Foundations of mathematics – week 8
November 27, 2009

Exercises

1. Does there exist an equivalence relation in $\mathbb{N}$ which has
   (a) two equivalence classes, each with 37 elements;
   (b) two equivalence classes with 17 elements, five equivalence classes with 33 elements and one infinite equivalence class;
   (c) infinitely many equivalence classes, each with an infinite number of elements;
   (d) one empty equivalence class and one infinite equivalence class?

2. Let $\mathbb{Z}[x]$ denote the set of polynomials in x with integer coefficients. Let r be a relation in $\mathbb{Z}[x]$ such that $\langle f, g\rangle \in r$ if and only if all coefficients in the difference $f - g$ are even.
   (a) Show that r is an equivalence relation.
   (b) Find equivalence class of the zero polynomial.
   (c) Find three different equivalence classes.
   (d) Is the set $\mathbb{Z}[x]_{/r}$ finite?
   (e) Is the set $\{W(x) \in \mathbb{Z}[x] \mid W(0) = 2\}$ an equivalence class of the relation?

3. Let $r \subseteq \mathbb{N} \times \mathbb{N}$ be an equivalence relation in $\mathbb{N}$ and let $f : \mathbb{N} \times \mathbb{N} \to P(\mathbb{N})$ be such that $f(\langle x, y\rangle) = [x]_r \cap [y]_r$.
   (a) Is f injective?
   (b) Is f onto $P(\mathbb{N})$?
   (c) Find $f^{-1}(\{[3]_r\})$.
   (d) Find $f(\mathbb{N} \times \mathbb{N} - r)$.

Homework

1. Let s be a relation in $\mathbb{Z}^{\mathbb{N}}$ such that $\langle f, g\rangle \in s$ if and only if the difference $f - g$ converges to 0.
   (a) Show that r is an equivalence relation.
   (b) Find three different equivalence classes.

2. Let $r \subseteq \mathbb{N} \times \mathbb{N}$ be an equivalence relation in $\mathbb{N}$ and let $f : \mathbb{N} \times \mathbb{N} \to P(\mathbb{N})$ be such that $f(\langle x, y\rangle) = [x]_r \cup [y]_r$.
   (a) Is f injective?
   (b) Is f onto $P(\mathbb{N})$?
   (c) Find $f^{-1}(\{[3]_r\})$.
   (d) Find $f(r)$.

3. Let $\mathcal{R}$ be a nonempty family of equivalence relations. Prove that $\bigcap \mathcal{R}$ is an equivalence relation.