Less Naive Type Theory

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Plan

How do we teach foundations of mathematics?

Set theory

- provides basic notions;
- unifies the language;
- builds on first principles;
- everyone does it.

Is set theory easy?

It SEEMS easy.

BUT it is not easy to learn.

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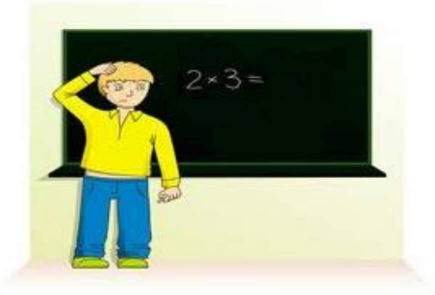
The confusion



$\in$ or $\subseteq$?

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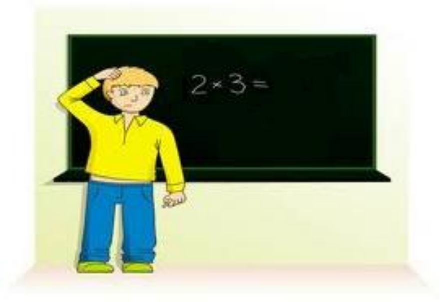

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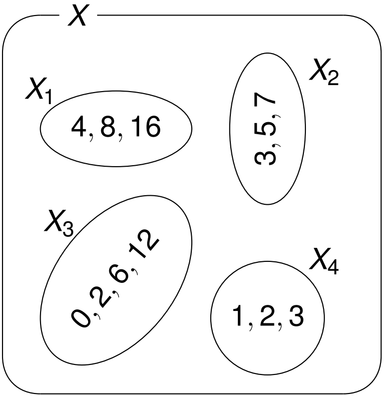

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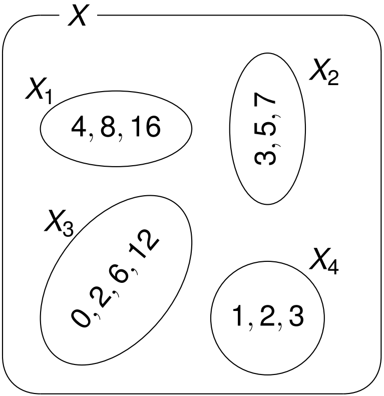

$$X \in P(P(\mathbb{N}))$$

$$X_i \in X \qquad X_i \in P(\mathbb{N})$$

$$n \in X_i \qquad n \in \mathbb{N}$$

You need to build a type system in your head.

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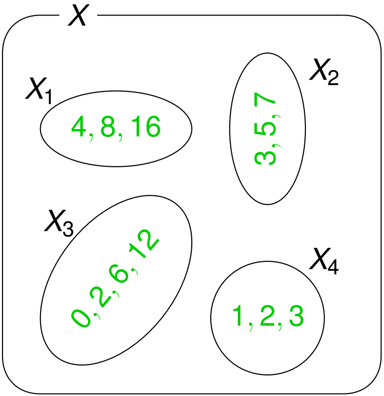

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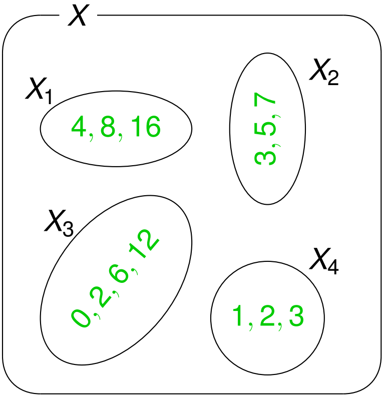

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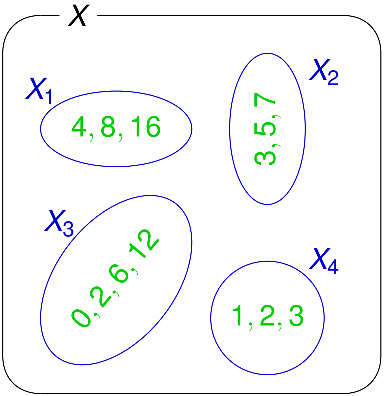

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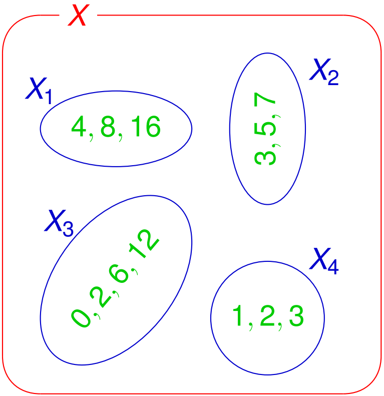

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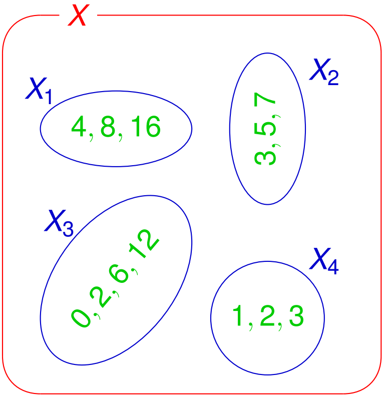

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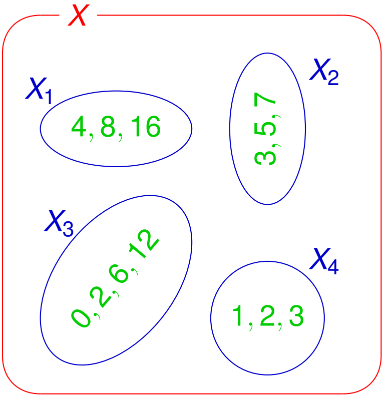

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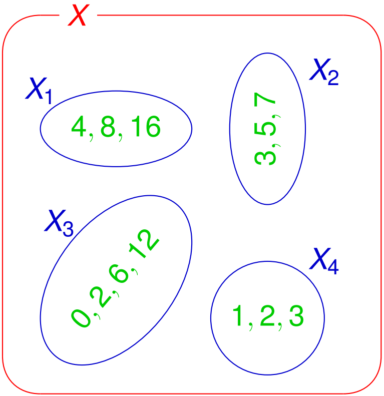

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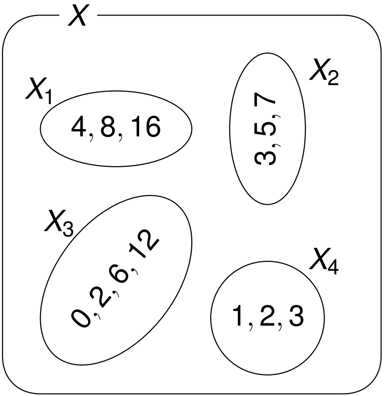

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You need to build a **type system** in your head.

What I want to do

The goal

Build a type theory which captures this informal type system.

Pure Type Systems (PTSs)

- formalism to talk about type systems
- commonly used
- parametric representation
- useful for comparing type systems

Less Naive Type Theory

Features of the system

- function space,
- dependent types,
- first order logic,
- higher order logic,
- subsets as objects.

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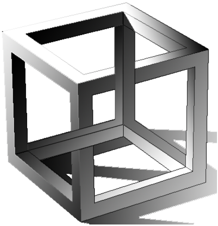

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Paradoxes

- Russell's paradox,
- Naive Type Theory.

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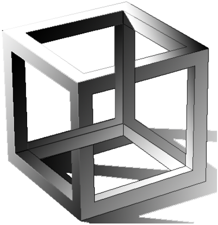

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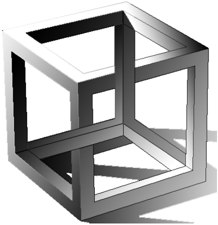

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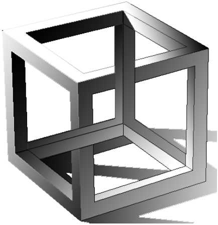

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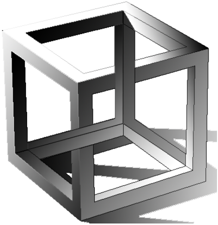

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Type assignment systems

$$\Gamma \vdash M : \tau$$

Environment
the assumptions

Object (term)

Type
classifies the object

Typing rules

Rules telling how to create typing assignments.

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Rules telling how to create typing assignments.

An example rule

function
from A to B

argument
of type A

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B}$$

application of a function

The application rule

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Does it look familiar?

An example rule

$$\frac{\Gamma \vdash A \to B \qquad \Gamma \vdash A}{\Gamma \vdash B}$$

The modus ponens rule

An example rule

proof of $A \to B$

proof of A

$$\frac{\Gamma \vdash p : A \to B \qquad \Gamma \vdash q : A}{\Gamma \vdash p(q) : B}$$

proof of B

The modus ponens rule

Eureka!

The Curry-Howard isomorphism!

The computations

$$f : \mathbb{N} \to \mathbb{N}, \qquad f = \lambda x : \mathbb{N}.x + 7$$

$$f(5) = (\lambda x : \mathbb{N}.x + 7)(5) \to 5 + 7 \to 12$$

reduction

normal term
term which
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non-normalizing term term which can be reduced infinitely

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$$Lemma_1 : A \to B,\ Lemma_2 : A$$

$$(\lambda lem_1 : A \to B \lambda lem_2 : A.lem_1(lem_2))Lemma_1 Lemma_2 : B$$

Suppose we have lem_1 which proves $A \to B$ and lem_2 which proves A. Then we can take lem_1 and apply it to lem_2 to obtain a proof of B. Now, note that $Lemma_1$ is a proof of $A \to B$ and $Lemma_2$ is a proof of A. Thus we have a proof of B.

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Strong normalization

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Every proof of false is non-normalizing.

Corollary

To prove consistency of a type system it is enough to prove that there are no non-normalizing terms.

Strong normalization property (Termination)

Every reduction sequence terminates = Every term is (strongly) normalizing.

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Theorem

Less Naive Type Theory is consistent.

Proof

Strong normalization via translation to the Calculus of Constructions.

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Work in progress

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Lets you define objects by induction and prove their properties.

Example

Natural numbers, lists, trees, graphs, . . .

Bad news

Much more complicated than pure type systems.

Technique of the proof

Girard's candidates

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Thank you.

Commercial

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