Foundations of mathematics – week 7
November 20, 2009

Exercises

1. Give an example of a function $f : A \to B$, $X \subseteq A$, $Y \subseteq B$ such that
   (a) $\vec{f}^{-1}(\vec{f}(X)) \neq X$;
   (b) $\vec{f}(\vec{f}^{-1}(X)) \neq X$;
   (c) $\vec{f}(C \cap D) \neq \vec{f}(C) \cap \vec{f}(D)$.
2. Let $f : P(\mathbb{N}) \times P(\mathbb{N}) \to P(\mathbb{N})$ be such that
   $$f(\langle C, D\rangle) = C \cap D.$$
   (a) Is f injective?
   (b) Is f onto $P(\mathbb{N})$?
   (c) Find $\vec{f}(P(B) \times P(B))$ for $B \subseteq \mathbb{N}$.
   (d) Find $\vec{f}^{-1}(\{\mathbb{N}\})$.
3. Show that the function $\varphi : P(A)^B \to P(A \times B)$ such that
   $$\varphi(f) = \{\langle a, b\rangle \in A \times B \mid a \in f(b)\}$$
   is injective and onto $P(A \times B)$.
4. Let $f \in T \to T$. Prove that $f \circ f = f$ if and only if $f|_{Rg(f)} = id_{Rg(f)}$.
5. Are the following equivalence relations:
   (a) $r \subseteq \mathbb{R}^2$, $\langle x, y\rangle \in r \Leftrightarrow x^2 \neq y^2$;
   (b) $r \subseteq \mathbb{R}^2$, $\langle x, y\rangle \in r \Leftrightarrow x^2 = y^2$;
   (c) $r \subseteq \mathbb{Z}^2$, $\langle x, y\rangle \in r \Leftrightarrow x \leqslant y$;
   (d) $r \subseteq P(\mathbb{N})^2$, $\langle x, y\rangle \in r \Leftrightarrow x \cap \mathbb{P} = y \cap \mathbb{P}$, (P) is the set of even numbers ?
6. Find equivalence class
   (b) $[1]_r$;
   (d) $[\{1\}]_r$.

Homework

1. Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be such that $\varphi(\langle n, k\rangle) = nk$. Check whether φ is injective and onto $\mathbb{N}$. Find $\vec{\varphi}(\mathbb{P} \times \mathbb{N} \setminus \mathbb{P})$, $\vec{\varphi}^{-1}(\{1\})$, $\vec{\varphi}^{-1}(\mathbb{N} \setminus \mathbb{P})$, $\vec{\varphi}^{-1}(\{2^n \mid n \in \mathbb{N} \setminus \{0\}\})$. Here, $\mathbb{P}$ is the set of all even numbers.
2. Let $f : \mathbb{N}^{\mathbb{N}} \to P(\mathbb{N})$ be such that $f(\varphi) = \varphi(\mathbb{N})$. Is f injective and is it onto $P(\mathbb{N})$? Find $f^{-1}(B)$ where B is the set of one-element subsets of $\mathbb{N}$.
3. Let $f : A \to A$ be such that $f^n = f$ for some $n > 1$. Prove that $f(Rg(f)) = Rg(f)$.