More on Light Graphs in 3-Connected Plane Graphs Without Triangular or Quadrangular Faces

Łukasz Kowalik*
Institute of Informatics, Warsaw University
Banacha 2, 02-097 Warsaw, Poland
kowalik@mimuw.edu.pl

Abstract

Intuitevly, a graph H is light in a class $\mathcal{G}$ of graphs when H is a subgraph of some member of $\mathcal{G}$ and for every $G \in \mathcal{G}$ if G contains H as a subgraph, there is also an occurence H' of H in G such that the degrees of vertices of H' in G are bounded by some constant. Characterizing light graphs in various subclasses of plane graphs attracted many researchers in recent years. We focus on the class $\mathcal{P}(3,5)$ of 3-connected plane graphs with faces of length at least 5. The class was previously studied by S. Jendrol' and P. J. Owens [6] who showed that every k-cycle, $k > 5$ and $k \neq 14$, is not light in $\mathcal{P}(3,5)$. They also proved that every block of any light graph in $\mathcal{P}(3,5)$ has at most 18 vertices.

In this paper we show that every block of a light graph in $\mathcal{P}(3,5)$ is a bridge, a cycle C_5 or a cycle C_{14}. Very recently, T. Madaras [9] showed that $K_{1,3}$ is light in $\mathcal{P}(3,5)$. This is an immediate corollary from more general result stating that in every planar graph with faces of length ≥ 5 and vertices of degree ≥ 3 there exists a vertex of degree at most 3 with neighbors a, b, c such that $\deg(a) = \deg(b) = 3$ and $\deg(c) \leq 4$. We give a new, much shorter proof of the latter result.

1 Introduction

One of the most well known facts concerning planar graphs states that every planar graph contains a vertex of degree at most 5. In 1955 Kotzig [7] showed that every 3-connected planar graph contains an edge for which the sum of degrees of its end-vertices is at most 13. Erdős conjectured that this property is valid also for planar graphs with vertices of degrees at least 3 and it turned out to be true due to Borodin [1]. We can put these results in a more general context. Consider the following definition.

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Definition. Let $\mathcal{G}$ be a class of graphs and let H be a graph isomorphic to a subgraph of at least one member of $\mathcal{G}$. We say that H is *light in class* $\mathcal{G}$ if there exists a contant c such that every graph $G \in \mathcal{G}$ which has a subgraph isomorphic to H also contains a subgraph H' isomorphic to H and such that, for every vertex $v \in V(H')$ we have $\deg_G(v) \leq c$. The smallest constant c for which this condition holds is denoted by $\varphi(H, \mathcal{G})$.

Thus, we can say that a vertex is light in the class of all planar graphs and an edge is light in the class of 3-connected planar graphs. Star $K_{1,n}$ shows that an edge is not light in planar graphs. Moreover, appart from a single vertex, there is no more connected light graphs in the class of all planar graphs. Surprisingly the situation is totally different in the class of 3-connected planar graphs. Fabrici and Jendrol' [3] showed that every path is light and any other graph is not light in this class . It is an interesting problem to characterize light graphs in other classes.

Among most natural subclasses of 3-connected planar graphs are classes $\mathcal{P}(\delta, \rho)$ of 3-connected planar graphs with minimum degree δ and minimum face size ρ. The classes are nonempty only for $(\delta, \rho) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}$. Except for paths, there is no more light graphs neither in class $\mathcal{P}(3, 4)$ [4] nor in class $\mathcal{P}(4, 3)$ [2]. The main open problem in this area is to characterize light graphs in $\mathcal{P}(3, 5)$ and $\mathcal{P}(5, 3)$. There are some partial results. Jendrol' and Madaras [5] proved that star $K_{1,r}$ is light in $\mathcal{P}(5, 3)$ if and only if $r \in \{3, 4\}$. For $\mathcal{P}(5, 3)$ there is a classical result of Lebesgue [8] stating that C_5 is light in this class. More precisely, he showed that every graph in $\mathcal{P}(5, 3)$ contains a face of length 5 with four vertices of degree 3 and one vertex of degree ≤ 5. Jendrol' and Owens [6] showed that every k-cycle, $k > 5$ and $k \neq 14$, is not light in $\mathcal{P}(3, 5)$. They also proved that every block of any light graph in $\mathcal{P}(3, 5)$ has at most 18 vertices. We continue their work by showing that every block of a light graph in $\mathcal{P}(3, 5)$ is a bridge, cycle C_5 or cycle C_{14}. Nevertheless, it is still not known whether C_{14} is light in $\mathcal{P}(3, 5)$. We also prove that $K_{1,3}$ is light in $\mathcal{P}(3, 5)$ and $\varphi(K_{1,3}, \mathcal{P}(3, 5)) = 4$. This is a corollary from a more general result stating that in every planar graph with faces of lengths ≥ 5 and vertices of degrees ≥ 3 there exists a vertex of degree at most 3 with neighbors a, b, c such that $\deg(a) = \deg(b) = 3$ and $\deg(c) \leq 4$. The latter result was proved independently by T. Madaras [9]. In this paper we give a new, much shorter proof. Let us note that the theorem is best possible in the sense that there is an example of a graph in $\mathcal{P}(3, 5)$ with every vertex of degree 3 adjacent to certain vertex of degree 4. Moreover it follows from the paper [6] that for every $r \geq 4$ the star $K_{1,r}$ is not light in our class.

2 On Blocks of Light Graphs in $\mathcal{P}(3, 5)$

In this section we prove the following theorem.

Theorem 1. *Let H be a connected light graph in $\mathcal{P}(3, 5)$. Every block in H is either a bridge, a cycle of length 5 or a cycle of length 14.*

The proof follows from lemmas below. We start from describing an operation of connecting a graph to a triangle. Let G be a plane graph with selected face f and let ABC

be a triangle. We start from putting G inside ABC. Then we add a new vertex X in the interior of f and add edges from X to all vertices of degree 2 in face f. Finally each of k vertices of degree 2 in the outer face of G is connected by an edge to one of vertices A, B or C in such a way that each of triangle vertices is connected to at least $\lfloor \frac{k}{3} \rfloor$ vertices of degree 2 and the resulting graph G' stays plane. We call G' as G *connected to a triangle*.

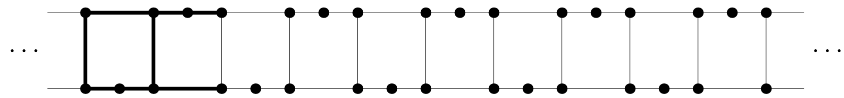

Figure 1: Infinite class of graphs $\{A_n\}$. Each graph A_n is compound of n copies of bold graph joined together to form a cycle (see A_{15} in Fig. 2).

Example. Figure 1 shows an infinite class of graphs $\{A_n\}$. Each graph A_n has two faces of length $3n$. One of them is the external face. We select the other face of length $3n$. Figure 2 shows a graph A_{15} from Fig. 1 connected the triangle.

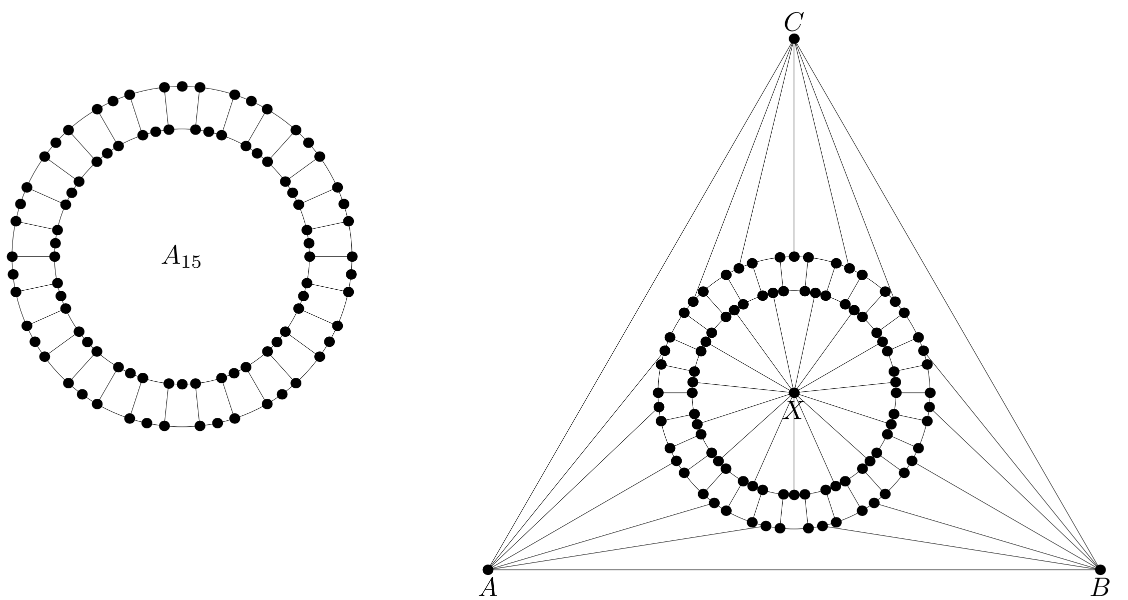

Figure 2: Connecting the graph A_{15} described in Fig. 1 to triangle ABC. Degrees of vertices A, B, C, X can be arbitrarily large when we take A_{3k} for k large enough.

Lemma 1. *Let H be a light subgraph in $\mathcal{P}(3,5)$. Every block of H is an outerplanar graph with every face of length $5 + 3k$ for certain $k = 0, 1, 2, \ldots$*

Proof. Let us choose an arbitrary $M > 3 \cdot \varphi(H, \mathcal{P}(3,5))$. Denote by T the graph A_M (see Fig. 1) connected to a triangle. Let H' be an arbitrary triangulation of H. After replacing each triangular face of H' by T we obtain a new graph G. Clearly $G \in \mathcal{P}(3,5)$. Subsequently H is a subgraph of the graph A_M. Moreover, $H \neq A_M$ since H is also a subgraph of A_k for any $k > M$. That ends the proof. □

Jendrol' and Owens showed that every block of a light graph in $\mathcal{P}(3,5)$ has at most 18 vertices. Now we show a little stronger result using similar methods:

Lemma 2. *Every block of a light subgraph in $\mathcal{P}(3,5)$ has at most 16 vertices.*

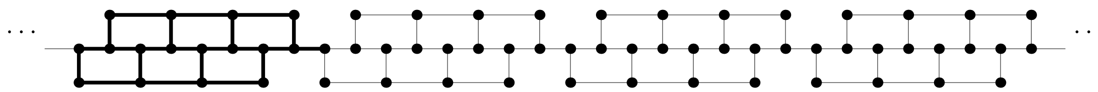

Figure 3: Infinite class of graphs $\{B_n\}$. Each graph B_n is compound of n copies of bold graph joined together to form a cycle.

Proof. Consider a family of graphs $\{B_n\}$ described in Fig. 3. We proceed similarly as in the proof of Lemma 1 but instead of graph A_M from Fig. 1 we use B_M from Fig. 3. As a result we get that every light graph in $\mathcal{P}(3,5)$ is a subgraph of B_M. Subsequently, every block of a light graph is a subgraph of the bold graph from Fig. 3 without the vertex of degree one. □

Lemma 3. *Light subgraphs in $\mathcal{P}(3,5)$ do not contain C_8.*

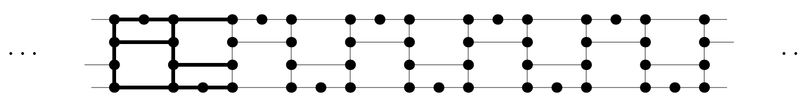

Figure 4: Infinite class of graphs $\{D_n\}$. Each graph D_n is compound of n copies of bold graph joined together to form a cycle.

Proof. Consider graphs D_n from Fig. 4. By replacing A_M by D_M in the proof of Lemma 1 we get that arbitrary light graph in $\mathcal{P}(3,5)$ is a subgraph of D_M. It suffices to observe that for any $M > 2$ graph D_M does not contain a cycle of length 8. □

Lemma 4. *Let H be a light subgraph in $\mathcal{P}(3,5)$. H does not contain C_{11}.*

Proof. The proof is actually the same as the proof of lemma 3, but instead graph D_M from Fig. 4 we use E_M from Fig. 5. For any $M > 3$ graph E_M does not contain a cycle of length 11.

□

It is easy to observe that Theorem 1 follows immediately from Lemmas 1, 2, 3 and 4.

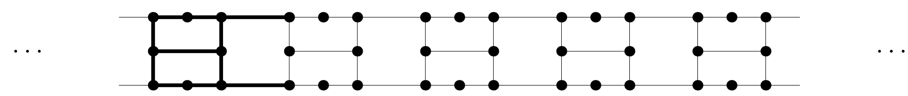

Figure 5: Infinite class of graphs $\{E_n\}$. Each graph E_n is compound of n copies of bold graph joined together to form a cycle.

3 Vertices of Low Degree with Neighbors of Low Degree

It is well known that every planar graph contains a vertex of degree at most 5. The following theorem shows that for planar graphs satisfying certain conditions much stronger invariant holds. The result was proved independently by T. Madaras [9]. Below we give much shorter proof.

Theorem 2. *Let G be a planar graph with every vertex of degree at least 3 and every face of length at least 5. Then G contains a vertex of degree at most 3 with neighbors a, b, c such that $\deg_G(a) = \deg_G(b) = 3$ and $\deg_G(c) \leq 4$.*

Proof. We use the well-known discharging technique. We put a *charge* of $\deg_G(v) - 4$ on every vertex v of G. Moreover, each face q of G obtains a charge of $|q| - 4$. Let n, m, f denote the number of vertices, edges and faces of graph G, respectively and let V, F be the set of vertices and faces of G respectively. Using Euler's formula we can easily calculate the total charge on G:

$$\sum_{v \in V}(\deg_G(v) - 4) + \sum_{q \in F}(|q| - 4) = 2m - 4n + 2m - 4f = -8$$

Now we move the charge from faces to vertices in such a way that each face q sends $\frac{|q|-4}{|q|}$ units of charge to every edge incident with q. If such an edge is incident with only one face it receives double charge. As $|q| \geq 5$, $\frac{|q|-4}{|q|} \geq \frac{1}{5}$. Then each edge divides its charge between its ends. Let $\phi(v)$ denote the amount of charge put so far on a vertex v. Subsequently, every vertex v has got $\phi(v) \geq \deg(v) - 4 + \frac{\deg(v)}{5}$ units of charge. We see that $\phi(v) \geq -\frac{2}{5}$ when v has degree 3 and $\phi(v) \geq \frac{4}{5}$ when v has degree at least 4. Finally, every vertex v with charge $\phi(v) > 0$ sends $\frac{\phi(v)}{\deg(v)}$ units of charge to each of its neighbors. Since $\frac{\phi(v)}{\deg(v)} = (\frac{6}{5}\deg(v) - 4)/\deg(v) = \frac{6}{5} - \frac{4}{\deg(v)}$ we see that every vertex of degree 4 sends at least $\frac{1}{5}$ units of charge and every vertex of degree at least 5 sends at least $\frac{2}{5}$ units of charge to each of its neighbors.

Let us assume that the desired vertex does not exist in G. Let u be an arbitrary vertex of degree 3 in G. Either u has two neighbors of degree at least 4 or u has a neighbor of degree at least 5. In the first case, u has got at least $-\frac{2}{5} + 2 \cdot \frac{1}{5} = 0$ units of charge. In the other case u stores at least $-\frac{2}{5} + \frac{2}{5} = 0$ units of charge. Thus, every vertex in G stores now nonnegative amount of charge – a contradiction. $\square$

Corollary 1. *$K_{1,3}$ is light in $\mathcal{P}(3,5)$.*

Let us note that T. Madaras [9] shows an example of a grap in $\mathcal{P}(3,5)$ with every vertex of degree 3 adjacent with a vertex of degree 4. It follows that $\varphi(K_{1,3}, \mathcal{P}(3,5)) = 4$ and that the above theorem is best possible, i.e. the constants 3, 4 cannot be improved. It is also worthwhile to point out that S. Jendrol' and P. J. Owens [6] showed that there is no light graph in $\mathcal{P}(3,5)$ containg a vertex of degree ≥ 4. In particular, for any $r \geq 4$ the star $K_{1,r}$ is not light in $\mathcal{P}(3,5)$.

Acknowledgments

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