An improved bound on the largest induced forests for triangle-free planar graphs *

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Abstract

We proved that every planar triangle-free graph of order n has a subset of vertices that induces a forest of size at least $(71n + 72)/128$. This improves the earlier work of Salavatipour [10]. We also pose some questions regarding planar graphs of higher girth.

1 Introduction

The maximum size of acyclic induced subgraphs is studied in several different ways. If only connected subgraphs are considered, the problem is to find the order of maximum induced tree of a graph G, denoted by $t(G)$. The problem was initiated by Erdős, Saks, and Sós in 1986 [5]. Some latest results are due to Matoušek and Šámal [8], and also due to Fox, Loh, and Sudakov [6].

On the other hand, if the maximum acyclic induced subgraph is not necessarily connected, the task is to find the maximum induced forest. There are two equivalent approaches to obtain the maximum forest of a graph. The former is determining the *decycling number* $\nabla(G)$ of a graph G, which is the least number of vertices whose deletion results in an induced forest. In [7] it was shown that determining this invariant is NP-hard even for planar graphs. An interested reader can find more results on the decycling number in a survey of Punnim [9].

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The latter approach is finding a maximum set S of vertices of graph G such that the graph $G[S]$ induced on S is a forest. The size of such a set S is denoted by $a(G)$ and it is referred to as a *forest number*. Note that $a(G) + \nabla(G) = |V(G)|$. We call the ratio between the forest number and the order of a graph a *forest ratio* and denote it by $\gamma(G)$. Large induced forests in graphs recently attracted attention in various graph classes. In 1979, Albertson and Berman [2] raised a conjecture regarding planar graphs and initiated the study of this topic:

Conjecture 1 (Albertson & Berman) *Every planar graph has an induced forest on at least half of its vertices.*

Notice that the conjecture implies that every planar graph of order n has an independent set of size at least $\frac{n}{4}$. This fact is known to be true only as a consequence of the Four Color Theorem. In 1987, Akiyama and Watanabe [1] posed a similar conjecture on bipartite planar graphs:

Conjecture 2 (Akiyama & Watanabe) *Every bipartite planar graph has an induced forest on at least $\frac{5}{8}$ of its vertices.*

Note that Conjectures 1 and 2, if true, are sharp by K_4 and Q_3. Motivated by Conjecture 2, Alon [3] proved the following result for sparse bipartite graphs:

Theorem 1 *There exists an absolute positive constant b such that for every bipartite graph G of order n and average degree at most d holds the inequality*

$$a(G) \geq (\frac{1}{2} + e^{-bd^2})\, n\,.$$

Additionally, Alon proved that the exponential dependence on d cannot be replaced by a polynomial one. In [4], Alon, Mubayi, and Thomas proved a result for triangle-free graphs:

Theorem 2 *Let G be a subcubic triangle-free graph of order n. Then $a(G) \geq \frac{5}{8}n$, and the bound is sharp whenever n is divisible by 8.*

Furthermore, Alon, Mubayi, and Thomas proved that for every triangle-free graph G of order n and size m the forest number is at least $n - \frac{m}{4}$. For planar triangle-free graphs this bound implies that the forest number is at least $\frac{n}{2} + 1$ due to Euler's formula. Salavatipour [10] improved the bound for planar graphs and showed that every triangle-free planar graph of order n has the forest number $a(G) \geq \lceil \frac{17n+24}{32} \rceil$. Using his approach, we improve this bound and prove the following theorem:

Theorem 3 *Let G be a planar graph of order n and size m with girth $g(G) \geq 4$. Then, $a(G) \geq \frac{119n-24m-24}{128}$.*

From Theorem 3 and the fact that there is at most $2n - 4$ edges in planar graphs of order n and girth at least 4, the following corollary immediatelly follows.

Corollary 4 *Let G be a planar graph of order n and girth $g(G) \geq 4$. Then, $a(G) \geq \lceil \frac{71n+72}{128} \rceil$.*

As mentioned above, the investigation of triangle-free graphs was motivated by Conjecture 2. However, one could also ask what is the forest number of graphs with girth at least 5. We pose the following problem:

Conjecture 3 *For every planar graph of order n and girth at least 5, the forest number $a(G)$ is at least $\frac{7}{10}n$.*

The conjecture, if true, is sharp by the dodecahedron, and it was inspired by the fact that the dodecahedron has the minimal edge to vertex ratio among all connected graphs of girth at least 5. By this fact, it is natural to ask:

Question 1 *Is the dodecahedron the only connected graph of girth at least five with forest ratio $\frac{7}{10}$?*

Now, we pose two simple observations regarding vertices of degree 2.

Proposition 5 *For every connected graph G distinct from a cycle, there exists a maximum induced forest which contains every vertex of degree 2 in G.*

Proof. Let F be a maximum induced forest in a graph G with as many vertices of degree 2 as possible. Suppose v is a vertex of degree 2 which is not in F. Obviously, v is an element of a cycle, since F is maximum. Let e_1 and e_2 be the edges incident with v. Let u_1 be a vertex of degree at least 3 such that the shortest path p_1 between v and u_1 contains e_1 and u_1 is the only vertex of degree at least 3 on p_1. Note that such u_1 is unique and it always exists, since G is not a cycle. Similarly, we define u_2 and the shortest path p_2 which should contain e_2. Notice that $u_1 = u_2$ is possible, but then v belongs to a cycle where $u_1 (= u_2)$ is the only vertex of degree at least 3.

Now, if $u_1 \neq u_2$, they are both contained in F, otherwise v could be added to F. In case when $u_1 = u_2$, similarly u_1 is in F. We now replace u_1 by v in F and obtain a contradiction to the choice of F. □

Before stating the next observation, we define graph G^* obtained from the graph G by contracting all 2-vertices. Notice that G^* may have parallel edges and loops.

Proposition 6 *For any graph G with n_2 vertices of degree 2 the following equality holds:*

$$a(G) = a(G^*) + n_2 \,.$$

Proof. First, we prove that $a(G) \leq a(G^*) + n_2$. Let V_2 be the set of all vertices of degree 2 in G, so $|V_2| = n_2$. By Proposition 5 there exists a maximum induced forest F in G that contains all vertices from V_2. Obviously, $F - V_2$ is an induced (not necessarily maximum) forest in G^*.

Now, we prove that $a(G) \geq a(G^*) + n_2$. Let F^* be the maximum induced forest in G^*. Obviously, F^* is an induced forest in G. Next, observe that by adding vertices from V_2 we do not introduce any cycles, thus $F^* \cup V_2$ is also an induced forest in G. □

The above observations imply that if the dodecahedron is the graph with the smallest forest ratio, it follows that when considering graphs of higher girth, the graphs with the smallest forest ratio are dodecahedra with some edges subdivided. In particular, we can easily state such graphs for girth 6, 7, and 8. Let M be the minimum set of edges in dodecahedron D, such that each face is incident with an edge in M. Note that $|M| = 6$. We define D_k to be a graph obtained by subdividing each edge in M by k-vertices. It is easy to see that D_k has girth $5 + k$ for $k \in \{1, 2, 3\}$, and $\gamma(D_k) = \frac{7+3k}{10+3k}$.

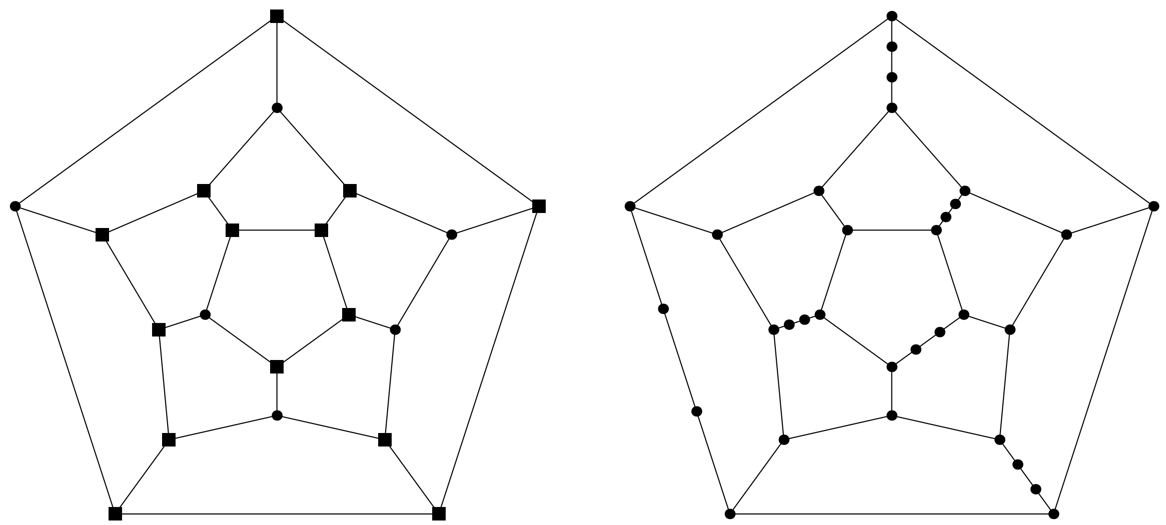

Figure 1: The dodecahedron with 14 square vertices that induce a forest, and the graph D_2.

Observe that the maximum induced forest in D_k contains all the vertices of degree 2 and the vertices which form the induced forest of the dodecahedron.

2 Proof of Theorem 3

In the proof, we mostly follow the notation from [10]. We call a vertex of degree k a *k-vertex*, and a neighbor of degree k of vertex v a *k-neighbor* of v. For a given cycle C in a plane embedding of graph G we define $\mathrm{int}(C)$ to be the graph induced by the vertices lying strictly in the interior of C. Similarly, $\mathrm{ext}(C)$ is the graph induced by the vertices lying strictly in the exterior of C. A *separating* cycle is a cycle C such that $\mathrm{int}(C) \neq \emptyset$ and $\mathrm{ext}(C) \neq \emptyset$.

We will prove Theorem 3 by contradiction, i.e. we will prove that the minimal counterexample G does not exist. It is easy to see that G is connected. In what follows, we prove several lemmas, which determine some structure of G. Note that by G we always mean a plane embedding of G, moreover n, m, and f are the order, size and the number of faces in G, respectively.

Lemma 7 *Graph G does not contain a bridge, i.e. it is 2-edge-connected.*

Proof. Assume G contains a bridge uv. Let G_u and G_v be the connected components that contain u and v, respectively, in $G - uv$. By the minimality of G, there is a set of vertices R_u in G_u and R_v in G_v, respectively, that induces a forest of size at least $\frac{119|V(G_u)|-24|E(G_u)|-24}{128}$ and $\frac{119|V(G_v)|-24|E(G_v)|-24}{128}$, respectively. Then $R_u \cup R_v$ induces a forest in G of size at least $\frac{119n-24(m-1)-48}{128} = \varphi$, which is a contradiction. $\square$

Lemma 8 *The maximum degree of G is at most 4, i.e. $\Delta(G) \leq 4$.*

Proof. Let v be a ≥ 5-vertex of G. By minimality of G, there is an induced forest of size at least $\frac{119(n-1)-24(m-5)-24}{128}$ in $G - v$, which is the required size φ for an induced forest of G. $\square$

Lemma 9 *The minimum degree of G is at least 3, i.e., $\delta(G) \geq 3$.*

Proof. Suppose that the lemma is false and that G has 2-vertices. First, we claim that *no 2-vertex is adjacent to a 4-vertex.* Let u be a 4-neighbor of a 2-vertex v. By minimality of G, we have that $G - \{u, v\}$ has an induced forest of size at least $\frac{119(n-2)-24(m-5)-24}{128} \geq \varphi - 1$ induced by a set R'. Then $R' \cup \{v\}$ induces a forest of size at least φ in G.

We now claim that *no 2-vertex has both a 2-neighbor and a 3-neighbor.* Let u be a 2-neighbor, and let w be a 3-neighbor of a 2-vertex v. Let R' be a subset of vertices in $G - \{u, v, w\}$ that induces a forest of size $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$. Then $R' \cup \{u, v\}$ induces a forest of size at least φ in G.

By the above two claims, we obtain that G is either a cycle or it does not contain a pair of adjacent 2-vertices. If G is a cycle, it has an induced tree of size $n - 1 \geq \varphi$. Hence, we conclude that *both neighbors of a 2-vertex in G are of degree 3.*

Now, we claim that *every 3-vertex in G has at most one 2-neighbor.* Let us consider a 3-vertex w adjacent to 2-vertices u and v. Let R' be a subset of vertices in $G - \{u, v, w\}$ that induces a forest of size at least $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$. By introducing vertices u and v to R', we obtain an induced forest of size at least φ in G, which establishes the claim.

Let v be one of the 2-vertices of G. By the third observation above, it has two 3-neighbors, say u and w. We will consider few possibilities regarding the number of their common neighbors, and each time we will obtain a contradiction. This will establish the lemma.

- *u and w have one common neighbor, namely v.* Let $G' = G + uw - v$. Observe that G' has girth at least 4, since the only 2-path between u and w in G contains v. By the minimality of G, there is a subset R' that induces a forest of size at least $\frac{119(n-1)-24(m-1)-24}{128} \geq \varphi - 1$ in G'. So, $R' \cup \{v\}$ induces a forest of size at least φ in G. Notice that adding v to the forest does not introduce a cycle, since u and w were adjacent in G', i.e. if $u, w \in R'$ then an edge in the forest is subdivided, and if at most one of u, w was in the forest, then v is a leaf or an isolated vertex.

- *u and w have precisely two common neighbors.* Let these two neighbors be v and z, and let x be the third neighbor of u. By the last claim, we know that z and x have degree at least 3. Since x and z are non-adjacent by the girth assumption, there are at most $m-9$ edges in $G-\{u,v,w,x,z\}$. By the induction, it has a subset R' of vertices that induces a forest of size at least $\frac{119(n-5)-24(m-9)-24}{128} \geq \varphi - 3$, and so, $R' \cup \{u,v,w\}$ induces a forest of size at least φ in G.

- *u and w have three common neighbors.* Let these three vertices be v, z, and x. Notice again that z and x are not adjacent due to the girth assumption. If one of z and x is a 4-vertex, similarly as above, we obtain an induced forest of size at least $\frac{119(n-5)-24(m-9)-24}{128} \geq \varphi - 3$ in $G-\{u,v,w,x,z\}$, and by adding vertices u, v, and w to the forest, we obtain a forest of size at least φ in G. So, we can assume that z and x are 3-vertices.

  Now, let y be the neighbor of z distinct from u and w (note that $x \neq y$). If y is a 2-neighbor of x, then G is a graph on six vertices, and the vertices u, v, x, z induce a forest of size $4 > \varphi = \frac{119 \cdot 6 - 24 \cdot 8 - 24}{128} = \frac{498}{128}$. However, if y is a $\geq$3-vertex, not necessarily adjacent to x, then there is a subset of vertices R' in $G-\{u,v,w,x,y,z\}$ that induces a forest of size at least $\frac{119(n-6)-24(m-9)-24}{128} \geq \varphi - 4$, and hence $R' \cup \{u,v,x,z\}$ induces a forest of size $\geq \varphi$ in G. Finally, if y is a 2-vertex not adjacent to x, then there are at most $m-9$ edges in $G' = G-\{u,v,w,x,y,z\}$, and G' has a forest F' of size $\varphi - 4$. So, by adding the vertices u, v, x, z to F' we infer an induced forest of size at least φ in G. $\square$

Lemma 10 *Let v be a 3-vertex adjacent to a 4-vertex u. Then the other two neighbors of v have a common neighbor distinct from v.*

Proof. Let w and z be the other two neighbors of v. Suppose that v is the only neighbor of w and z. Consider the graph $G' = G + wz - \{u,v\}$. Note that G' has girth at least 4. By minimality of G, there is a subset of vertices R' in G' that induce a forest of size at least $\frac{119(n-2)-24(m-5)-24}{128} \geq \varphi - 1$ in G', thus $R' \cup \{v\}$ induces a forest of size at least φ in G. $\square$

In the following two lemmas the indices are considered modulo 4.

Lemma 11 *G does not contain a 4-cycle $C = v_0v_1v_2v_3$ which has at least two 4-vertices and a 3-vertex v_i such that*

(a) *v_{i+2} is a 3-vertex; or*

(b) *v_{i+2} is connected to both $\mathrm{int}(C)$ and $\mathrm{ext}(C)$.*

Proof. By minimality of G, there is a set R' of size at least $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$ which induces a forest F' in $G - V(C)$. Note that v_i has at most one neighbor in R' and v_{i+2} has either at most one neighbor in R' (when $\deg(v_{i+2}) = 3$) or it is connected to two distinct trees in F' (when $\deg(v_{i+2}) = 4$). It follows that $R' \cup \{v_i, v_{i+2}\}$ induces a forest of size at least φ in G. $\square$

Lemma 12 *There is no separating 4-cycle $C = v_0v_1v_2v_3$ which has* (*a*) *at least two* 3*-vertices; or*

(*b*) *precisely one* 3*-vertex* v_i *and precisely one neighbor of* v_{i+2} *in* $\mathrm{int}(C)$.

Proof. Let $C = v_0v_1v_2v_3$ be a separating 4-cycle. Note that (*b*) follows from Lemma 11. We split the proof of (*a*) in several cases regarding the number of 3-vertices of C.

Case 1: *C has at least three* 3*-vertices.* Let v_0, v_1, and v_2 be three such vertices.

Suppose first that vertices v_0, v_1, and v_2 are all connected to one of $\mathrm{ext}(C)$ and $\mathrm{int}(C)$, say $\mathrm{int}(C)$. Then by 2-edge-connectivity of G and the fact that C is a separating cycle, we have that v_3 is a 4-vertex connected to $\mathrm{ext}(C)$ with two edges. Let u_0, u_1, and u_2 be the neighbors of v_0, v_1, and v_2 from $\mathrm{int}(C)$, respectively. By Lemma 10 we infer that $u_0u_1, u_2u_1 \in E(G)$, and from that, by the girth assumption, $u_0u_2 \notin E(G)$. Notice that u_0 and u_2 may coincide. In that case, the graph $G' = G - V(C) - u_0$ has an induced forest F of size at least $\frac{119(n-5)-24(m-10)-24}{128} \geq \varphi - 3$, so by adding v_0, v_1, v_2 to F, we obtain an induced forest of size at least φ in G. Hence, we may assume $u_0 \neq u_2$.

If all three vertices u_0, u_1, u_2 are of degree 3, then we add u_0, u_1, and v_0 to an induced forest F' in $G - \{v_0, v_1, u_0, u_1, u_2\}$ of size at least $\frac{119(n-5)-24(m-10)-24}{128} \geq \varphi - 3$. So the induced forest in G is of size at least φ. On the other hand, if at least one of these three neighbors is of degree 4, the induced forest in G is obtained from a forest in the graph $G - V(C) - \{u_0, u_1, u_2\}$ of size at least $\frac{119(n-7)-24(m-14)-24}{128} \geq \varphi - 4$ by introducing vertices v_0, v_1, v_2, and u_1.

Suppose now that not all of v_0, v_1, and v_2 are connected to $\mathrm{ext}(C)$ or to $\mathrm{int}(C)$. Without loss of generality, two vertices from $\{v_0, v_1, v_2\}$ are connected to $\mathrm{int}(C)$ and the third one is connected to $\mathrm{ext}(C)$. By symmetry, we can assume v_0 is connected to $\mathrm{int}(C)$, and just one of v_1, v_2 with $\mathrm{ext}(C)$. Let u be a neighbor of v_0 in $\mathrm{int}(C)$. There is an induced forest F' of size at least $\frac{119(n-5)-24(m-10)-24}{128} \geq \varphi - 3$ in the graph $G - V(C) - u$. By adding v_0, v_1, and v_2 to F', we obtain an induced forest F of size at least φ in G. Note that F is acyclic, since the path $v_0v_1v_2$ is connected to F' by at most two edges which are not incident with the same tree in F'.

Case 2: *C has exactly two* 3*-vertices.* Note that by Lemma 11 we can assume that the two 3-vertices of C are consecutive, say v_0 and v_1. Let u_0 be the neighbor of v_0 distinct from v_1 and v_3. By symmetry we can assume that $u_0 \in \mathrm{ext}(C)$ for otherwise one can change the plane embedding of G. By Lemma 10, the vertices v_1 and u_0 have another common neighbor u_1, beside the vertex v_0.

Observe that by Lemma 11 and the fact that C is a separating cycle, at least one of the vertices v_2 and v_3 has two neighbors in $\mathrm{int}(C)$. By symmetry, we can assume that v_2 has two neighbors in $\mathrm{int}(C)$. By Lemma 11, v_3 has either both neighbors in $\mathrm{int}(C)$ or both in $\mathrm{ext}(C)$. In the former case consider the graph $G - \{v_0, v_1, v_3, u_0\}$. It has at most $m - 10$ edges, so it has an induced a forest F' of size at least $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$. By adding the vertices v_0 and v_1 to F, we obtain an induced forest of size at least φ in G. It is easy to see that no cycle is introduced, since $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ are not connected. It follows that v_3 is connected to $\mathrm{ext}(C)$ with two edges.

Now we note that if v_3 is adjacent with u_1 then $\deg(u_1) = 4$ for otherwise $u_1v_1v_0v_3$ is a separating 4-cycle with three 3-vertices and we can proceed as in Case 1. It follows that the set of vertices $S = \{u_1, v_1, v_0, v_3\}$ is always incident with at least 10 edges. Hence in the graph $G' = G - S$ there is a subset of vertices R' which induces a forest of size at least $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$. In graph G' there is no path from v_2 to u_0 since in G vertex v_2 has two neighbors in $\mathrm{int}(C)$. It follows that $R' \cup \{v_0, v_1\}$ induces a forest of size at least φ in G, even if both v_2 and u_0 are in R'. □

Lemma 13 *The graph G has no 4-face with four 3-vertices.*

Proof. Let $C = v_0v_1v_2v_3$ be a 4-face in G incident only with 3-vertices. If v_0 and v_2 have a common neighbor u in $G - V(C)$, then we have a separating 4-cycle $v_0v_1v_2u$ with at least three 3-vertices, which is a contradiction by Lemma 12. A similar argument applies if v_1 and v_3 have a common neighbor. Thus, each vertex of C has a distinct neighbor in $G - V(C)$. Let u_0, u_1, u_2, and u_3 be the third neighbors of v_0, v_1, v_2, and v_3, respectively. As we argued, u_0, u_1, u_2, and u_3 are pairwise distinct. Note also that by the planarity, at most one of u_0u_2 and u_1u_3 is in G.

Suppose first that at least two consecutive edges, v_iv_{i+1} and $v_{i+1}v_{i+2}$, of C are incident to ≥ 5-faces (indices are considered modulo 4). Note that in case, when u_i and u_{i+2} are adjacent, at least one of $v_{i+2}v_{i+3}$ and $v_{i+3}v_i$ is incident to a ≥ 5-face (and u_{i+1}, u_{i+3} are non-adjacent as stated above), due to the girth assumption. Hence, there always exist edges v_jv_{j+1}, $v_{j+1}v_{j+2}$ incident to ≥ 5-faces such that u_j and u_{j+2} are non-adjacent. Say v_0v_1 and v_1v_2 are such edges. Now, consider the graph G' obtained from G by removing vertices of C, and adding a new vertex x. As u_0 and u_2 are non-adjacent, let x be adjacent to u_0, u_1, and u_2, i.e. $G' = G \cup \{x, xu_0, xu_1, xu_2\} - V(C)$. The resulting graph G' has girth at least 4, since u_0, u_1, and u_2 are pairwise non-adjacent by Lemma 12. By minimality of G, there is a vertex set R' that induces a forest of size at least $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$ in G'. If $x \notin R'$, then $R' \cup \{v_1, v_3\}$ induces a forest of size at least φ in G. On the other hand, if $x \in R'$, we consider the vertex set $R' \setminus \{x\} \cup \{v_0, v_1, v_2\}$, inducing a forest of size at least φ in G.

Now, we may assume that C has at most two edges incident to $\geq$5-faces, and they are non-consecutive on C. We will consider several cases regarding the number of 3-vertices in $U = \{u_0, u_1, u_2, u_3\}$. Note that any two vertices of U incident with the same $\geq$5-face, which is also incident with C, are non-adjacent due to Lemma 12.

Case 1: *All the vertices in U are of degree 3 and C is incident with at least three 4-faces.* In case when C is incident with four 4-faces, G is the cube, so there is an induced forest of size $\varphi = \frac{119 \cdot 8 - 24 \cdot 12 - 24}{128} = 5$.

Suppose now that C is incident to precisely three 4-faces, and let u_0 and u_1 be the two vertices incident with the only ≥ 5-face. Since G is not a cube, there exists a vertex x adjacent to u_0 and distinct from v_0, u_1, u_3. By the minimality, there is a subset of vertices R' in $G - V(C) - \{u_0, u_1, u_2, u_3, x\}$ which induces a forest in G' of size at least $\frac{119(n-9)-24(m-14)-24}{128} \geq \varphi - 6$. Consider the set $R = R' \cup \{u_0, u_1, u_2, v_0, v_1, v_3\}$. The tree induced by $\{u_0, u_1, u_2, v_0, v_1, v_3\}$ might be connected with R' only by one edge that is incident with u_1. So, R induces a forest of size at least φ in G.

Case 2: *U has one 4-vertex and C is incident with at least three 4-faces.* First, note that C cannot be incident with four 4-faces due to 2-edge-connectivity, so we may assume that C is incident to precisely one ≥ 5-face f. Then U must have a 3-vertex x which is incident to two of the three 4-faces, and which has a 4-neighbor y in U. We may assume $x = u_0$. Observe that u_2 is a 3-vertex. Also note that the two vertices of U incident to f are not adjacent by Lemma 12. Now, by minimality of G there is a set of vertices R' in $G - V(C) - \{u_0, u_1, u_2, u_3\}$ that induces a forest of size at least $\frac{119(n-8)-24(m-14)-24}{128} \geq \varphi - 5$. The set $R' \cup \{u_0, u_2, v_0, v_1, v_3\}$ induces a forest of size at least φ in G. Obviously, no cycles are introduced, since the vertices u_0, v_0, v_1, v_3 induce a tree which is not connected to R' and u_2 has at most one neighbor in R'.

Case 3: *U has at most one 4-vertex and C is incident with precisely two 4-faces.* Recall that these two faces are not consecutive around C, so we may assume the 4-faces incident with C are bounded by the cycles $v_0u_0u_1v_1$ and $v_2u_2u_3v_3$. Note that $u_0u_2, u_1u_3 \notin E(G)$ due to 2-edge-connectivity of G. By symmetry, we may also assume that the possible 4-vertex in U is u_1. Consider the graph $G - V(C) - \{u_0, u_1, u_2, u_3\}$ and its vertex set R' that induces the forest F' of size at least $\frac{119(n-8)-24(m-14)-24}{128} \geq \varphi - 5$. The set of vertices $R' \cup \{u_0, u_2, v_0, v_1, v_3\}$ induces a forest of size at least φ in G.

Case 4: *U has two 4-vertices.* If C is incident to four 4-faces, then there is a separating 4-cycle $u_0u_1u_2u_3$, which is reducible by Lemma 12.

Suppose now C is incident with precisely one ≥ 5-face f. We consider two possibilities. First, let both 4-vertices of U be incident to f. We may assume that these two vertices are u_0 and u_1. Note that u_0 and u_1 are non-adjacent by Lemma 12. Again, there exists an induced forest of size at least $\frac{119(n-8)-24(m-15)-24}{128} \geq \varphi - 5$ in $G - V(C) - \{u_0, u_1, u_2, u_3\}$. It is easy to see that inserting the vertices u_2, u_3, v_0, v_1, and v_3 into the forest does not introduce any cycles, so we obtain a forest of size at least φ in G. Thus we can assume that at least one 3-vertex from U is incident to f, say u_0. There exists a vertex set R' that induces a forest of size at least $\frac{119(n-7)-24(m-14)-24}{128} \geq \varphi - 4$ in $G - V(C) - \{u_1, u_2, u_3\}$. By inserting vertices u_2, v_1, v_2, and v_3 to R' we obtain an induced forest of size at least φ in G.

Finally, suppose C is incident with two (non-consecutive) 4-faces. Again, we may assume that u_0 and u_1 are incident with the same ≥ 5-face, and u_0 is a 3-vertex. Let R' be the vertex set that induces a forest in $G' = G - V(C) - \{u_0, u_1, u_3\}$. By the minimality, this set is of size at least $\frac{119(n-7)-24(m-14)-24}{128} \geq \varphi - 4$ unless u_1 and u_3 are adjacent. But in the exceptional case, u_0 and u_2 are not adjacent by the planarity, so we redefine $G' = G - V(C) - \{u_0, u_1, u_2\}$ to obtain an induced forest of size at least $\frac{119(n-7)-24(m-14)-24}{128} \geq \varphi - 4$. Now, we add vertices u_0, v_0, v_1, and v_3 (resp. u_1, v_0, v_1, and v_2) to R'. We obtain an induced forest in G of size at least φ.

Case 5: *U has three 4-vertices.* By Lemma 12, we infer that there are no four 4-faces incident with C, since $u_0u_1u_2u_3$ is a separating 4-cycle with one 3-vertex and its opposite vertex is adjacent with the internal and external component.

Suppose now that there are three 4-faces incident with C. By symmetry, assume u_0 is the 4-vertex not incident to a ≥ 5-face. Next, let R' be the vertex set which induces the forest of size at least $\frac{119(n-7)-24(m-14)-24}{128} \geq \varphi - 4$ in $G - V(C) - \{u_0, u_1, u_3\}$. To obtain an induced forest of size at least φ in G, just introduce the vertices u_0, v_0, v_1 and v_3 to

R'. Again, the introduced claw could be connected to the forest only by u_0, thus no cycle is introduced.

Finally, suppose that C is incident to two 4-faces. Here, let u_0 be the only 3-vertex in U. It is easy to see that by defining R' as in the case above and by adding the vertices u_0, v_0, v_1, and v_3 to R' we obtain the induced forest of size at least φ in G.

Case 6: *U has four 4-vertices.* Note first that by planarity if u_0 and u_2 are adjacent then u_1 and u_3 are non-adjacent. By symmetry, we may assume $u_0u_2 \notin E(G)$. Next, let R' be the vertex set that induces a forest of size at least $\frac{119(n-6)-24(m-14)-24}{128} \geq \varphi - 3$ in $G - V(C) - \{u_0, u_2\}$. It is easy to see that $R' \cup \{v_0, v_1, v_2\}$ induces a forest of size at least φ in G. □

Lemma 14 *The graph G has no 4-face with precisely two 3-vertices.*

Proof. Let the cycle $C = v_0v_1v_2v_3$ be a 4-face with precisely two 3-vertices. By Lemma 11 we obtain that the 3-vertices are adjacent, so we can assume v_0 and v_1 are of degree 3. Let u_0 be the third neighbor of v_0, and let u_1 be the third neighbor of v_1. By Lemma 10, we have that u_0 and u_1 are adjacent, moreover, Lemma 12 implies that the cycle $v_0v_1u_1u_0$ bounds a face and that $u_0v_2, u_1v_3 \notin E(G)$. Moreover, at least one of u_0 and u_1 is a 4-vertex, otherwise $v_0v_1u_1u_0$ is a 4-face with four 3-vertices, which is reducible by Lemma 13. By symmetry, we may assume that $\deg(u_0) = 4$.

If u_1 is a 3-vertex, there is a set of vertices R' of size at least $\frac{119(n-6)-24(m-14)-24}{128} \geq \varphi - 3$ in $G - V(C) - \{u_0, u_1\}$. The set $R' \cup \{u_1, v_1, v_0\}$ induces a forest of size at least φ in G. It is easy to see that no cycle is introduced by this set.

Finally, assume u_1 is a 4-vertex. Note that if u_0 and v_2 have a common neighbor, then by the planarity and the girth assumption, u_1 and v_3 do not have a common neighbor. By symmetry, we may assume that u_0 and v_2 have no common neighbor (and recall that u_0 and v_2 are non-adjacent). Then there is a subset of vertices R' in $G + u_0v_2 - \{u_1, v_0, v_1, v_3\}$ which induces a forest of size at least $\frac{119(n-4)-24(m-10)-24}{128} \geq \varphi - 2$. The vertex set $R' \cup \{v_0, v_1\}$ induces a forest of size at least φ in G; observe that by adding these two vertices no cycle is introduced, since if $\{u_0, v_2\} \subseteq R'$, we only subdivide an edge in the forest. □

Lemma 15 *The graph G has no 5-face incident only with 3-vertices.*

Proof. Assume $C = v_0v_1v_2v_3v_4$ is a 5-face incident only with 3-vertices. Let u_i be the third neighbor of v_i, where $i \in \{0, 1, 2, 3, 4\}$. By minimality of G, there is a subset of vertices R' in a graph G' obtained by removing C and adding the vertices x, y and the edges xu_0, xu_1, xy, yu_2, and yu_3 to G, i.e. $G' = G - V(C) + \{x, y, xu_0, xu_1, xy, yu_2, yu_3\}$, that induces a forest F' of size $\frac{119(n-3)-24(m-5)-24}{128} \geq \varphi - 2$. Note that by adding these edges we do not violate the girth assumption, since there is no 4-cycle with two 3-vertices in G by Lemma 11.

Now, we distinguish few possibilities regarding whether x and y are in R'. If none of them is in R', then adding v_0 and v_2 to F' does not introduce a cycle and the size of the resulting forest is at least φ in G. If precisely one of x and y is in R', say x, we need to add three vertices to F' to assure that its size is at least φ, since x is not a vertex of G.

However, as x is in the forest, adding v_0 and v_1 to $F' - x$ does not introduce a cycle. The third vertex, v_3, is connected to the forest with at most one edge.

Finally, if x and y are both in R', then we replace them with four vertices of C. Namely, we add vertices v_0, v_1, v_2, and v_3. It is easy to see that no cycle is introduced, so we obtain an induced forest of size at least φ in G. □

Lemma 16 *The graph G is not cubic.*

Proof. Suppose for a sake of contradiction that G is cubic. By the girth assumption and Lemmas 13 and 15, we infer that faces in G have length at least 6. On the other hand, Lemma 9 implies that the minimum degree in G is at least 3, so by Euler's formula G contains a face of length at most 5, a contradiction. □

Lemma 17 *Every 3-vertex of G has three 4-neighbors.*

Proof. Suppose the claim is false. By Lemma 16, G is not cubic, thus it has at least one 4-vertex. Therefore if there is any 3-vertex in G, there is also a 3-vertex v adjacent to at least one 4-vertex u. Let w and z be the other two neighbors of v. By Lemma 10, we know that w and z have a common neighbor x distinct from v. However, if w or z is a 3-vertex, a separating 4-cycle or a 4-face with two 3-vertices is introduced, but such configurations are reducible by Lemmas 12 and 14. □

Lemma 18 *The graph G does not contain 4-faces with precisely one 3-vertex.*

Proof. Suppose for a sake of contradiction that the cycle $C = v_0v_1v_2v_3$ is a 4-face in G with exactly one 3-vertex v_0. Let u_0 be the neighbor of v_0 distinct from v_1 and v_3. By Lemma 17, it follows that u_0 is a 4-vertex. Moreover, by Lemma 10, u_0 and v_1 have a common neighbor x distint from v_0, and also u_0 and v_3 have a common neighbor y distint from v_0. Lemmas 12 and 14 imply that $d(x) = d(y) = 4$. Now we show in few steps that the vertices of Fig. 2 are all pairwise distinct.

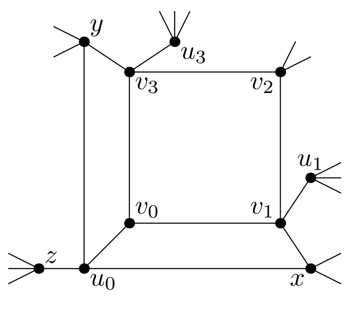

Figure 2: The neighborhood of vertices v_0, v_1, v_2, and v_3.

First we claim that $x \neq v_2$. Suppose contrary that $x = v_2$. Let u_2 be the neighbor of v_2 which is distinct from v_1, v_3, and u_0. Let C_1 be the separating 4-cycle $v_0v_3v_2u_0$. If u_2 is not in the same component of $G - V(C_1)$ as v_1, then we have a separating 4-cycle with one 3-vertex and its opposite vertex adjacent to two components. Such a cycle is reducible by Lemma 12. On the other hand, if u_2 is in the same component, consider instead the reducible separating 4-cycle $v_0v_1v_2u_0$, a contradiction. We similarly show that $y \neq v_2$.

Now, we claim that $x \neq y$. Again, suppose contrary that $x = y$. Let x_1 be the neighbor of x distinct from u_0, v_1, and v_3. Consider the 4-cycles $C_1 = xu_0v_0v_1$, $C_2 = xu_0v_0v_3$, and $C_3 = xv_3v_0v_1$. Note that among these three cycles we can always choose a cycle C_i such that x_1 and the neighbor of x not incident to C_i are in different parts $\text{int}(C_i)$ and $\text{ext}(C_i)$ (see Fig. 3). Hence, C_i is a separating cycle. Since C_1, C_2, and C_3 all contain a 3-vertex v_0, C_i is reducible by Lemma 12(b), a contradiction. This establishes the claim.

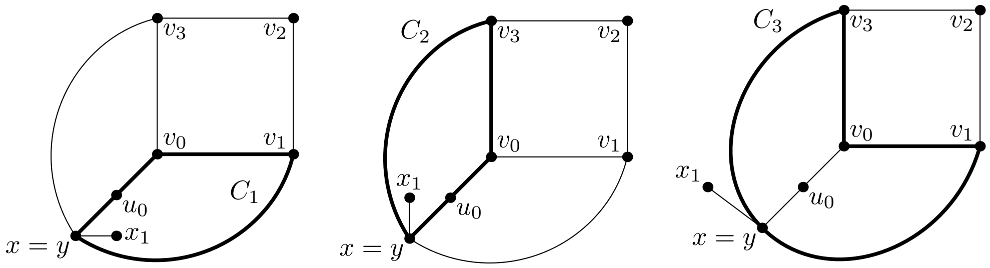

Figure 3: We can always find such a 4-cycle C_i that x_1 and the other neighbor of x not in incident with C_i are in different parts $\text{int}(C_i)$ and $\text{ext}(C_i)$.

Consider now the neighbor z of u_0 distinct from v_0, x, and y. We claim that $z \neq v_2$, for otherwise we consider the separating 4-cycles $u_0v_0v_3v_2$ and $u_0v_0v_1v_2$. At least one of them satisfies the assumptions of Lemma 12(b), a contradiction.

Next, let u_1 be the neighbor of v_1 distinct from x, v_0, and v_2. Also, let u_3 be the neighbor of v_3 distinct from y, v_0, and v_2. We claim that $u_1 \neq u_3$. Suppose contrary that $u_1 = u_3$. Note that xu_1 and yu_1 are not the edges in G, due to the girth assumption. Let R' be the subset of vertices that induce the forest in $G - V(C) - \{u_0, u_1, x, y\}$ of size at least $\frac{119(n-8)-24(m-19)-24}{128} \geq \varphi - 4$. Now, the set $R' \cup \{u_0, v_0, v_1, v_3\}$ induces a forest of size at least φ in G. This establishes the claim that u_1 and u_3 are distinct.

We also claim that $x \neq u_3$. Suppose contrary that $x = u_3$. By planarity, $y = u_1$. Let $G' = G - \{v_0, v_1, v_3, u_0, x, y\}$. There exists a set of vertices R' in G' that induces the forest of size at least $\frac{119(n-6)-24(m-15)-24}{128} \geq \varphi - 3$. Inserting the vertices v_0, v_3, and u_0 in R' infers a forest of size at least φ in G. Observe that no cycle is introduced, since z and v_2 are in different parts of the plane regarding the separating cycle $v_0v_1xv_3$. We show similarly that $y \neq u_1$.

As we established that all vertices from Fig. 2 are distinct, we continue by considering the adjacency of the vertices z, u_1, and u_3. If neither of them are adjacent, then there exists a set of vertices R' of size at least $\frac{119(n-6)-24(m-15)-24}{128} \geq \varphi - 3$ in $G \cup \{w, wz, wu_1, wu_3\} - \{v_0, v_1, v_2, v_3, u_0, x, y\}$, where w is a new vertex. If $w \notin R'$, then the vertices v_1, v_3, and u_0 are added to R' to induce a forest F of size at least φ. On the other hand, if $w \in R'$, then such a forest is induced by adding v_0, v_1, v_3, and u_0 to F. Obviously in both cases no cycle is introduced.

By the above paragraph, we may assume that some two vertices from $\{z, u_1, u_3\}$ are adjacent, however, there exist a pair which is not, due to the girth assumption. Observe that without loss of generality, we may assume that $zu_3 \notin E(G)$ and $u_1u_3 \in E(G)$, since we may use the symmetry of the neighborhood of vertex v (see Fig. 4).

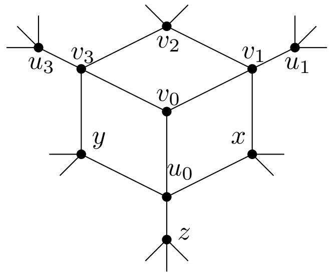

Figure 4: The symmetric neighborhood of vertex v_0.

Next, we claim that both of the vertices u_1 and u_3 are of degree 4. Suppose for a contradiction that u_1 is a 3-vertex. Then, by Lemma 17, u_3 is a 4-vertex, and by Lemma 10, u_3 is adjacent to x. So, consider the separating 5-cycle $g = u_1v_1v_0v_3u_3$ and an induced forest F' of $G - V(g) - y$. It is of size at least $\frac{119(n-6)-24(m-16)-24}{128} \geq \varphi - 3$. Observe that after adding vertices v_0, v_3, and u_1 to F' no cycles are introduced, thus we obtain an induced forest of size at least φ in G, a contradiction.

Finally, consider two subcases regarding the degree of z:

- *z is a 4-vertex.* In this case, by planarity at most one of the edges u_1y, and u_3x exists. By symmetry, suppose that u_3x does not. Then there is a set of vertices R' in $G - V(C) - \{u_0, u_3, z, x, y\}$ which induces a forest of size at least $\frac{119(n-9)-24(m-24)-24}{128} \geq \varphi - 4$. The vertex set $R' \cup \{u_0, v_0, v_1, v_3\}$ induces a forest of size at least φ in G.

- *z is a 3-vertex.* Let s and t be the neighbors of z distinct from u_0. By Lemma 10 we infer that s and t have a common neighbor p. In addition, by Lemma 10, we may assume that $ty, sx \in E(G)$. Note that $t \neq u_3$ and $s \neq u_1$ by the girth assumption. Moreover, if $t = u_1$, then either $p = y$ or $p = u_3$. Both cases are violating the planarity, thus $t \neq u_1$. Similarly we show that $s \neq u_3$.

  Let q be the neighbor of x distinct from v_1, u_0, and s, and similarly let r be the neighbor of y distinct from v_3, u_0, and t. Note that $q = r$ is possible only when q is a 4-vertex. Otherwise, if q is a 3-vertex, we obtain a separating 4-cycle u_0xqy which is reducible by Lemma 12(b).

  Next, by the planarity at least one of the edges qt and rs does not exists, say $qt \notin E(G)$. Then, there exists a set R' in $G - \{v_0, v_1, v_3, u_0, x, y, z, s, t, q\}$ which induces a forest F' of size at least $\frac{119(n-10)-24(m-24)-24}{128} \geq \varphi - 5$. By introducing vertices u_0, v_0, x, y, and z to R', we obtain a set of vertices which induces a forest of size at least φ in G. It is easy to see that the new vertices do not introduce any cycles, since only the edge yu_0 could be incident with F'.

This analysis establishes the lemma. □

From Lemmas 13, 14, and 16–18 immediately follows the corollary below:

Corollary 19 *Every 4-face (resp. 5-face) of G is incident with four (resp. three) 4-vertices.*

Finally we are ready to establish the theorem with the following short application of Euler's formula.

Proof (of Theorem 3). Let n_d be the number of vertices of degree d and let f_l be the number of faces of length l in G. By Corollary 19 we infer

$$4n_4 \geq 4f_4 + 3f_5\,.$$

Using this inequality and Euler's Formula we obtain

$$\begin{aligned}
-12 &= 6m - 6n - 6f \\
&= 2\sum_{v\in V(G)} d(v) + \sum_{f\in F(G)} l(f) - 6n - 6f \\
&= \sum_{d\geq 3}(2d-6)n_d + \sum_{l\geq 4}(l-6)f_l \\
&\geq 2n_4 - 2f_4 - f_5 \\
&\geq 0\,.
\end{aligned}$$

Hence, we obtain a contradiction which shows that the minimal counter-example does not exist and establish Theorem 3. □

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References

[1] J. Akiyama, M. Watanabe, *Maximum induced forests of planar graphs*, Graphs Combin. **3** (1987), 201–202.

[2] M. O. Albertson, D. M. Berman, *A conjecture on planar graphs*, Bondy, J.A., Murty, U.S.R. (eds) Graph Theory and Related Topics. Academic Press (1979), 357.

[3] N. Alon, *Problems and results in Extremal Combinatorics I*, Discrete Math. **273** (2003), 31–53.

[4] N. Alon, D. Mubayi, R. Thomas, *Large induced forests in sparse graphs*, J. Graph Theory **38** (2001), 113–123.

[5] P. Erdős, M. Saks, V. T. Sós, *Maximum induced trees in graphs*, J. Combin. Theory Ser. B **41** (1986), 61–79.

[6] J. Fox, P. S. Loh, B. Sudakov, *Large induced trees in K_r-free graphs*, J. Combin. Theory Ser. B **99** (2009), 494–501.

[7] R. M. Karp, *Reducibility among combinatorial problems*, Complexity of Computer Computations (R. E. Miller, J. W. Thatcher, ed.), Plenum Press, New York – London (1972), 85–103.

[8] J. Matoušek, R. Šámal, *Induced trees in triangle-free graphs*, Electron. J. Combin. **15** (2008), 7–14.

[9] N. Punnim, *The decycling number of regular graphs*, Thai J. Math. **4** (2008), 145–161.

[10] M. R. Salavatipour, *Large Induced Forests in Triangle-free Planar Graphs*, Graphs Combin. **22** (2006), 113–126.