hindman's Finite Sums Theorem. For example, we show that Hindman's Theorem for sums of length at most 2 and 4 colors implies ACA_0. An emerging *leitmotiv* is that the known lower bounds for Hindman's Theorem and for its restriction to sums of at most 2 elements are already valid for a number of restricted versions which have simple proofs and better computability- and proof-theoretic upper bounds than the known upper bound for the full version of the theorem. We highlight the role of a sparsity-like condition on the solution set, which we call apartness.

1 Introduction and Motivation

The Finite Sums Theorem by Neil Hindman [14] (henceforth denoted HT) is a celebrated result in Ramsey Theory stating that for every finite coloring of the positive integers there exists an infinite set such that all the finite non-empty sums of distinct elements from it have the same color. Thirty years ago Blass, Hirst and Simpson proved in [2] that *all* computable instances of HT have *some* solutions computable in $\emptyset^{(\omega+1)}$ and that for *some* computable instances of HT *all* solutions compute $\emptyset'$. In terms of Reverse Mathematics, they showed that $\mathsf{ACA}_0^+ \vdash \mathsf{HT}$ and that $\mathsf{RCA}_0 \vdash \mathsf{HT} \to \mathsf{ACA}_0$. Both bounds hold for the particular case of colorings in two colors. Closing the gap between the upper and lower bound is one of the major open problems in Computable and Reverse Mathematics (see, e.g., [19]).

Blass advocated the study of restrictions of Hindman's Theorem in which a bound is put on the length (i.e., number of distinct terms) of sums for which monochromaticity is guaranteed [1], conjecturing that the complexity of Hindman's Theorem grows as a function of the length of sums. Recently Dzhafarov, Jockusch, Solomon and Westrick showed in [12] that the known $\emptyset'$ (ACA_0) lower bound on Hindman's Theorem holds for the restriction to sums of at most 3 terms (with no repetitions, as is the case throughout the paper), and 3 colors (henceforth denoted by $\mathsf{HT}^{\leq 3}_3$). They also established that the restriction to sums of at most 2 terms, and 2 colors (denoted $\mathsf{HT}_2^{\leq 2}$), is unprovable in RCA_0 and implies SRT_2^2 (the Stable Ramsey's Theorem for pairs and 2 colors) over $\mathsf{RCA}_0 + B\Sigma_2^0$. This prompted the first author (see [5,3]) to look into direct combinatorial reductions yielding, e.g., a direct implication from $\mathsf{HT}_5^{\leq 2}$ to IPT_2^2 (the Increasing Polarized Ramsey's Theorem for pairs of Dzhafarov and Hirst [11]) over RCA_0. Note that IPT_2^2 is strictly stronger than SRT_2^2 (see *infra* for details).

It should be stressed that no upper bound other than the $\emptyset^{(\omega+1)}$ (ACA_0^+) upper bound on the full Finite Sums Theorem is known to hold for the restrictions of the theorem to sums of length (i.e., number of terms) ≤ 2 or ≤ 3. It is indeed a long-standing open question in Combinatorics whether the latter restrictions admit a proof that does not establish the full Finite Sums Theorem (see, e.g., [15], Question 12). On the other hand, Hirst investigated in [17] an apparently slight variant of the Finite Sums Theorem and proved it *equivalent* to $B\Sigma_2^0$. This prompted the first author to investigate versions of HT for which an upper bound better than $\emptyset^{(\omega+1)}$ (ACA_0^+) could be established, while retaining as strong a lower bound as possible. In [4] (resp. [3]) such restrictions were isolated and proved to attain the known lower bounds for HT (resp. $\mathsf{HT}_2^{\leq 2}$), while being provable from ACA_0 (resp. RT_2^2). All these principles have a built-in sparsity-like condition on the solution set. This condition, called apartness in [4], is crucial yet was not given a name in earlier work ([14,12]).

We present new results along these lines of research. In Section 3 we prove an ACA_0 lower bound for $\mathsf{HT}_4^{\leq 2}$, and an equivalence with ACA_0 for Hindman's theorem restricted to sums of exactly 3 terms ($\mathsf{HT}_2^{=3}$), with an apartness condition, and for some principles from [4]. In Section 4 we establish combinatorial implications from other restrictions of Hindman's Theorem to the Increasing Polarized Ramsey's Theorem for Pairs. In particular we show that the latter principle is implied by (and indeed strongly computably reducible to) $\mathsf{HT}_2^{=2}$ with an apartness condition.

2 Restricted Hindman and the Apartness Condition

Let us fix some notation. For technical convenience and to avoid trivial cases we will deal with colorings of the positive integers. We use $\mathbf{N}$ to denote the positive integers. If $a \in \mathbf{N}$ and B is a set we denote by $FS^{\leq a}(B)$ (resp. $FS^{=a}(B)$) the set of non-empty sums of at most (resp. exactly) a-many distinct elements from B. More generally, if A and B are sets we denote by $FS^A(B)$ the set of all sums of j-many distinct terms from B, for all $j \in A$. By $FS(B)$ we denote $FS^{\mathbf{N}}(B)$. If X is the set $\{x_1, x_2, x_3, \dots\}$ we write $\{x_1, x_2, x_3, \dots\}_<$ to indicate that $x_1 < x_2 < x_3 < \dots$. Let us recall the statement of Hindman's Finite Sums Theorem [14].

Definition 1 (Hindman's Finite Sums Theorem). *Let $k \geq 2$. HT_k is the following assertion: For every coloring $f : \mathbf{N} \to k$ there exists an infinite set $H \subseteq \mathbf{N}$ such that $FS(H)$ is monochromatic for f. HT denotes $\forall k \mathsf{HT}_k$.*

It was proved in [2] that $\mathsf{ACA}_0^+ \vdash \mathsf{HT}$ and that $\mathsf{RCA}_0 \vdash \mathsf{HT}_2 \to \mathsf{ACA}_0$. We define below two restrictions of Hindman's Theorem that will feature prominently in the present paper. We then discuss a sparsity-like condition that will be central to our results.

2.1 Hindman's Theorem with bounded-length sums

We define two natural types of restrictions of Hindman's Theorem based on bounding the length of sums for which homogeneity is guaranteed.

Definition 2 (Hindman's Theorem with bounded-length sums). *Fix $n, k \geq 1$.*

1. $\mathsf{HT}^{\leq n}_k$ *is the following principle: For every coloring $f : \mathbf{N} \to k$ there exists an infinite set $H \subseteq \mathbf{N}$ such that $FS^{\leq n}(H)$ is monochromatic for f.*
2. $\mathsf{HT}^{=n}_k$*) is the following principle: For every coloring $f : \mathbf{N} \to k$ there exists an infinite set $H \subseteq \mathbf{N}$ such that $FS^{=n}(H)$ is monochromatic for f.*

The principles $\mathsf{HT}^{\leq n}_k$ were discussed in [1] (albeit phrased in terms of finite unions instead of sums) and first studied from the perspective of Computable and Reverse Mathematics in [12], where the principles $\mathsf{HT}^{=n}_k$ were also defined.

The principle $\mathsf{HT}^{\leq 2}_2$ is the topic of a long-standing open question in Combinatorics: Question 12 of [15] asks whether there exists a proof of $\mathsf{HT}^{\leq 2}_2$ that does not also prove the full Finite Sums Theorem. On the other hand, the principle $\mathsf{HT}^{=2}_2$ easily follows from Ramsey's Theorem for pairs: given an instance $f : \mathbf{N} \to 2$ of $\mathsf{HT}^{=2}_2$, define $g : [\mathbf{N}]^2 \to 2$ by setting $g(x, y) := f(x+y)$. Dzhafarov, Jockusch, Solomon and Westrick recently proved in [12] that $\mathsf{HT}^{\leq 3}_3$ implies ACA_0 over RCA_0 and that $\mathsf{HT}^{\leq 2}_2$ is unprovable in RCA_0. They also proved that $\mathsf{HT}^{\leq 2}_2$ implies SRT^2_2 (the Stable Ramsey's Theorem for pairs) over $\mathsf{RCA}_0 + B\Sigma^0_2$. The first author proved that $\mathsf{HT}^{\leq 2}_5$ implies IPT^2_2 (the Increasing Polarized Ramsey's Theorem for pairs) over RCA_0 (see [5]).

2.2 The Apartness Condition

We discuss a property of the solution set – which we call the apartness condition – that is crucial in Hindman's original proof and in the proofs of the $\emptyset'$ (ACA_0) lower bounds in [2,12,4]. We use the following notation: Fix a base $t \geq 2$. For $n \in \mathbf{N}$ we denote by $\lambda_t(n)$ the least exponent of n written in base t, by $\mu_t(n)$ the largest exponent of n written in base t, and by $i_t(n)$ the coefficient of the least term of n written in base t. We will drop the subscript when clear from context.

Definition 3 (Apartness Condition). *Fix $t \geq 2$. We say that a set $X \subseteq \mathbf{N}$ satisfies the t-apartness condition (or* is t-apart*) if for all $x, x' \in X$, if $x < x'$ then $\mu_t(x) < \lambda_t(x')$.*

Our results are in terms of 2-apartness except in one case (Lemma 1 below) where we have to use 3-apartness for technical reasons. For a Hindman-type principle P, let "P *with t-apartness*" denote the corresponding version in which the solution set is required to satisfy the t-apartness condition. Note that the apartness condition is inherited by subsets. In Hindman's original proof 2-apartness can be ensured (Lemma 2.2 in [14]) by a simple counting argument (Lemma 2.2 in [13]), under the assumption that we have a solution to the Finite Sums Theorem, i.e. an infinite H such that $FS(H)$ is monochromatic. In our terminology, we have that, for each $k \in \mathbf{N}$, HT_k is equivalent to HT_k with 2-apartness, over RCA_0.

As will be observed below, it is significantly easier to prove lower bounds on P with t-apartness than on P in all the cases we consider. Moreover, for *all* restrictions of Hindman's Theorem for which a proof is available that does not also establish the full theorem, the t-apartness condition (for $t > 1$) can be guaranteed by construction (see, e.g., [3,4]). This is the case, e.g., for the principle $\mathsf{HT}^{=2}_2$: the proof from Ramsey's Theorem for pairs sketched above yields t-apartness for any $t > 1$ simply by applying Ramsey's Theorem relative to an infinite t-apart set. In *some* cases the apartness condition can be ensured at the cost of increasing the number of colors. This is the case of $\mathsf{HT}^{\leq n}_k$ as illustrated by the next lemma. The idea of the proof is from the first part of the proof of Theorem 3.1 in [12], with some needed adjustments.

Lemma 1 (RCA_0)**.** *For all $n \geq 2$, for all $d \geq 1$, $\mathsf{HT}^{\leq n}_{2d}$ implies $\mathsf{HT}^{\leq n}_d$ with 3-apartness.*

Proof. We work in base 3. Let $f : \mathbf{N} \to d$ be given. Define $g : \mathbf{N} \to 2d$ as follows.

$$g(n) := \begin{cases} f(n) & \text{if } i(n) = 1, \\ d + f(n) & \text{if } i(n) = 2. \end{cases}$$

Let H be an infinite set such that $FS^{\leq n}(H)$ is homogeneous for g of color k. For $h, h' \in FS^{\leq n}(H)$ we have $i(h) = i(h')$. Then we claim that for each $k \geq 0$ there is at most one $h \in H$ such that $\lambda(h) = k$. Suppose otherwise, by way of contradiction, as witnessed by $h, h' \in H$. Then $i(h) = i(h')$ and $\lambda(h) = \lambda(h')$. Therefore $i(h + h') \neq i(h)$, but $h + h' \in FS^{\leq n}(H)$. Contradiction. Therefore we can computably obtain a 3-apart infinite subset of H. $\square$

3 Restricted Hindman and Arithmetical Comprehension

We prove a new ACA_0 lower bound and a new ACA_0 equivalence result for restrictions of Hindman's Theorem. The lower bound proof is in the spirit of Blass-Hirst-Simpson's proof that Hindman's Theorem implies ACA_0 – on which the proof of Theorem 3.1 of [12] is also based – with extra care to work with sums of length at most two. The upper bound proof is in the spirit of [4].

3.1 $\mathsf{HT}_4^{\leq 2}$ implies ACA_0

We show that $\mathsf{HT}_4^{\leq 2}$ implies ACA_0 over RCA_0. This is to be compared with Corollary 2.3 and Corollary 3.4 of [12], showing, respectively, that $\mathsf{RCA}_0 \nvdash \mathsf{HT}_2^{\leq 2}$ and that $\mathsf{RCA}_0 \vdash \mathsf{HT}_3^{\leq 3} \to \mathsf{ACA}_0$. Blass, towards the end of [1], states without giving details that inspection of the proof of the $\emptyset'$ lower bound for HT in [2] shows that these bounds are true for the restriction of the Finite Unions Theorem to unions of at most two sets.[4] Note that the Finite Unions Theorem has a built-in apartness condition. Blass indicates in Remark 12 of [1] that things might be different for restrictions of the Finite Sums Theorem, as those considered in this paper. Also note that the proof of Theorem 3.1 in [12], which stays relatively close to the argument in [2], requires sums of length 3.

Proposition 1 (RCA_0)**.** *For any fixed $t \geq 2$, $\mathsf{HT}_2^{\leq 2}$ with t-apartness implies* ACA_0.

Proof. We write the proof for $t = 2$. Assume $\mathsf{HT}_2^{\leq 2}$ with 2-apartness and consider an injective function $f\colon \mathbf{N} \to \mathbf{N}$. We have to prove that the range of f exists.

For a number n, written as $2^{n_0} + \cdots + 2^{n_r}$ in base 2 notation, we call $j \in \{0, \ldots, s\}$ *important in* n if some value of $f{\restriction}[n_{j-1}, n_j)$ is below n_0. Here $n_{-1} = 0$. The coloring $c\colon \mathbf{N} \to 2$ is defined by

$$c(n) := \mathrm{card}\{i : i \text{ is important in } n\} \bmod 2.$$

By $\mathsf{HT}_2^{\leq 2}$ with 2-apartness, there exists an infinite set $H \subseteq \mathbf{N}$ such that H is 2-apart and $\mathrm{FS}^{\leq 2}(H)$ is monochromatic w.r.t. c. We claim that for each $n \in H$ and each $x < \lambda(n)$, $x \in \mathrm{rg}(f)$ if and only if $x \in \mathrm{rg}(f{\restriction}\mu(n))$. This will give us a computable definition of $\mathrm{rg}(f)$: given x, find the smallest $n \in H$ such that $x < \lambda(n)$ and check whether x is in $\mathrm{rg}(f{\restriction}\mu(n))$.

It remains to prove the claim. In order to do this, consider $n \in H$ and assume that there is some element below $n_0 = \lambda(n)$ in $\mathrm{rg}(f) \setminus \mathrm{rg}(f{\restriction}\mu(n))$. By the consequence of Σ^0_1-induction known as *strong Σ^0_1-collection*, there is a number ℓ such that for any $x < \lambda(n)$, $x \in \mathrm{rg}(f)$ if and only if $x \in \mathrm{rg}(f{\restriction}\ell)$. By 2-apartness, there is $m \in H$ with $\lambda(m) \geq \ell > \mu(n)$. Write $n + m$ in base 2 notation,

$$n + m = 2^{n_0} + \cdots + 2^{n_r} + 2^{n_{r+1}} + \cdots + 2^{n_s},$$

where $n_0 = \lambda(n) = \lambda(n + m)$, $n_r = \mu(n)$, and $n_{r+1} = \lambda(m)$. Clearly, $j \leq s$ is important in $n + m$ if and only if either $j \leq r$ and j is important in n or $j = r + 1$; hence, $c(n) \neq c(n + m)$. This contradicts the assumption that $\mathrm{FS}^{\leq 2}(H)$ is monochromatic, thus proving the claim. □

[4] The Finite Unions Theorem states that every coloring of the finite non-empty sets of $\mathbf{N}$ admits an infinite and pairwise unmeshed set H of finite non-empty sets H such that every finite non-empty sum of elements of H is of the same color. Two finite non-empty subsets x, y of $\mathbf{N}$ are unmeshed if either $\max x < \min y$ or $\max y < \min x$. Note that Hindman's Theorem is equivalent to the Finite Unions Theorem only if the pairwise unmeshed condition is present.

Theorem 1 (RCA_0). $\mathsf{HT}_4^{\leq 2}$ *implies* ACA_0.

Proof. By Proposition 1 and Lemma 1. $\square$

3.2 Restrictions of HT equivalent to ACA_0

We prove that some restrictions of HT, including $\mathsf{HT}_2^{=3}$ with 2-apartness, are equivalent to ACA_0. The first examples of this kind were given in [4], where a family of natural restrictions of Hindman's Theorem was isolated such that each of its members admits a simple combinatorial proof, yet each member of a non-trivial sub-family implies ACA_0.

The weakest principle proved in [4] to be equivalent to ACA_0 is the following, called the Hindman-Brauer Theorem (with 2-apartness): Whenever $\mathbf{N}$ is 2-colored there is an infinite and 2-apart set $H \subseteq \mathbf{N}$ and *there exist* positive integers a, b such that $FS^{\{a,b,a+b,a+2b\}}(H)$ is monochromatic.

We improve on [4] by showing that some apparently weak restrictions of Hindman's Theorem provable from Ramsey's Theorem are equivalent to ACA_0.

We first show that the same holds for the following apparently weaker principle.

Definition 4. $\mathsf{HT}_2^{\exists\{a<b\}}$ *is the following principle: For every coloring $f : \mathbf{N} \to 2$ there is an infinite set $H \subseteq \mathbf{N}$ and positive integers $a < b$ such that $FS^{\{a,b\}}(H)$ is monochromatic.*

Theorem 2. $\mathsf{HT}_2^{\exists\{a<b\}}$ *with 2-apartness is equivalent to* ACA_0 *over* RCA_0.

Proof. We first prove the upper bound. Given $c : \mathbf{N} \to 2$ let $g : [\mathbf{N}]^3 \to 8$ be defined as follows:

$$g(x_1, x_2, x_3) := \langle c(x_1), c(x_1 + x_2), c(x_1 + x_2 + x_3)\rangle.$$

Fix an infinite and 2-apart set $H_0 \subseteq \mathbf{N}$. By RT_8^3 relativized to H_0 we get an infinite (and 2-apart) set $H \subseteq H_0$ monochromatic for g. Let the color be $\sigma = (c_1, c_2, c_3)$, a binary sequence of length 3. Then, for each $i \in \{1, 2, 3\}$, g restricted to $FS^{=i}(H)$ is monochromatic of color c_i. Obviously for some $3 \geq b > a > 0$ it must be that $c_a = c_b$. Then $FS^{\{a,b\}}(H)$ is monochromatic of color c_a.

The lower bound is proved by a minor adaptation of the proof of Proposition 1. As the n in that proof take an a-term sum. Then take a $(b - a)$-term sum as the m. $\square$

The same proof yields that the following Hindman-Schur Theorem with 2-apartness from [4] implies ACA_0: Whenever $\mathbf{N}$ is 2-colored there is an infinite 2-apart set H and *there exist* positive integers a, b such that $FS^{\{a,b,a+b\}}(H)$ is monochromatic. Indeed, the latter principle implies $\mathsf{HT}^{\exists\{a<b\}}$. Provability in ACA_0 is shown in [4].

We next adapt the proof of Proposition 1 to show that $\mathsf{HT}_2^{=3}$ with 2-apartness implies ACA_0. Since $\mathsf{HT}_2^{=3}$ with 2-apartness is also easily deducible from RT_2^3, we obtain an equivalence. Note that no lower bounds on $\mathsf{HT}_2^{=3}$ without apartness are known.

Theorem 3 (RCA_0)**.** $\mathsf{HT}_2^{=3}$ *with* 2*-apartness is equivalent to* ACA_0.

Proof. We first prove the lower bound. We work in $\mathsf{RCA}_0+\mathsf{HT}_2^{=3}$ with 2-apartness and consider an injective function $f\colon \mathbf{N}\to\mathbf{N}$. We have to prove that the range of f exists. The relation j *is important in* n and the computable $c:\mathbf{N}\to 2$ are defined as in the proof of Proposition 1.

By $\mathsf{HT}_2^{=3}$ with 2-apartness, there exists an infinite set H such that H is 2-apart and $FS^{=3}(H)$ is monochromatic w.r.t. c. Let $r<2$ be the color of $FS^{=3}(H)$ under c. We describe a method of deciding the range of f with H given as an oracle.

Claim. For each $n,k\in H$. If $n<k$ and $c(n+k)=r$ then for each $x<\lambda(n)$,

$$x\in \mathrm{rg}(f) \iff x\in \mathrm{rg}(f{\restriction}\mu(k)).$$

Proof. Let $n,k\in H$ be such that $n<k$ and $c(n+k)=r$. By strong Σ^0_1 collection, let ℓ be such that for all $x<\lambda(n)$,

$$x\in \mathrm{rg}(f) \iff x\in \mathrm{rg}(f{\restriction}\ell)$$

and take $m\in H$ such that $\lambda(m)>\ell$. Now, if for some $x<\lambda(n)$, $x\in \mathrm{rg}(f)\setminus \mathrm{rg}(f{\restriction}\mu(k))$, then the number of important digits in $n+k+m$ is greater by one than the number of important digits in $n+k$. Then, $c(n+k+m)=1-c(n+k)=1-r$ which contradicts the fact that r is the color of $FS^{=3}(H)$.

Claim. For each $n\in H$ there exists $k\in H$ such that $n<k$ and $c(n+k)=r$.

Proof. Let us fix n and, again, let ℓ be such that for all $x<\lambda(n)$,

$$x\in \mathrm{rg}(f) \iff x\in \mathrm{rg}(f{\restriction}\ell).$$

Then, we define $k\in H$ as any element such that $\lambda(k)>\ell$. Now, for any $m\in H$, if $k<m$, then $c(n+k)=c(n+k+m)=r$.

We now describe an algorithm for deciding $\mathrm{rg}(f)$. Given x, find $n\in H$ such that $x<\lambda(n)$. Then, find $k\in H$ such that $n<k$ and $c(n+k)=r$. By Lemma 3.2 this part of computation ends successfully. Finally, check whether $x\in \mathrm{rg}(f{\restriction}\mu(k))$. By Lemma 3.2 this is equivalent to $x\in\mathrm{rg}(f)$.

The upper bound follows by observing that, for $n,k\geq 1$ and $t\geq 2$, $\mathsf{RCA}_0\vdash \mathsf{RT}^n_k\to\mathsf{HT}_k^{=n}$ with t-apartness. Given $c:\mathbf{N}\to k$ define $f:[\mathbf{N}]^n\to k$ as $f(x_1,\dots,x_n)=c(x_1+\cdots+x_n)$. Let X be an infinite t-apart set. Apply RT^n_k to f relative to X. □

Let us observe that the above argument works in the case of $\mathsf{HT}_2^{=a}$ with apartness, for any fixed $a\geq 3$. In the proof above take a sum of $a-2$ elements in place of n. Indeed, a in $\mathsf{HT}_2^{=a}$ could also be nonstandard. This leads us to the following definition and corollary.

Definition 5. *Let* $\mathsf{HT}_2^{\exists\{a\geq 3\}}$ *be the following principle: For every coloring* $f:\mathbf{N}\to 2$ *there exists an infinite set* $H\subseteq\mathbf{N}$ *and there exists a number* $a\geq 3$ *such that* $FS^{\{a\}}(H)$ *is monochromatic for* f.

Theorem 4 (RCA_0)**.** $\mathsf{HT}_2^{\exists\{a\geq 3\}}$ *with 2-apartness is equivalent to* ACA_0.

3.3 Hindman's Theorem for Exactly Large Sums

We consider a restriction of Hindman's Theorem to exactly large sums. A finite set $S \subseteq \mathbf{N}$ is *exactly large*, or $!\omega$-large, if $|S| = \min(S) + 1$. Large sets play a prominent role in the study of unprovability results for first-order theories of arithmetic. We denote by $[X]^{!\omega}$ the set of exactly large subsets of X and by $FS^{!\omega}(X)$ the set of positive integers that can be obtained as sums of terms of an exactly large subset of X. We call sums of this type *exactly large sums* (from X). Ramsey's Theorem for exactly large sums ($\mathsf{RT}^{!\omega}_2$) asserts that every 2-coloring f of the exactly large subsets of an infinite set $X \subseteq \mathbf{N}$ admits an infinite set $H \subseteq X$ such that f is constant on $[H]^{!\omega}$. It was studied in [6] and there proved equivalent to ACA_0^+. We introduce an analogue for Hindman's Theorem.

Definition 6 (Hindman's Theorem for Large Sums). $\mathsf{HT}^{!\omega}_2$ *denotes the following principle: For every coloring $c : \mathbf{N} \to 2$ there exists an infinite set $H \subseteq \mathbf{N}$ such that $FS^{!\omega}(H)$ is monochromatic under c.*

$\mathsf{HT}^{!\omega}_2$ (with t-apartness, for any $t > 1$) follows easily from $\mathsf{RT}^{!\omega}_2$. Given $c : \mathbf{N} \to 2$ just set $f(S) := c(\sum S)$, for S an exactly large set (to get t-apartness, restrict f to an infinite t-apart set). By results from [6] this reduction yields an upper bound of $\emptyset^{(\omega)}$ on $\mathsf{HT}^{!\omega}_2$.

The argument of Theorem 3 can be easily adapted to show that $\mathsf{HT}^{!\omega}_2$ with 2-apartness implies ACA_0. In the proof take, instead of n, an almost exactly large sum $n_0 + n_1 + \cdots + n_{n_0-2}$ of elements of H. The argument then proceeds unchanged.

Proposition 2 (RCA_0)**.** $\mathsf{HT}^{!\omega}_2$ *with 2-apartness implies* ACA_0.

Some further results on $\mathsf{HT}^{!\omega}_2$ will be proved in Section 4.

4 Restricted Hindman and Polarized Ramsey

In this section we establish new lower bounds for restricted versions of Hindman's Theorem by reduction to the Increasing Polarized Ramsey's Theorem for pairs [11]. In particular we obtain unprovability in WKL_0. Most of the restrictions of HT considered in this section do not imply ACA_0 and are therefore provably weaker than HT. All proofs of an implication to IPT^2_2 in the present section yield a strong computable reduction in the sense of [10], not just an implication. P is *strongly computably reducible* to Q, written $\mathsf{P} \leq_{sc} \mathsf{Q}$, if every instance X of P computes an instance X^* of Q, such that if Y^* is any solution to X^* then there is a solution Y to X computable from Y^*.

Definition 7 (Increasing Polarized Ramsey's Theorem). *Fix $n, k \geq 1$. IPT^n_k is the following principle: For every $f : [\mathbf{N}]^n \to k$ there exists a sequence $(H_1, \dots, H_n)$ of infinite sets such that all increasing tuples $(x_1, \dots, x_n) \in H_1 \times \cdots \times H_n$ have the same color under f. The sequence $(H_1, \dots, H_n)$ is called increasing polarized homogeneous (or increasing p-homogeneous) for f.*

Note that IPT^2_2 is strictly stronger than SRT^2_2. On the one hand, $\mathsf{RCA}_0 \vdash \mathsf{IPT}^2_2 \to \mathsf{D}^2_2$ by Proposition 3.5 of [11], and $\mathsf{RCA}_0 \vdash \mathsf{D}^2_2 \to \mathsf{SRT}^2_2$ by Theorem 1.4 of [8].[5] On the other hand, $\mathsf{RCA}_0 + \mathsf{SRT}^2_2 \nvdash \mathsf{IPT}^2_2$: Theorem 2.2 in [9] showed that there is a non-standard model of $\mathsf{SRT}^2_2 + B\Sigma^0_2$ having only low sets in the sense of the model. Lemma 2.5 in [11] can be formalized in RCA_0 and shows that no model of IPT^2_2 can contain only Δ^0_2 sets.[6]

4.1 $\mathsf{HT}^{=2}_2$ with 2-apartness implies IPT^2_2

We show that $\mathsf{HT}^{=2}_2$ with 2-apartness implies IPT^2_2 by a combinatorial proof establishing a strong computable reduction. This should be contrasted with the fact that no lower bounds on $\mathsf{HT}^{=2}_2$ without apartness are known.

Theorem 5 (RCA_0)**.** $\mathsf{HT}^{=2}_2$ *with 2-apartness implies* IPT^2_2*.*

Proof. Let $f : [\mathbf{N}]^2 \to 2$ be given. Define $g : \mathbf{N} \to 2$ as follows.

$$g(n) := \begin{cases} 0 & \text{if } n = 2^m, \\ f(\lambda(n), \mu(n)) & \text{if } n \neq 2^m. \end{cases}$$

Note that g is well-defined since $\lambda(n) < \mu(n)$ if n is not a power of 2. Let $H = \{h_1, h_2, \dots\}_<$ witness $\mathsf{HT}^{=2}_2$ with 2-apartness for g. Let the color be $k < 2$. Let

$$H_1 := \{\lambda(h_{2i-1}) \,:\, i \in \mathbf{N}\},\ H_2 := \{\mu(h_{2i}) \,:\, i \in \mathbf{N}\}.$$

We claim that (H_1, H_2) is increasing p-homogeneous for f.

First observe that we have

$$H_1 = \{\lambda(h_1), \lambda(h_3), \lambda(h_5), \dots\}_<,\ H_2 = \{\mu(h_2), \mu(h_4), \mu(h_6), \dots\}_<.$$

This is so because $\lambda(h_1) \leq \mu(h_1) < \lambda(h_2) \leq \mu(h_2) < \dots$ by the 2-apartness condition. We claim that $f(x_1, x_2) = k$ for every increasing pair $(x_1, x_2) \in H_1 \times H_2$. Note that $(x_1, x_2) = (\lambda(h_i), \mu(h_j))$ for some $i < j$ (the case $i = j$ is impossible by construction of H_1 and H_2). We have

$$k = g(h_i + h_j) = f(\lambda(h_i + h_j), \mu(h_i + h_j)) = f(\lambda(h_i), \mu(h_j)) = f(x_1, x_2),$$

since $FS^{=2}(H)$ is monochromatic for g with color k. This shows that (H_1, H_2) is increasing p-homogeneous of color k for f. $\square$

With minor adjustments the proof of Theorem 5 yields that $\mathsf{IPT}^2_2 \leq_{\mathrm{sc}} \mathsf{HT}^{\leq 2}_4$ (for a self-contained proof see [5]).

[5] Note that the diagram in [11] does not take into account the latter result. D^2_2, defined in [7], is the following assertion: For every 0, 1-valued function $f(x, s)$ for which a $\lim_{s\to\infty} f(x, s)$ exists for each x, there is an infinite set H and a $k < 2$ such that for all $h \in H$ we have $\lim_{s\to\infty} f(h, s) = k$.

[6] We thank Ludovic Patey for pointing out to us the results implying strictness.

4.2 IPT^2_2 and the Increasing Polarized Hindman's Theorem

We define a (increasing) polarized version of Hindman's Theorem. We prove that its version for pairs and 2 colors with an appropriately defined notion of 2-apartness is equivalent to IPT^2_2.

Definition 8 ((Increasing) Polarized Hindman's Theorem). *Fix $n \geq 1$. PHT^n_2 (resp. IPHT^n_2) is the following principle: For every $f : \mathbf{N} \to 2$ there exists a sequence $(H_1, \ldots, H_n)$ of infinite sets such that for some color $k < 2$, for all (resp. increasing) $(x_1, \ldots, x_n) \in H_1 \times \cdots \times H_n$, $f(x_1 + \cdots + x_n) = k$.*

We impose a t-apartness condition on a solution $(H_1, \ldots, H_n)$ of IPHT^n_2 by requiring that the union $H_1 \cup \cdots \cup H_n$ is t-apart. We denote by "IPHT^n_2 with t-apartness" the principle IPHT^n_2 with this t-apartness condition on the solution set.

Theorem 6. *IPT^2_2 and IPHT^2_2 with 2-apartness are equivalent over RCA_0.*

Proof. We first prove that IPT^2_2 implies IPHT^2_2 with 2-apartness. Given $c : \mathbf{N} \to 2$ define $f : [\mathbf{N}]^2 \to 2$ in the obvious way setting $f(x, y) := c(x + y)$. Fix two infinite disjoint sets S_1, S_2 such that $S_1 \cup S_2$ is 2-apart. By Lemma 4.3 of [11], IPT^2_2 implies over RCA_0 its own relativization: there exists an increasing p-homogeneous sequence (H_1, H_2) for f such that $H_i \subseteq S_i$. Therefore $H_1 \cup H_2$ is 2-apart by construction. Let the color be $k < 2$. Obviously we have that for any increasing pair $(x_1, x_2) \in H_1 \times H_2$, $c(x_1 + x_2) = f(x_1, x_2) = k$. Therefore (H_1, H_2) is an increasing p-homogeneous pair for c.

Next we prove that IPHT^2_2 with 2-apartness implies IPT^2_2. Let $f : [\mathbf{N}]^2 \to 2$ be given. Define $c : \mathbf{N} \to 2$ by setting $c(n) := f(\lambda(n), \mu(n))$ if n is not a power of 2 and $c(n) = 0$ otherwise. Let (H_1, H_2) be a 2-apart solution to IPHT^2_2 for c, of color $k < 2$. Then set $H_1^+ := \{\lambda(h_{2i-1}) : i \in \mathbf{N}\}$ and $H_2^+ := \{\mu(h_{2i}) : i \in \mathbf{N}\}$. We claim that (H_1^+, H_2^+) is a solution to IPT^2_2 for f. Let $(x_1, x_2) \in (H_1^+, H_2^+)$ be an increasing pair. Then for some $h \in H_1$ and $h' \in H_2$ such that $h < h'$ we have $\lambda(h) = x_1$ and $\mu(h') = x_2$. Therefore

$$k = c(h + h') = f(\lambda(h + h'), \mu(h + h')) = f(\lambda(h), \mu(h')) = f(x_1, x_2).$$

$\square$

4.3 $\mathsf{HT}^{!\omega}_2$ and IPT^2_2

Proposition 3 (RCA_0)**.** IPHT^2_2 *with 2-apartness* $\leq_{\mathrm{sc}}$ $\mathsf{HT}^{!\omega}_2$.

Proof. Let $f : \mathbf{N} \to 2$ be given, and let $H = \{h_0, h_1, h_2, \ldots\}_<$ be an infinite 2-apart set such that $FS^{!\omega}(H)$ is monochromatic for f of color $k < 2$. Let $S_1, S_2, S_3, \ldots$ be such that each S_i is an exactly large subset of H, $\bigcup_{i \in \mathbf{N}} S_i = H$, and $\max S_i < \min S_{i+1}$, for each $i \in \mathbf{N}$. Let $s_i = \sum S_i$. Let $H_s := \{s_1, s_2, \ldots\}$. H_s is 2-apart and consists of the consecutive exactly large sums of elements of H. Let $H_t = \{t_1, t_2, \ldots\}_<$ be the set consisting of the of elements from H_s minus their largest term (when written as $!\omega$-sums). Note that distinct elements of H_s share no term, because H_s is 2-apart. Let $H_1 := H_t$ and $H_2 := \{s_i - t_i : i \in \mathbf{N}\}$. Then (H_1, H_2) is a 2-apart solution for IPHT^2_2: $\square$

Other results on $\mathsf{HT}^{!\omega}_2$ were proved by the third author in his BSc. thesis, e.g., that the following implications hold over RCA_0: $\mathsf{HT}^{!\omega}_2$ implies $\mathsf{HT}^{=2}_2$, $\mathsf{HT}^{!\omega}_2$ with 2-apartness implies $\forall n \mathsf{HT}^{=2^n}_2$, $\mathsf{HT}^{!\omega}_2$ implies $\forall n \mathsf{PHT}^n_2$.

5 Conclusion

Table 1. Summary of main results

Principles	Lower Bounds	Upper Bounds
HT	ACA_0 ([2])	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\leq 2}_2$	$\mathsf{RCA}_0 \nvdash$ ([12])	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\leq 2}_2 + B\Sigma^0_2$	SRT^2_2 ([12])	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\leq 2}_2$ with 2-apartness	ACA_0 (Proposition 1)	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\leq 2}_4$	ACA_0 (Theorem 1)	$\emptyset^{(\omega+1)}$, ACA_0^+ ([2])
$\mathsf{HT}^{\exists\{a<b\}}_2$	?	ACA_0 (Theorem 2)
$\mathsf{HT}^{\exists\{a<b\}}_2$ with 2-apartness	ACA_0 (Theorem 3)	ACA_0 (Theorem 2)
$\mathsf{HT}^{=3}_2$	?	ACA_0 (folklore)
$\mathsf{HT}^{=3}_2$ with 2-apartness	ACA_0 (Theorem 3)	ACA_0 (folklore)
$\mathsf{HT}^{!\omega}_2$	?	$\emptyset^{(\omega)}$, ACA_0^+ ([6])
$\mathsf{HT}^{!\omega}_2$ with 2-apartness	ACA_0 (Proposition 2)	$\emptyset^{(\omega)}$, ACA_0^+ ([6])
$\mathsf{HT}^{=2}_2$	?	RT^2_2 (folklore)
$\mathsf{HT}^{=2}_2$ with 2-apartness	IPT^2_2 (Theorem 5)	RT^2_2 (folklore)
IPHT^2_2 with 2-apartness	IPT^2_2 (Theorem 6)	IPT^2_2 (Theorem 6)

Our results are summarized in Table 1, along with previously known results. In many cases they confirm that the known lower bounds on Hindman's Theorem hold for restricted versions for which — contrary to the restrictions studied in [12] — the upper bound lies strictly below $\emptyset^{(\omega+1)}$ (most being provable in ACA_0 or even in RT^2_2). Our results also highlight the role of the apartness condition on the solution set.

References

1. A. Blass. Some questions arising from Hindman's Theorem. *Scientiae Mathematicae Japonicae*, 62 (2005), 331–334.

2. A. R. Blass, J.L. Hirst, S. G. Simpson. Logical analysis of some theorems of combinatorics and topological dynamics. In: *Logic and combinatorics* (Arcata, Calif., 1985), Contemp. Math., vol. 65, pp. 125–156. Amer. Math. Soc., Providence, RI (1987).
3. L. Carlucci. A weak variant of Hindman's Theorem stronger than Hilbert's Theorem. Preprint, 2016, available at `https://arxiv.org/abs/1610.05445`.
4. L. Carlucci. Weak Yet Strong restrictions of Hindman's Finite Sums Theorem. Preprint, 2016, available at `https://arxiv.org/abs/1610.07500`.
5. L. Carlucci. Bounded Hindman's Theorem and Increasing Polarized Ramsey's Theorem. In André Nies (editor), Logic Blog, 2016, Part 4, Section 9, available at `https://www.dropbox.com/s/ksj4xyno7ze1ucm/Blog2016.pdf?dl=1`.
6. L. Carlucci, K. Zdanowski, The strength of Ramsey's Theorem for coloring relatively large sets, *Journal of Symbolic Logic*, 79:1, (2014), 89–102.
7. P. A. Cholak, C. G. Jockusch, and T. A. Slaman. On the strength of Ramsey's theorem for pairs. *Journal of Symbolic Logic*, 66(1):1–55, 2001.
8. C. T. Chong, S. Lempp, and Y. Yang, On the role of the collection principle for Σ^0_2 formulas in second-order reverse mathematics, *Proceedings of the American Mathematical Society*, 138 (2010), 1093–1100.
9. C. T. Chong, T. A. Slaman, and Y. Yang, The metamathematics of the Stable Ramsey's Theorem for Pairs, *Journal of the American Mathematical Society*, 27 (2014), 863–892.
10. D. D. Dzhafarov. Cohesive avoidance and strong reductions. *Proceedings of the American Mathematical Society*, 143:869–876, 2015.
11. D. D. Dzhafarov and J. L. Hirst. The polarized Ramsey's theorem. *Archive for Mathematical Logic*, 48(2):141–157, 2011.
12. D. Dzhafarov, C. Jockusch, R. Solomon, L. B. Westrick. Effectiveness of Hindman's Theorem for bounded sums. In A. Day, M. Fellows, N. Greenberg, B. Khoussainov, and A. Melnikov, eds., *Proceedings of the International Symposium on Computability and Complexity* (in honour of Rod Downey's 60th birthday), Lecture Notes in Computer Science, Springer, to appear.
13. N. Hindman. The existence of certain ultrafilters on N and a conjecture of Graham and Rothschild. *Proceedings of the American Mathematical Society* 36(2) (1972), 341–346.
14. N. Hindman. Finite sums from sequences within cells of a partition of N. *Journal of Combinatorial Theory Series A* 17 (1974), 1–11.
15. N. Hindman, I. Leader, and D. Strauss. Open problems in partition regularity. *Combinatorics Probability and Computing* 12 (2003), 571–583.
16. D. R. Hirschfeldt *Slicing the Truth (On the Computable and Reverse Mathematics of Combinatorial Principles)*. Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore: Volume 28.
17. J. Hirst. Hilbert vs. Hindman. *Archive for Mathematical Logic 51*, (1-2) (2012), 123–125.
18. C. G. Jockusch. Ramsey's theorem and recursion theory, *Journal of Symbolic Logic* 37 (1972), 268–280.
19. A. Montalbán. Open questions in Reverse Mathematics. *Bulletin of Symbolic Logic*, 17:3 (2011), 431–454.