Tight Lower Bounds for List Edge Coloring

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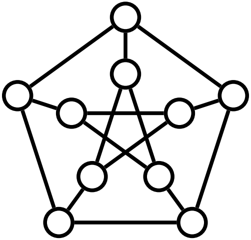
Edge Coloring

Assign colors to edges so that incident edges get distinct colors.



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Question: Does G admit edge coloring in k colors?

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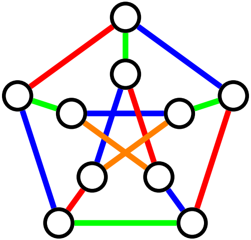

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Can we do better than $2^m = 2^{O(n^2)}$?

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- What about **dense** graphs?
- Algorithm runs in $2^{O(n^2)}$.
- **Huge gap!**

Open problem

Is there an algorithm for EDGE COLORING running in time

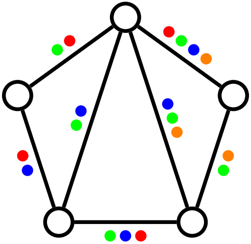
$$2^{o(n^2)} \quad ?$$

(Assuming ETH or other well-justified assumption.)

List Edge Coloring

LIST EDGE COLORING (decision version)

Input: Graph $G = (V, E)$, function $L : E \to 2^{\mathbb{N}}$



For $e \in E$, we call $L(e)$ a **list** of e.

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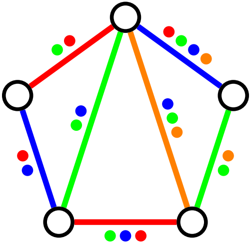
LIST EDGE COLORING (decision version)

Input: Graph $G = (V, E)$, function $L : E \to 2^{\mathbb{N}}$
Question: Does G admit an edge coloring c such that $c(e) \in L(e)$ for every edge $e \in E$?



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More general problem, same algorithms

Algorithms for Edge Coloring can be easily adapted to solve List Edge Coloring

- $2^m n^{O(1)}$ time exp space (Björklund, Husfeldt, Koivisto 2006)
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Even more general, both work for multigraphs.

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Can we do better? $2^{o(n^2)}$ **running time?**

Our results

Theorem 1

If there is an algorithm for LIST EDGE COLORING *for simple graphs that runs in time* $2^{o(n^2)}$*, then Exponential Time Hypothesis fails.*

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Note

Holds even if lists are of length a most 6.

A consequence for EDGE COLORING

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A simple way of verifying if a new idea for a faster algorithm for Edge Coloring works:

if it applies to the list version as well, there is no hope for it.

Proof for multigraphs

Theorem 2

If there is a function $f : \mathbb{N} \to \mathbb{N}$ such that List Edge Coloring *can be solved for multigraphs in time $f(n) \cdot m^{O(1)}$ for any input graph on n vertices and m edges, then $P = NP$.*

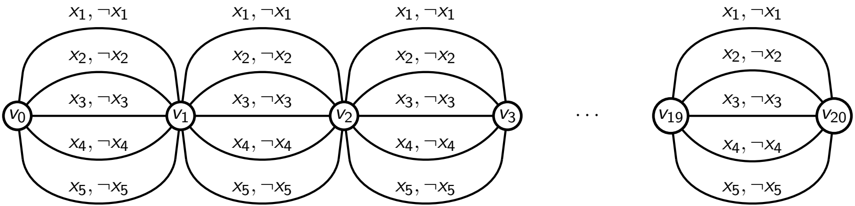
Proof plan

- $(3,4)$-SAT is 3-SAT restricted to formulas with every variable appearing in at most 4 clauses.
- $(3,4)$-SAT is NP-complete [Tovey 1984].
- Let φ be an instance of $(3,4)$-SAT.
- Reduce φ in polynomial time to an instance (G, L) of List Edge Coloring such that $|V(G)| = O(1)$.

literals as colors

Main idea

- Colors form the set $\{x_i, \neg x_i \mid i = 1, \ldots, n\}$
- In every coloring c, for every $i = 1, \ldots, n$, $c^{-1}(\{x_i, \neg x_i\})$ forms a path P_i with alternating colors.
- For multigraphs, $V = \{v_0, \ldots, v_{20}\}$ and $P_i = v_0, v_1, \ldots, v_{20}$.



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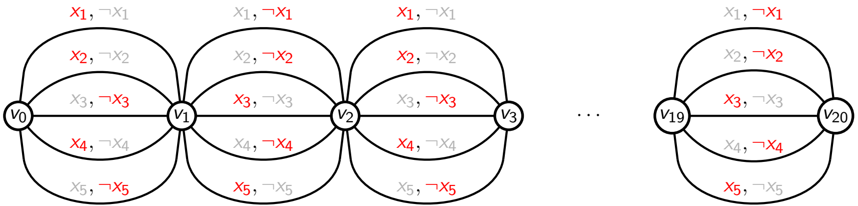
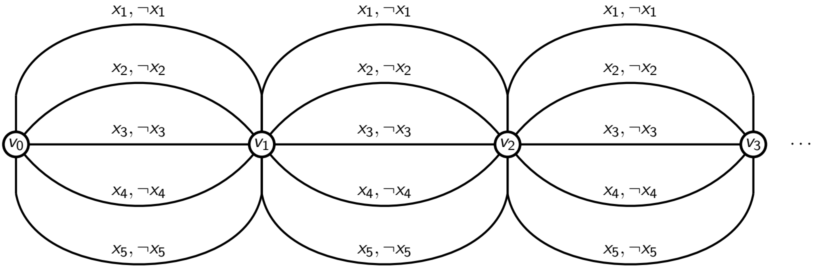


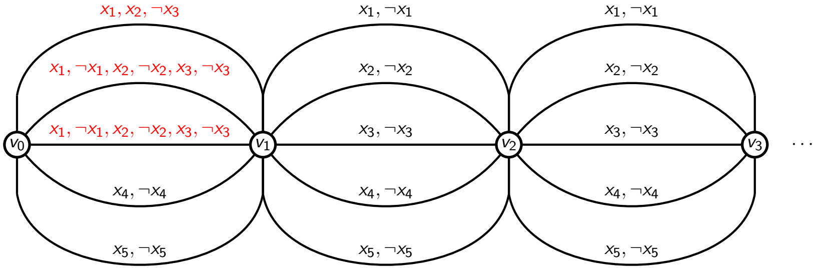
- Colors on edges $v_0 v_1$ (also $v_2 v_3, v_4 v_5, \ldots$) define a boolean assignment: $x_1 = T$, $x_2 = T$, $x_3 = F$, $x_4 = T$, $x_5 = F$.

Representing clauses



Representing clauses

- Add clause $x_1 \vee x_2 \vee \neg x_3$



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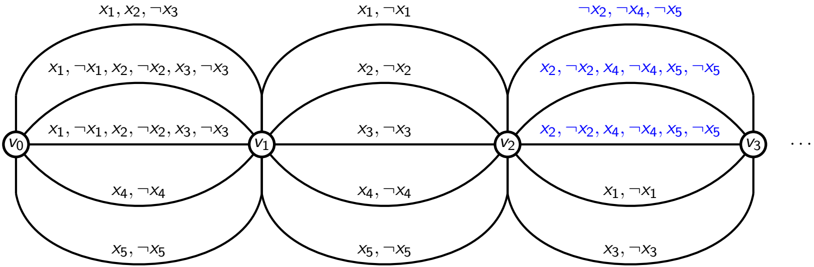

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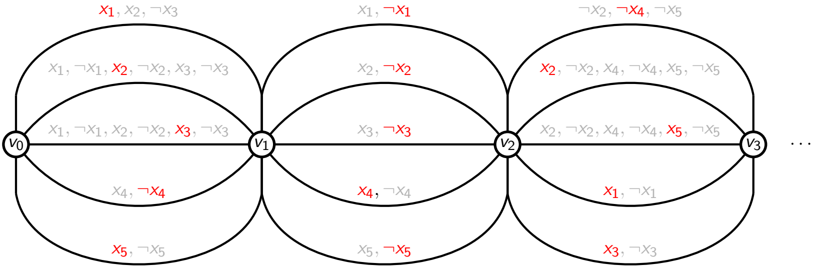

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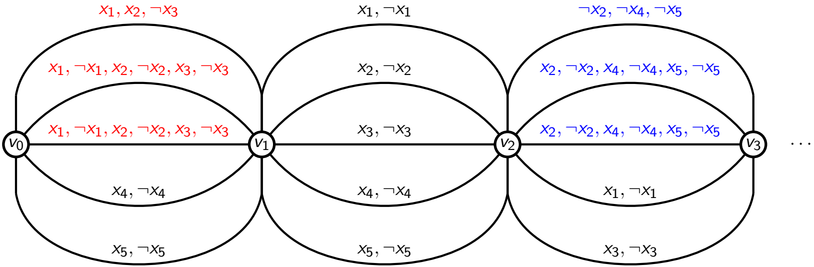

Observation

Two clauses must be disjoint to use edges with the same endpoints

Partitioning clauses into disjoint groups

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- For every i clauses in C_i disjoint.
- Every clause in C_i corresponds to three edges $x_{2i}x_{2i+1}$

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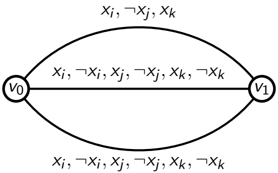
- Instead of $O(1)$ vertices, use $O(1)$ layers of $\Theta(\sqrt{n})$ vertices.
- Then indeed solving the output instance in time $2^{o(|V|^2)} = 2^{o(n)}$ refutes ETH.

Avoiding parallel edges: a new clause gadget

Previous clause gadget (for $x_i \vee \neg x_j, x_k$):



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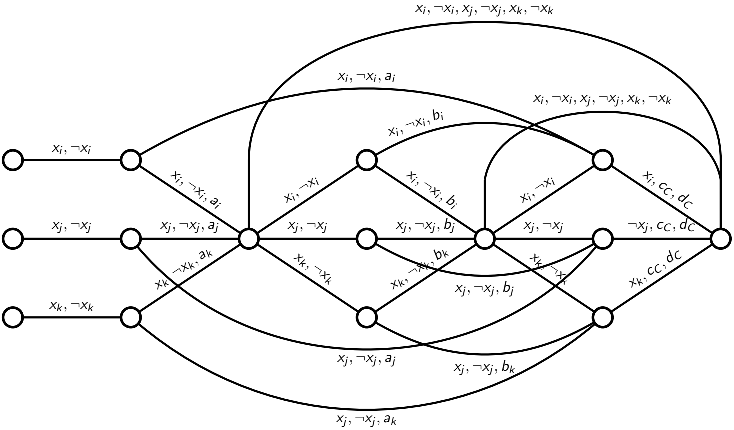

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