Approximating the maximum 3- and 4-edge-colorable subgraph

Marcin Kamiński[1] and Łukasz Kowalik (speaker)[2]

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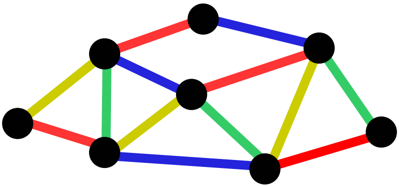
Bergen, 23.06.2010

(Regular) Edge-Coloring

Assign colors to edges so that incident edges get distinct colors.



What is known? ($\Delta = \max_{v \in V(G)} \deg(v)$)

- Δ colors needed (trivial)
- For simple graphs, $\Delta + 1$ colors suffice (Vizing)
- For simple graphs, deciding "$\Delta/(\Delta + 1)$" is NP-hard even for $\Delta = 3$.

Maximum k-Edge-Colorable Subgraph (k-ECS)

Problem: Given graph G find a k-edge-colorable subgraph $H \subseteq G$ so as to maximize $|E(H)|$.
OPT will denote the optimal H or the optimal $|E(H)|$.

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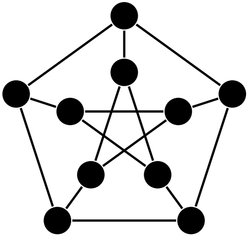

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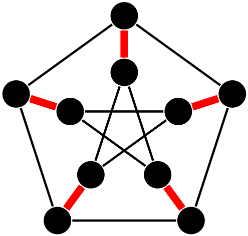

- $k = 1$: a maximum matching. Here: $OPT = 5$.

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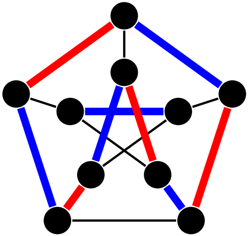

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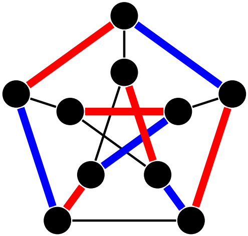

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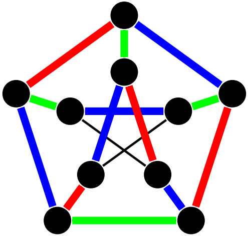

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- $k = 2$: paths and even cycles. Here: $OPT = 9$.
- $k = 3$: no special structure. Here: $OPT = 13$.

Maximum k-ECS: Complexity

- Poly-time for $k = 1$,
- NP-hard for $k \geq 2$ [Holyer 1981, Feige, Ofek, Wieder 2002]

In this talk we are interested in polynomial-time **approximation algorithms**.

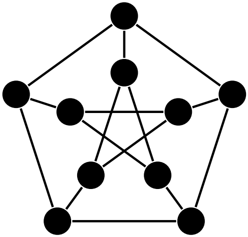
α-approximation

Algorithm A is a α-approximation algorithm for the Maximum k-ECS Problem when for any input graph G it always returns a k-edge-colorable subgraph of G with $\geq \alpha \cdot \mathrm{OPT}$ edges, where $\mathrm{OPT} = s_k(G)$.

Maximum k-ECS: Hardness of Approximation

The problem is APX-hard for $k \geq 2$ [Feige et al. 2002]
i.e. no $(1+\varepsilon)$-approximation for some $\varepsilon > 0$ unless $\mathrm{P} = \mathrm{NP}$.

A simple approach [Feige et al. 2002]



A k-matching in G is a subgraph $F \subseteq G$ such that for any $v \in V(F)$ we have $\deg_F(v) \leq k$.

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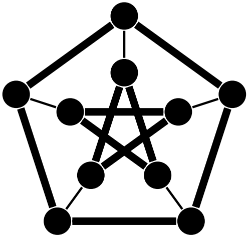

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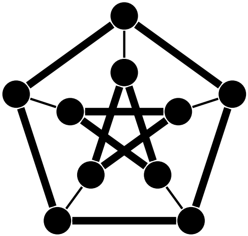

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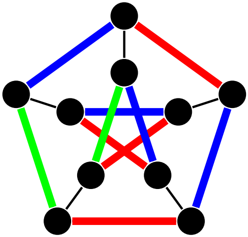

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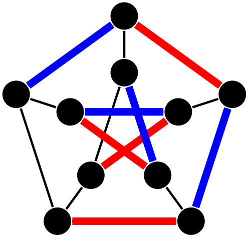

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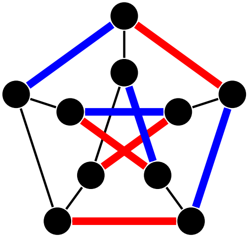

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Result: approximation ratio of $|U|/\mathrm{OPT} \geq |U|/|F|$.

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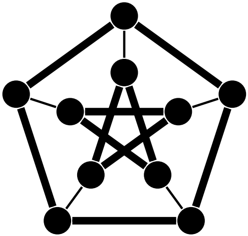

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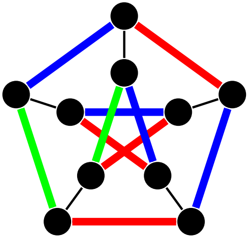

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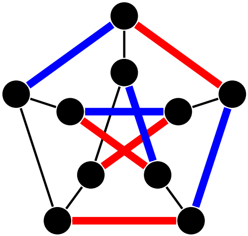

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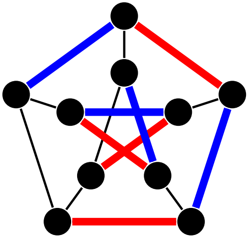

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Result: approximation ratio of $|U|/|F| \geq (\frac{k}{k+1}|F|)/F = \frac{k}{k+1}$.

Maximum k-ECS: Previous results

for simple graphs:

- $\frac{5}{6}$-approximation for 2-ECS [Kosowski 2009],
- $\frac{6}{7}$-approximation for 3-ECS [Rizzi 2009],
- $\frac{k}{k+1}$-approximation for k-ECS [Feige et al + Vizing]
  Note that $\lim_{k\to\infty} \frac{k}{k+1} = 1$.

for multigraphs:

- $\frac{10}{13}$-approximation for 2-ECS [Feige et al. 2002],
- $\frac{2}{3}$-approximation for k-ECS [Feige et al. + Shannon],
- $\frac{k}{k+\mu}$-approximation for k-ECS [Feige et al. + Vizing],
- $\xi(k)$-approximation for k-ECS [Feige et al. + Sanders & Steurer '08],
  where $\xi(k) = k / \left\lceil k + 2 + \sqrt{k+1} + \sqrt{\frac{9}{2}(k+2+\sqrt{k+1})} \right\rceil$
  Note that $\lim_{k\to\infty} \xi(k) = 1$.

Maximum k-ECS: Our results

for simple graphs:

- $\frac{13}{15}$-approximation for 3-ECS,
- $\frac{9}{11}$-approximation for 4-ECS.

for multigraphs:

- $\frac{7}{9}$-approximation for 3-ECS.

Improving the simple approach

Two ways of improving:

1. Improve the lower bound for OPT (find something better than $|F| \geq \mathrm{OPT}$), or
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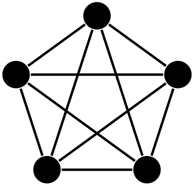
2. Improve the coloring phase. $\longleftarrow$ Let's start from this

Can we beat Vizing? (Even case)



Observation

For every even $k > 0$ in $G = K_{k+1}$ every k-ECS H has size $\leq \frac{k}{k+1}|E(G)|$.

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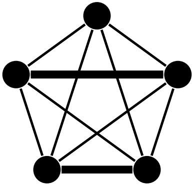

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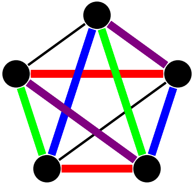
- Every color has $\leq k/2$ edges,



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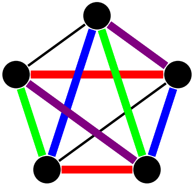

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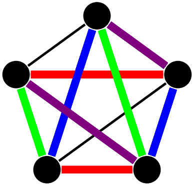
Can we beat Vizing? (Even case) No!



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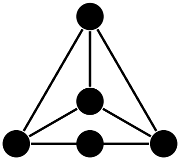
- $|E(G)| = \binom{k+1}{2} = (k+1)k/2$,
- hence $|E(H)|/|E(G)| \leq \frac{k}{k+1}$.

Can we beat Vizing? (Odd case)

$\widetilde{K}_p := K_p$ with one edge subdivided.



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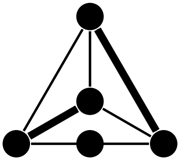

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Can we beat Vizing? (Odd case) Maybe...

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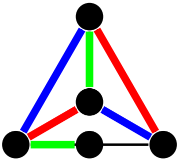

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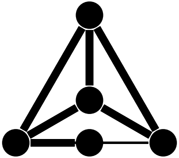
- So all colors have $|E(H)| \leq k\frac{k+1}{2} = |E(G)| - 1$ edges.

Can we beat Vizing? ($k = 3$ case: Yes, we can!)

Theorem [Rizzi 2009]

Every simple graph G of max degree 3 has a 3-ECS with $\geq \frac{6}{7}|E(G)|$ edges.

Tight by $\widetilde{K}_4$:



Corollary [Rizzi 2009]

There is a $\frac{6}{7}$-approximation for the max 3-ECS problem in simple graphs.

Can we beat Vizing? ($k = 3$ case, subclasses: even more!)

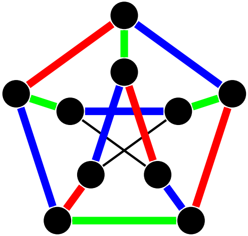
Theorem [Albertson and Haas 1996]

Every simple **3-regular** graph G has a 3-ECS with $\geq \frac{13}{15}|E(G)|$ edges.

Theorem [Rizzi 2009]

Every simple **triangle-free** graph G of max degree 3 has a 3-ECS with $\geq \frac{13}{15}|E(G)|$ edges.

Both tight by the Petersen graph:



Question and Answer

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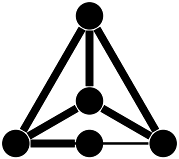
Is there any other bottleneck than $\widetilde{K}_4$ for general graps of maximum degree 3?



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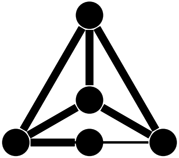

Our Answer

No!

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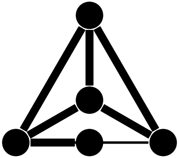


Our Answer

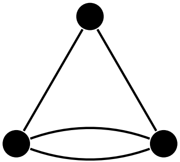
Every multigraph G of max degree 3 has a 3-ECS with $\geq \frac{13}{15}|E(G)|$ edges, unless $G = \widetilde{K}_4$.

Some more answers: cubic multigraphs

Theorem (Vizing)

Every multigraph G of max degree 3 has a 3-ECS with $\geq \frac{3}{4}|E(G)|$ edgess.

Tight by the following graph, call it G_3:



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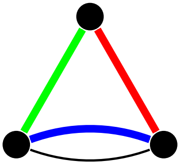
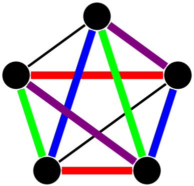


Our result

Every multigraph G of max degree 3 has a 3-ECS with $\geq \frac{7}{9}|E(G)|$ edges, unless $G = G_3$.

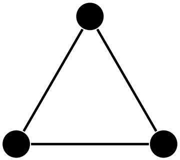
One more answer



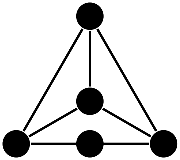
Our result

Every simple graph G of max degree 4 has a 3-ECS with $\geq \frac{5}{6}|E(G)|$ edges, unless $G = \widetilde{K}_5$.

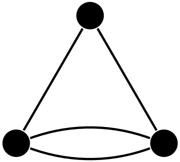
Annoying bottlenecks



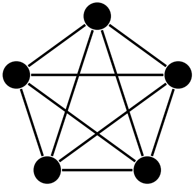
$k = 2$
simple graphs
ratio $\frac{2}{3}$



$k = 3$
simple graphs
ratio $\frac{6}{7}$



$k = 3$
multigraphs
ratio $\frac{3}{4}$



$k = 4$
simple graphs
ratio $\frac{4}{5}$

Improving the lower bound for $k = 2$

For the $k = 2$ case the bottleneck in the $\frac{2}{3}$-approximation is **a triangle**.

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For a **simple** graph G one can find a maximum **triangle-free** 2-matching in G in polynomial time.

(immediate) Corollary [Feige et al.]

A $\frac{4}{5}$-approximation for simple graphs.

Now the bottleneck is...

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(immediate) Corollary [Feige et al.]

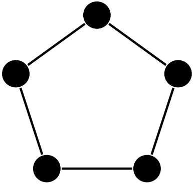
A $\frac{4}{5}$-approximation for simple graphs.

Now the bottleneck is... **a pentagon**.

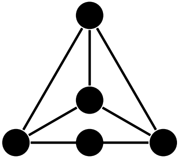
Can we repeat the trick?

- It is not known whether finding a maximum k-matching without odd cycles of length ≤ 5 is in P.
- For some $\ell > 0$, finding a maximum k-matching without odd cycles of length $\leq \ell$ is NP-hard.

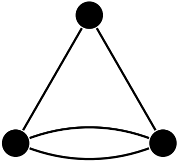
Annoying bottlenecks



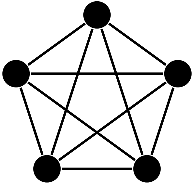
$k = 2$
simple graphs
ratio $\frac{4}{5}$



$k = 3$
simple graphs
ratio $\frac{6}{7}$



$k = 3$
multigraphs
ratio $\frac{3}{4}$



$k = 4$
simple graphs
ratio $\frac{4}{5}$

Improving the lower bound for $k = 2$ [Kosowski 2009]

Observation

Consider a pentagon C and a fixed optimal solution OPT.

- If OPT has no edge xy with $x \in V(C)$, $y \notin V(C)$ then $|E(OPT[V(C)]|) \leq 4$ — i.e. C is very good for us: locally we get approximation ratio 1.

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- Otherwise, there is an edge in G connecting C and another connected component in F. We can use these edges to form super-components, which have larger 2-edge-colorable sugraphs than $\frac{4}{5}$ of their edges.

Theorem [Kosowski]

This leads to a $\frac{5}{6}$-approximation for 2-ECS in simple graphs

Theorem

- Let $\mathcal{G}$ be a family of graphs.

The approach of Kosowski generalized

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- Assume that whenever $F \notin \mathcal{B}$ then A colors $\geq (\alpha + \epsilon)|E(F)|$ edges.
- Then, (if $\mathcal{B}$ has some nice properties), we can get approximation ratio better than α for the family $\mathcal{G}$.

Corollary

- $\frac{13}{15}$-approximation for 3-ECS in simple graphs ($\mathcal{B} = \{\widetilde{K}_4\}$),
- $\frac{7}{9}$-approximation for 3-ECS in multigraphs ($\mathcal{B} = \{G_4\}$).
- $\frac{9}{11}$-approximation for 4-ECS in simple graphs ($\mathcal{B} = \{K_5\}$).

We conjecture...

Conjecture 1

For any simple graph G and odd number k, there is an $\epsilon > 0$ such that $\frac{s(G)}{|E(G)|} \geq \frac{k}{k+1} + \epsilon$.

Conjecture 2

For any simple graph G and even number k, there is an $\epsilon > 0$ such that $\frac{s(G)}{|E(G)|} \geq \frac{k}{k+1} + \epsilon$, unless $G = K_{k+1}$.

Verified for $k = 3, 4$ (this work).

The end

Thank you for your attention!