Nonblocker in H-minor free graphs: kernelization meets discharging

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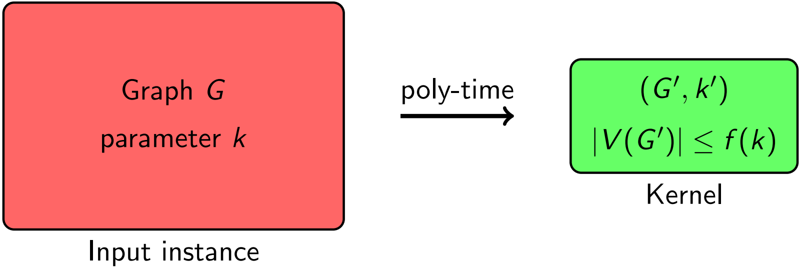
Kernelization (of graph problems)

Let (G, k) be an instance of a decision problem (k is a parameter).



- (G, k) is a YES-instance iff (G', k') is a YES-instance.
- $k' \leq k$,
- $|V(G')| \leq f(k)$.

Some examples of kernels

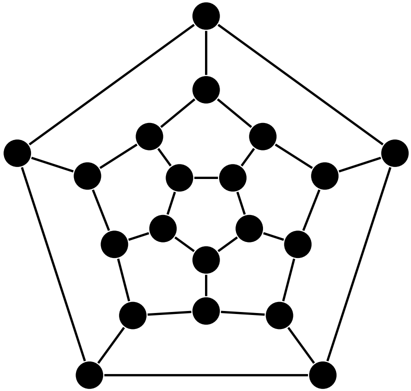
General graphs:

- Vertex Cover $2k$,
- Feedback Vertex Set $O(k^2)$,
- Odd Cycle Transversal $k^{O(1)}$,
- ...

Planar graphs:

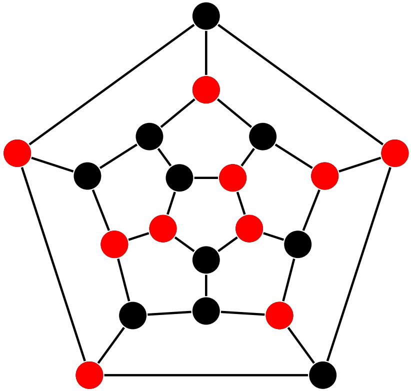
- Dominating Set $335k \rightarrow 67k$,
- Feedback Vertex Set $112k \rightarrow 97k$,
- Induced Matching $40k \rightarrow 28k$,
- Connected Vertex Cover $14k \rightarrow \frac{11}{3}k$,
- ...

Dominating Set



Dominating Set

C is a dominating set (●) in graph $G = (V, E)$ when every vertex $v \in V$ has **at least one neighbor** in C.



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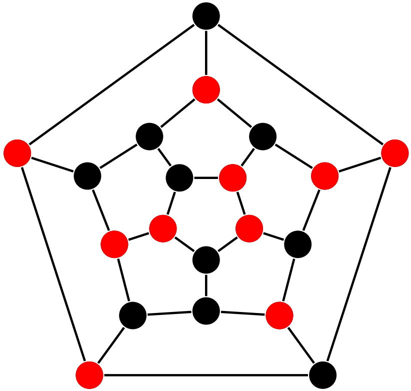
DOMINATING SET

INSTANCE: $G = (V, E)$, $k \in \mathbb{N}$

PARAMETER: $k \in \mathbb{N}$

QUESTION: Does G contain a dominating set of size k?



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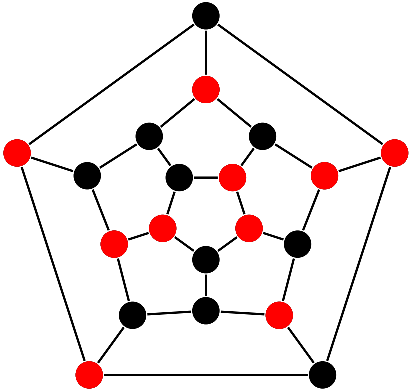

Dominating Set

Instance: $G = (V, E)$, $k \in \mathbb{N}$
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Nonblocker

Instance: $G = (V, E)$, $k \in \mathbb{N}$
Parameter: $k \in \mathbb{N}$
Question: Does G contain a dominating set of size $n - k$?

Parametric Duality

Dominating Set

Instance: $G = (V, E)$, $k \in \mathbb{N}$

Question: Does G contain a dominating set of size k?

Nonblocker

Instance: Graph $G = (V, E)$, $k \in \mathbb{N}$

Question: Does G contain a dominating set of size $n - k$?

- We can treat Nonblocker as Dominating Set with $|V| - k$ as a parameter.
- Then we say that Nonblocker is a **parametric dual** of Dominating Set.
- Other pairs of parametric duals: Vertex Cover and Independent Set, Max Leaf and Connected Dominating Set, ...
- **Note:** a small kernel for one problem does not give a small kernel for another.

General graphs (NONBLOCKER)

- $(\frac{5}{3}k + 3)$-kernel for general graphs,
- **but** the kernelization procedure does not preserve planarity.

Planar graphs (PLANAR NONBLOCKER)

A (trivial) $2k$-kernel for planar graphs:

- while there is an isolated vertex, remove it and decrease k by one.
- if $k \leq |V|/2$, i.e. $|V| - k \geq |V|/2$, answer YES: pick a spanning forest, 2-color it, choose the larger color class.
- Otherwise $k > |V|/2$, so $|V| < 2k$ and G is a kernel.

Our results

Main Result

There is a $\frac{7}{4}k$-kernel for PLANAR NONBLOCKER.

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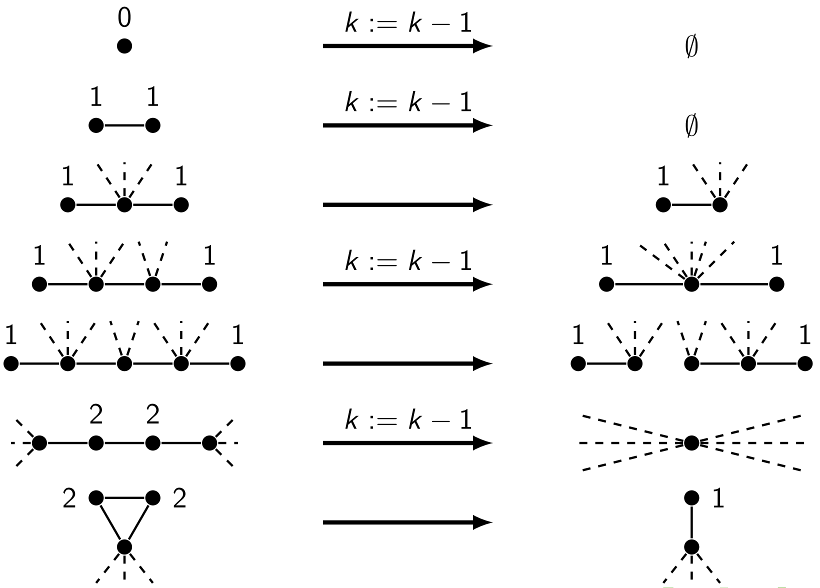
Note

In the above results planar graphs can be replaced by any H-minor-free graph family (without changing the constants).

Our (planarity preserving) rules



Kernel bound

Graph after applying our rules

- No isolated vertices,
- Every pair of degree 1 vertices is at distance at least 5,
- Every pair of degree 2 vertices is at distance at least 2.

Key Theorem

Every graph as above has a dominating set of size at least $3/7|V|$.

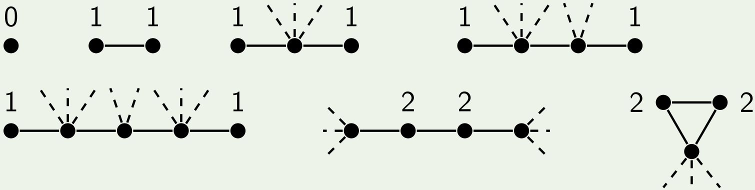
Final Rule

Let (G', k') be the instance after applying our rules.
If $k' \leq 4/7|V(G')|$ answer YES.
(Otherwise, $|V(G')| \leq 7/4k' \leq 7/4k$, so we've got a 7/4-kernel.)

Key theorem: a familiar scheme

Key theorem

In a graph without any of **reducible configurations**:

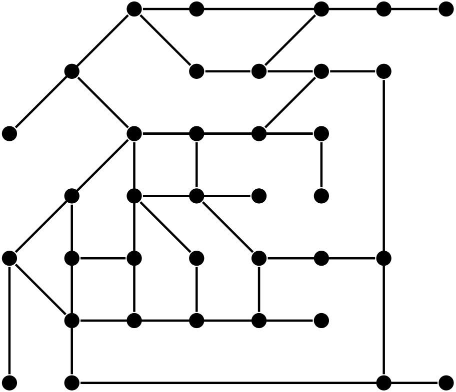


there is a dominating set of size $\geq \frac{3}{7}n$.

Proof of Four Color Theorem

Let G be an internally 6-connected triangulation.
Assign a charge: $ch(v) = \deg(v) - 6$. By Euler Formula, $\sum_v ch(v) < 0$.
If G does not contain any of (... 633 reducible configurations ...) then we can redistribute the charge so that for every $v \in V$ we have $ch(v) \geq 0$ (a contradiction).

Dominating set and discharging



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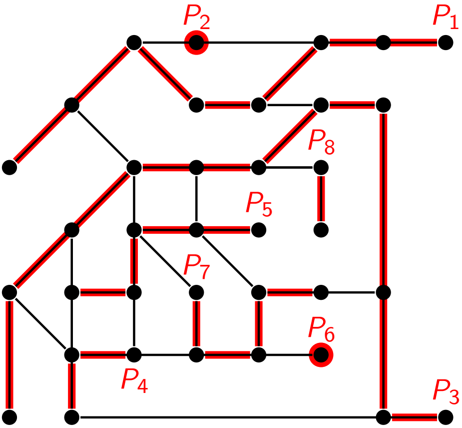

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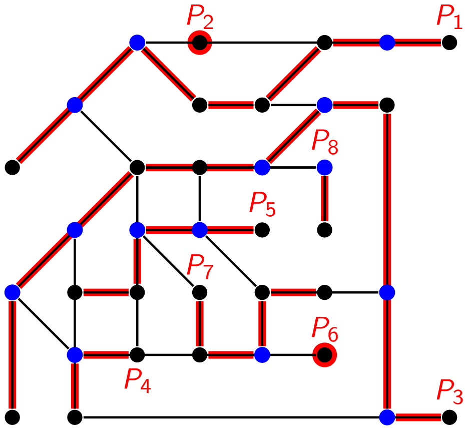

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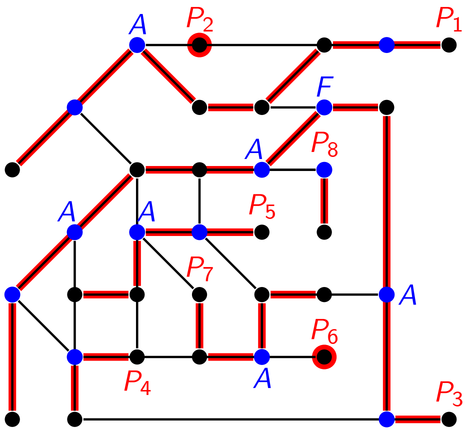

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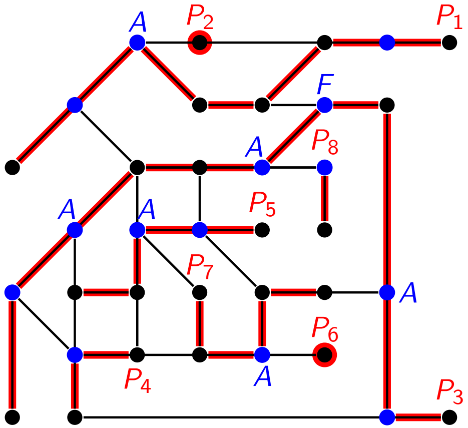

$$P_1 : \frac{3}{8} < \frac{3}{7} \quad P_2 : 0 \quad P_3 : \frac{6}{11} > \frac{3}{7} \quad P_4 : \frac{1}{3} < \frac{3}{7}$$
$$P_5 : \frac{2}{5} < \frac{3}{7} \quad P_6 : 0 \quad P_7 : \frac{1}{5} < \frac{3}{7} \quad P_8 : \frac{1}{2} > \frac{3}{7}$$

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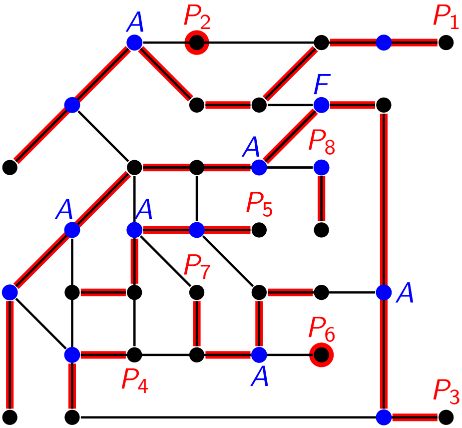

Idea

- Obs: some paths are overloaded, some are underloaded.
- Introduce rules which move charge between paths, so that for each path P_i, $|P_i \cap D| + \text{ch}(P_i) \leq \frac{3}{7}$
- Since $\sum_i \text{ch}(P_i) = 0$ we have:

$$|D| = \sum_i (|P_i \cap D| + \text{ch}(P_i)) \leq \tfrac{3}{7}|V|.$$

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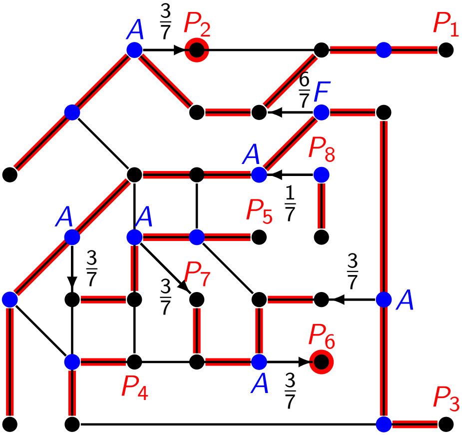

Discharging rules (simplified)

(D1) (...) Every $w \in A$ sends $\frac{3}{7}$ to every vertex it accepts.

(D2) (...)

(D3) Every 2-vertex path with a degree 1 endpoint sends $\frac{1}{7}$ to every neighboring path.

(D4) Every $w \in F$ sends $\frac{6}{7}$ to every neighboring weak path.

$P_1 : \frac{3-\frac{3}{7}+\frac{6}{7}}{8} = \frac{9}{28} < \frac{3}{7}$ $\quad P_2 : \frac{3}{7}$ $\quad P_3 : \frac{6-2\times\frac{3}{7}+\frac{1}{7}-\frac{6}{7}}{11} = \frac{31}{77} < \frac{3}{7}$ $\quad P_4 : \frac{1}{3} < \frac{3}{7}$

$P_5 : \frac{2-\frac{3}{7}+\frac{3}{7}}{5} = \frac{2}{5} < \frac{3}{7}$ $\quad P_6 : \frac{3}{7}$ $\quad P_7 : \frac{1+2\times\frac{3}{7}-\frac{3}{7}}{5} = \frac{10}{35} < \frac{3}{7}$ $\quad P_8 : \frac{1-\frac{1}{7}}{2} = \frac{3}{7}$

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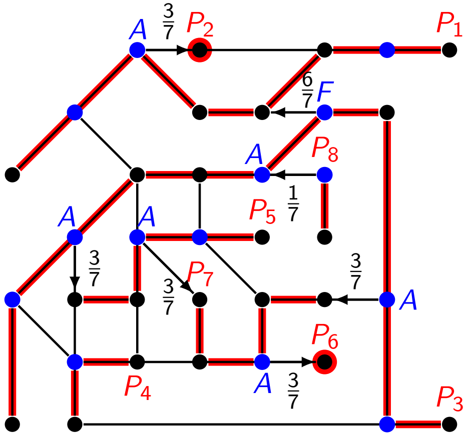

Note

- The most technical part of the proof: showing that for every i, $|P_i \cap D| + \mathrm{ch}(P_i) \leq \frac{3}{7}$.
- To prove this we use (among other arguments) the nonexistence of the reducible configurations
- E.g., P_3 would be overloaded if more paths like P_8 were attached to it (each sends $\frac{1}{7}$).

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Conclusions

- Observation: as an effect of many kernelization algorithms we get a set of reducible configurations.
- Approach: bound a global parameter (size of the dominating set) by analyzing **local** structures.
- It may happen that the parameter is locally bad.
- Using the reducible configurations we can show that locally bad structures are surrounded by locally good structures.
- A convenient way of proving that globally we get a good bound: **discharging**.

Further work

- Improve kernels for other problems using the discharging approach.
- In particular: improve the $\frac{5}{3}k$-kernel for Nonblocker in general graphs.
- Can we use planarity to improve the bound from this work?

Thank you for your attention!