Two approximation algorithms for ATSP with strengthened triangle inequality*

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Abstract. In this paper, we study the asymmetric traveling salesman problem (ATSP) with strengthened triangle inequality, i.e. for some $\gamma \in [\frac{1}{2}, 1)$ the edge weights satisfy $w(u, v) \leq \gamma(w(u, x) + w(x, v))$ for all distinct vertices u, v, x.
We present two approximation algorithms for this problem. The first one is very simple and has approximation ratio $\frac{1}{2(1-\gamma)}$, which is better than all previous results for all $\gamma \in (\frac{1}{2}, 1)$. The second algorithm is more involved but it also gives a much better approximation ratio: $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$ when $\gamma > \gamma_0$, and $\frac{1}{2}(1+\gamma)^2 + \epsilon$ for any $\epsilon > 0$ when $\gamma \leq \gamma_0$, where $\gamma_0 \approx 0.7003$.

1 Introduction

The Traveling Salesman Problem is one of the most researched NP-hard problems. In its classical version, given a set of vertices V and a symmetric weight function $w : V^2 \to \mathbb{R}_{\geq 0}$ one has to find a Hamiltonian cycle of minimum weight. Asymmetric Traveling Salesman Problem (ATSP) is a natural generalization where the weight function w does not need to be symmetric. Both TSP and ATSP without additional assumptions do not allow for any reasonable polynomial-time approximation algorithm, i.e. they are NPO-complete problems. A natural assumption, ofter appearing in applications, is the triangle inequality, i.e. $w(u, v) \leq w(u, x) + w(x, v)$, for all distinct vertices u, v, x. With this assumption, TSP has a 3/2-approximation by the well-known algorithm of Christofides [5]. On the other hand, no constant-factor polynomial time algorithm is known for ATSP with triangle inequality and the best algorithm up to date, due to Feige and Singh [6], has approximation ratio $\frac{2}{3}\log_2 n$.

1.1 γ-Parameterized Triangle Inequality and Previous Results

The TSP and ATSP problems have also been studied under the γ-parameterized triangle inequality, i.e. for some constant γ, and for all distinct vertices u, v, and x,

$$w(u, v) \leq \gamma(w(u, x) + w(x, v)).$$

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It can be easily seen that only values $\gamma \geq \frac{1}{2}$ make sense, since otherwise all edge weights would need to be 0, see e.g. [3]. In the first work on ATSP problem with the γ-parameterized triangle inequality, Chandran and Ram [3] showed a $\gamma/(1-\gamma)$-approximation algorithm for any $\gamma \in [\frac{1}{2}, 1)$. Note that for any fixed γ the approximation ratio is bounded. Next, Bläser [1] announced an algorithm with approximation ratio $1/(1-\frac{1}{2}(\gamma+\gamma^3))$, which is an improvement for $\gamma \in [0.66, 1)$. Later, it was extended by Bläser, Manthey and Sgall [2] to $(1+\gamma)/(2-\gamma-\gamma^3)$ which otperforms the earlier algorithms for $\gamma \in [0.55, 1)$. Finally, there is a very recent algorithm of Zhang, Li and Li [11] which is better than the previous methods for $\gamma \in [0.59, 0.72]$.

In their work Chandran and Ram [3] were also interested in bounding the ratio $\mathrm{ATSP}(G)/\mathrm{AP}(G)$ where $\mathrm{ATSP}(G)$ and $\mathrm{AP}(G)$ are the minimum weight of a Hamiltonian cycle in graph G and the minimum weight of a cycle cover in graph G, respectively. Analysis of their algorithm implies that $\mathrm{ATSP}(G)/\mathrm{AP}(G) \leq \gamma/(1-\gamma)$. On the other hand they show an infinite family of graphs for which $\mathrm{ATSP}(G)/\mathrm{AP}(G) = \frac{1}{2(1-\gamma)}$.

1.2 Our Results

In Section 2 we describe a very simple algorithm, using methods similar to those used by Kostochka and Serdyukov [8] for the max-ATSP with triangle inequality. We show that its approximation ratio is $\frac{1}{2(1-\gamma)}$, which is better than all previous results for any $\gamma \in (\frac{1}{2}, 1)$. This result implies that $\mathrm{ATSP}(G)/\mathrm{AP}(G) \leq \frac{1}{2(1-\gamma)}$ for any graph G, which is tight.

In Section 3 we present an even more efficent method, which outperforms our first algorithm for $\gamma \geq 0.619$. There is a constant $\gamma_0 \approx 0.7003$, such that for $\gamma \in (\gamma_0, 1)$ the second method gives the approximation ratio of $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$, while for $\gamma \in [\frac{1}{2}, \gamma_0)$ it can achieve approximation ratio of $\frac{1}{2}(1+\gamma)^2 + \epsilon$ for any $\epsilon > 0$ (see Figure 1 for comparison with previous results).

We show that for $\gamma \in (\gamma_0, 1)$ our approximation factor is essentially optimal w.r.t. the relaxation used.

1.3 Notation

Throughtout the paper, V is the vertex set of the input complete graph and $w : V^2 \to \mathbb{R}_{\geq 0}$ is a weight function which satisfies the γ-parameterized triangle inequality and such that $w(v, v) = 0$ for any $v \in V$. The vertex sets of all the graphs and multigraphs in the paper are subsets of V and w naturally induces weights on their edges.

For any (multi)set of edges S we define $w(S) = \sum_{(x,y)\in S} w(x, y)$. We will also write $w(S)$ when S is a (multi)graph, a cycle, a walk etc. always meaning the corresponding (multi)set of edges.

We will say that a directed graph G is *connected* when the underlying undirected graph is connected. Similarly, a *connected component* of G is a inclusion-wise maximal subgraph of G that is connected.

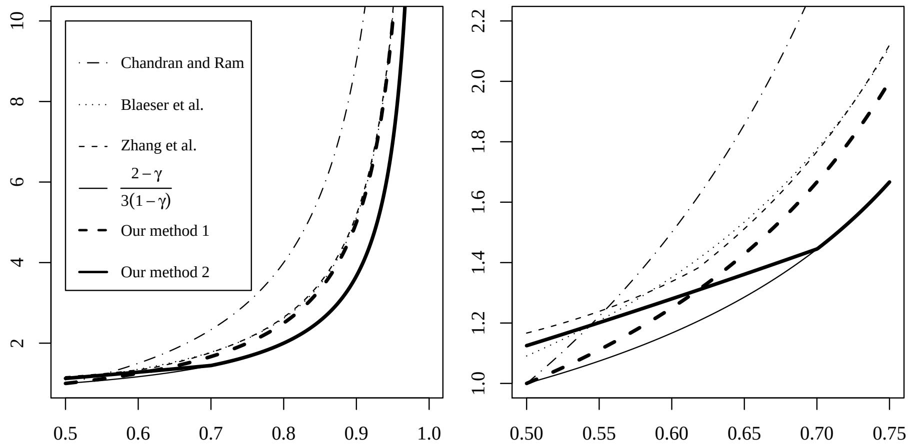

Fig. 1. Comparision of our methods with previous results.

2 A simple $\frac{1}{2(1-\gamma)}$-approximation algorithm

In this section we give a very simple algorithm that achieves an approximation ratio of $\frac{1}{2(1-\gamma)}$. Our algorithm starts, as is typical when approximating variants of TSP, by finding a minimum weight cycle cover of the input graph. Such a cover can be found in polynomial time. The cycles are then broken and patched to form a Hamiltonian cycle. The technique we use to guarantee that the patching edges have low weight is very similar to the one introduced by Kostochka and Serdyukov [8] for a variant of max-TSP. Interestingly, this technique has not so far been used to solve min-TSP problems.

2.1 A Randomized Algorithm

We begin by showing a randomized version of our algorithm since it is more natural this way.

Throughout this paper we will use the following lemma.

Lemma 1. *Let $W = (v_1, v_2, \ldots, v_k, v_1)$ be a closed walk and let x be a vertex not visited by W. Then,*

$$\sum_{j=1}^{k} w(v_j, x) \leq \frac{\gamma}{1-\gamma} w(W).$$

Proof. From the γ-parametrized triangle inequality it follows that

$$w(v_j, x) \leq \gamma \left(w(v_j, v_{j+1}) + w(v_{j+1}, x) \right)$$

for $j = 1, \ldots, k$ (we let $v_{k+1} = v_1$ to avoid having to consider it separately). By summing the above inequality over all $j = 1, \ldots, k$, we get

$$\sum_{j=1}^{k} w(v_j, x) \leq \gamma \left(w(W) + \sum_{j=1}^{k} w(v_j, x) \right),$$

which easily implies the claim of the lemma. □

Theorem 1. *Let C be a directed cycle of length l, and let P be a path created by randomly removing a single edge from C. Also let u be the last vertex on P, and let v be a vertex not contained in C. Then*

$$E[w(P) + w(uv)] \leq \frac{l - 1 - (l - 2)\gamma}{l(1 - \gamma)} w(C),$$

Proof. Let $C = v_1 v_2 \ldots v_l v_1$. We have

$$E[w(P) + w(uv)] = w(C) - \frac{w(C)}{l} + \frac{\sum_{i=1}^{l} w(v_i v)}{l}. \quad (1)$$

From Lemma 1 we get

$$E[w(P) + w(uv)] \leq \left(1 - \frac{1}{l} + \frac{\gamma}{l(1 - \gamma)}\right) w(C) = \left(\frac{l - 1 - (l - 2)\gamma}{l(1 - \gamma)}\right) w(C). \quad (2)$$

□

Since $\frac{l-1-(l-2)\gamma}{l(1-\gamma)} = 1 + \frac{2\gamma-1}{l(1-\gamma)}$ is a monotonically decreasing function of l, and $l \geq 2$ for all cycles, it follows that

Corollary 1. *With C, P, u and v as in Theorem 1, we have*

$$E[w(P) + w(uv)] \leq \frac{1}{2(1 - \gamma)} w(C).$$

We are now ready to present our basic algorithm.

Theorem 2. *Let $\mathcal{C} = \{C_1, \ldots, C_k\}$ be a directed cycle cover of graph G. Then a Hamiltonian cycle H in G with expected weight $E[w(H)] \leq \frac{1}{2(1-\gamma)} w(\mathcal{C})$ can be found in polynomial time.*

Proof. Remove a random edge from each cycle C_i thus turning it into a directed path P_i. Connect the last vertex of P_i with the first vertex of P_{i+1} for $i = 1, \ldots, k - 1$ and connect the last vertex of P_k with the first vertex of P_1. Let H be the resulting Hamiltonian cycle. We claim that $E[w(H)] \leq \frac{1}{2(1-\gamma)} w(\mathcal{C})$.

Consider any cycle in $\mathcal{C}$, say C_i, and assume that we have already removed a random edge from all the other cycles, and in particular from C_{i+1}. Let us now randomly break C_i (creating P_i) and let e_i be the edge connecting the last vertex of P_i with the first vertex of P_{i+1}. It follows from Corollary 1 that

$$E[w(P_i) + w(e_i)] \leq \frac{1}{2(1 - \gamma)} w(C_i).$$

This holds for cycle C_i regardless of the random choices made for the other cycles, so it also holds when all cycles are broken randomly. Summing this inequality over all cycles proves the theorem. □

Since the minimum weight of a cycle cover lowerbounds the minimum weight of a Hamiltonian cycle we get the following.

Corollary 2. *For any $\gamma \in [\frac{1}{2}, 1)$ there is a randomized polynomial time algorithm for ATSP with γ-parameterized triangle inequality, which has expected approximation ratio of $\frac{1}{2(1-\gamma)}$.*

2.2 A deterministic algorithm

The algorithm presented in the previous section can be derandomized using a generic conditional expected value approach. The resulting algorithm is particularly simple, but we defer its explicit description to the journal version due to space limitations.

Here, we give only a deterministic version of Theorem 1 and Corollary 1, as they will be used in a further section.

Corollary 3. *Let C be a directed cycle of length l and let v be a vertex not contained in C. Then, an edge can be removed from C so that the resulting directed path P satisfies*

$$w(P) + w(uv) \leq \frac{l - 1 - (l-2)\gamma}{l(1-\gamma)} w(C),$$

where u is the last vertex of P. In particular,

$$w(P) + w(uv) \leq \frac{1}{2(1-\gamma)} w(C).$$

Proof. Remove an edge that gives the lowest value of $w(P) + w(uv)$. The first bound follows easily from Theorem 1. It implies the second one since $l \geq 2$. □

Derandomizing the algorithm of Theorem 2 is a bit harder (we skip it in this conference version).

Theorem 3. *Let $\mathcal{C} = \{C_1, \dots, C_k\}$ be a directed cycle cover of graph G. Then, a Hamiltonian cycle H in G with weight $w(H) \leq \frac{1}{2(1-\gamma)} w(\mathcal{C})$ can be found in polynomial time.*

Remark 1. Note that the above analysis implies that $\mathrm{ATSP}(G)/\mathrm{AP}(G) \leq \frac{1}{2(1-\gamma)}$ for any graph G, which is tight by the result of Chandran and Ram [3]. In particular the approximiation ratio of our algorithm is optimal w.r.t. minimum weight cycle cover relaxation.

3 Improved approximation using double cycle covers

In the algorithm from Section 2 the 2-cycles are the obvious bottleneck, i.e. if we were able to find the minimum weight cycle cover with no 2-cycles, the algorithm would have a significantly better approximation ratio of $\frac{2-\gamma}{3(1-\gamma)}$ (see Fig 1). Unfortunately, finding such a cover is APX-hard even when w has exactly two values as shown by Manthey [10], and when these values are 1 and 2γ the γ-parameterized triangle inequality holds. A similar phenomenon occurs for some other min-ATSP and max-ATSP variants. Nevertheless, Kaplan, Lewenstein, Shafrir and Sviridenko [7] obtained a major progress for three ATSP variants and motivated a series of other improvements (see [6, 4, 9]) by proving the following.

Theorem 4 (Kaplan et al. [7]). *Let G be a directed weighted graph. One can find in polynomial time a pair of cycle covers $\mathcal{C}_1$, $\mathcal{C}_2$ such that (i) $\mathcal{C}_1$ and $\mathcal{C}_2$ share no 2-cycles, (ii) total weight $w(\mathcal{C}_1) + w(\mathcal{C}_2)$ of the two covers is at most* 2OPT*, where* OPT *is the weight of the minimum weight Hamiltonian cycle in G.*

In this section we will show that the double cycle covers can be also used for ATSP with γ-parameterized triangle inequality. The resulting algoritm has approximation ratio of $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$ for $\gamma \in (\gamma_0, 1)$, $\gamma_0 \approx 0.7003$.

The sketch of our algorithm is as follows. We begin by finding a pair of cycle covers $\mathcal{C}_1$, $\mathcal{C}_2$ described in Theorem 4. The union of these covers corresponds to a 2-regular directed graph M. We then replace the connected components of M by paths which eventually form a Hamiltonian cycle. We present two methods of doing it.

In the first method, each connected component Q of M is replaced by a low-weight cycle which contains all vertices of Q. These cycles are then joined to form a single path using the method from Section 2, Corollary 3. This approach is efficient enough provided that the components are *big* i.e. they have size at least $f(\gamma)$, for certain function f, to be defined later.

For small components we use a different method. Using another deep result of Kaplan et al. we show that a component Q can be replaced by a collection of paths of total weight $\frac{2-\gamma}{3(1-\gamma)} w(Q)$, provided that we have $|V(Q)|$ previously constructed paths to work with. Guaranteeing that is a technical detail, and it increases the final approximation ratio by $O(\frac{1}{n})$.

In the next two subsections we show how to deal with small and big connected components of M. Then we give a more detailed decription of the complete approximation algorithm.

3.1 Dealing with small components

A directed graph which is a union of vertex-disjoint paths will be called a *path graph*. The following theorem is due to Kaplan et al. [7] (Theorem 5.1).

Theorem 5. *Let G be a directed 2-regular multigraph that contains neither two copies of the same 2-cycle, nor two copies, oppositely oriented, of the same 3-cycle. Then G is a union of three path graphs, and such a decomposition can be found in polynomial time.*

We will need the following corollary from Lemma 1.

Corollary 4. *Let Q be a 2-regular subgraph in a directed multigraph G and let x be a vertex not in $V(Q)$. Then,*

$$\sum_{v \in V(Q)} w(v,x) \le \frac{\gamma}{2(1-\gamma)} w(Q).$$

Proof. We apply Lemma 1 with W being an Eulerian cycle in Q. Since each vertex of Q is visited by W exactly twice, the claimed bound follows. ☐

Lemma 2. *Let Q be a connected component in a directed multigraph G such that Q is 2-regular and has t vertices, $t \ge 3$. Let X be a set of k vertices of G, disjoint with $V(Q)$, $k \ge t$. Then one can find in polynomial time a set of k vertex-disjoint paths $\mathcal{P}$ such that*

(i) *paths in $\mathcal{P}$ end in vertices of X,*
(ii) *$V(\mathcal{P}) = V(Q) \cup X$, and*
(iii) *$w(\mathcal{P}) \le \frac{2-\gamma}{6(1-\gamma)} w(Q)$.*

Proof. We begin by decomposing Q into three path graphs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ using Theorem 5. This is possible unless Q is a union of two oppositely oriented 3-cycles, in which case we replace the heavier of these cycles by another copy of the lighter one and then apply Theorem 5. We assume that for each $i \in \{1,2,3\}$, $V(\mathcal{P}_i) = V(Q)$. If that is not the case, we add the missing vertices (treated as paths of length 0).

Consider any vertex v in Q. Since Q is 2-regular, $\text{outdeg}_{\mathcal{P}_1}(v) + \text{outdeg}_{\mathcal{P}_2}(v) + \text{outdeg}_{\mathcal{P}_3}(v) = 2$. Hence $\text{outdeg}_{\mathcal{P}_i}(v) = 0$ for exactly one $i \in \{1,2,3\}$. It follows that for every vertex v of Q there is exactly one path in $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ that ends in v. In particular, graphs $\mathcal{P}_i$ contain exactly t paths in total. Now, we are going to choose a distinct vertex x of X for each such path, and connect its end v to x by a new edge. In order to guarantee that the total weight of the added edges is small, we consider t ways of assigning vertices to paths and choose the best one.

More precisely, for each $i = 0, \dots, t-1$ we construct three path graphs $\mathcal{P}_{i,1}$, $\mathcal{P}_{i,2}$, and $\mathcal{P}_{i,3}$, each with the vertex set $V(Q) \cup X$. Let $\{x_0, x_1, \dots, x_{t-1}\}$ be a set of t vertices of X (arbitrarily selected) and let $V(Q) = \{v_0, \dots, v_{t-1}\}$. For any $i = 0, \dots, t-1$, $q = 1,2,3$, the graph $\mathcal{P}_{i,q}$ is obtained from $\mathcal{P}_q$ by extending each of its paths with an edge — if a path ends in v_j we extend it with $(v_j, x_{(j+i) \bmod t})$.

Our algorithm returns the lightest among the path graphs $\mathcal{P}_{i,q}$. Denote it by $\mathcal{P}$. Let $A = \{(v_j, x_{(j+i) \bmod t}) \ : \ i,j = 0, \dots, t-1\}$ be the set of edges added to the paths of $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. Then,

$$w(\mathcal{P}) \le \frac{\sum_{i=0}^{t-1} \sum_{q=1}^{3} w(\mathcal{P}_{i,q})}{3t} = \frac{t \cdot w(Q) + w(A)}{3t}. \tag{3}$$

It suffices to bound $w(A)$. We proceed as follows.

$$w(A) = \sum_{j=0}^{t-1}\sum_{i=0}^{t-1} w(v_j, x_{(j+i) \bmod t}) = \sum_{j=0}^{t-1}\sum_{\ell=0}^{t-1} w(v_j, x_\ell) = \sum_{\ell=0}^{t-1}\sum_{j=0}^{t-1} w(v_j, x_\ell) \leq$$

$$\leq \sum_{\ell=0}^{t-1} \frac{\gamma}{2(1-\gamma)} w(Q) = \frac{t\gamma}{2(1-\gamma)} w(Q), \tag{4}$$

where the ineaquality follows from Corollary 4. After plugging (4) to (3) we get the claimed bound on $w(\mathcal{P})$. □

3.2 Dealing with big components

An easy proof of the following lemma is deferred to a journal version because of space limitations.

Lemma 3. *For $k \geq 3$, let $W = (x_1, x_2, \ldots, x_k)$ be a walk such that each vertex appears at most twice in W. Then,*

$$w(x_1 x_k) \leq \gamma w(x_1 x_2) + \gamma^2 \sum_{i=2}^{k-2} w(x_i x_{i+1}) + \gamma w(x_{k-1} x_k).$$

Lemma 4. *Let Q be a connected 2-regular directed multigraph. Then there is a randomized polynomial time algorithm which finds in Q a Hamiltonian cycle of expected weight at most $\frac{1}{4}(1+\gamma)^2 w(Q)$.*

Proof. Let t be the number of vertices in Q. Our algorithm begins by finding an Eulerian cycle $\mathcal{E} = (v_0, v_1, \ldots, v_{2t-1}, v_0)$ in Q. For each pair of indices i, j such that $v_i = v_j$, our algorithm chooses one index uniformly at random, and the chosen index will be called a *stop point*. Observe that every vertex of Q appears in $\mathcal{E}$ precisely twice, so for each vertex x there is exactly one stop point i such that $v_i = x$. Let $i_0, i_1, \ldots, i_{t-1}$ be the sequence of stop points such that $i_j < i_{j+1}$ for $j = 0, \ldots, t-2$. Assume w.l.o.g. that $i_0 = 0$. Finally, the algorithm returns the Hamiltonian cycle $C = v_{i_0}, v_{i_1}, \ldots, v_{i_{t-1}}$.

Now we are going to bound $E[w(C)]$. Consider arbitrary choice of stop points. In what follows, indices at v_q and i_p are modulo $2t$ and t, respectively. We have

$$w(C) = \sum_{p=0}^{t-1} w(v_{i_p} v_{i_{p+1}}).$$

Using Lemma 3 we can bound the values of $w(v_{i_p} v_{i_{p+1}})$ for $p = 0, \ldots, t-1$. Then we get

$$w(C) \leq \sum_{q=0}^{2t-1} \text{contrib}(v_q v_{q+1}) w(v_q v_{q+1}),$$

where for any q such that $i_p \le q < i_{p+1}$

$$\text{contrib}(v_q v_{q+1}) = \begin{cases} 1 & \text{when both } q \text{ and } q+1 \text{ are stop points },\\ \gamma & \text{when exactly one of } q,\, q+1 \text{ is a stop point},\\ \gamma^2 & \text{when neither } q \text{ nor } q+1 \text{ is a stop point.} \end{cases}$$

It follows that

$$E[\text{contrib}(v_q v_{q+1})] \le \frac{1}{4} + \frac{1}{2}\gamma + \frac{1}{4}\gamma^2 = \frac{1}{4}\left(1+\gamma\right)^2. \tag{5}$$

The lemma follows by the linearity of expectation. □

Note that the algorithm in Lemma 4 can be easily derandomized using the method of conditional expectation.

Lemma 5. *Let Q be as before. Then there is a deterministic polynomial-time algorithm which finds in Q a Hamiltonian cycle of weight at most $\frac{1}{4}(1+\gamma)^2 w(Q)$.* □

Lemma 5 and Corollary 3 immediately imply the following.

Corollary 5. *Let Q be a connected component in a directed multigraph G such that Q is 2-regular and has t vertices, $t \ge 3$. Let x be a vertex of G which does not belong to $V(Q)$. Then one can find in polynomial time a path P such that*

(i) path P ends in x,
(ii) $V(P) = V(Q) \cup \{x\}$, and
(iii) $w(P) \le \frac{1}{4}(1+\gamma)^2 \cdot \frac{t-1-(t-2)\gamma}{t(1-\gamma)} w(Q)$. □

3.3 The complete algorithm

Now we are going to combine the ingredients developed in the two previous sections in a complete approximation algorithm. Let K be an integer constant whose value will be determined later, $K \ge 1$. (K depends on the constant γ only). Intuitively, K is the maximum size of what we call a small component.

Let G be the input complete graph. Our algorithm begins by computing for each vertex $v \in V$ the value of $D(v) = \max_{x \in V} w(v,x)$, i.e. the maximum weight of an edge in G that leaves v. Let X_0 be a set of K vertices of V with smallest values of $D(v)$. The algorithm finds in $G[V \setminus X_0]$ the two cycle covers described in Theorem 4. Let M be the corresponding 2-regular multigraph.

Our algorithm builds a collection $\mathcal{H}$ of K vertex-disjoint directed paths which end in the vertices of X_0. Initially we put $\mathcal{H} = X_0$ (regarded as a collection of paths of length 0). For each connected component Q of M the algorithm extends the paths in $\mathcal{H}$ using the paths returned by the algorithm from Lemma 2 if Q is of size at most K or from Corollary 5 otherwise. We use the first vertices of the paths in $\mathcal{H}$ as the set X in Lemma 2, and x in Corollary 5 is the first vertex of any of these paths. Note that after each such extension $\mathcal{H}$ is still a collection of K vertex-disjoint paths ending in X_0.

After processing all the connected components of M, $\mathcal{H}$ is a collection of vertex-disjoint paths incident to all vertices of V. The algorithm finishes by patching $\mathcal{H}$ arbitrarily to a Hamiltonian cycle H.

It is clear that our algorithm returns a Hamiltonian cycle. Now we are going to bound its weight. Let $\mathcal{H}$ be the collection of K paths just before patching them to a Hamiltonian cycle. Let M_{small} (resp. M_{big}) be the union of the components of M that have size at most K (resp. at least $K+1$). By Lemma 2 and Corollary 5 we get (recall that $\frac{t-1-(t-2)\gamma}{t(1-\gamma)}$ is a decreasing function of t)

$$w(\mathcal{H}) \leq \frac{2-\gamma}{6(1-\gamma)} w(M_{\text{small}}) + \frac{1}{4}(1+\gamma)^2 \frac{K-(K-1)\gamma}{(K+1)(1-\gamma)} w(M_{\text{big}}).$$

Hence,

$$w(\mathcal{H}) \leq \max\left\{ \frac{2-\gamma}{6(1-\gamma)}, \frac{1}{4}(1+\gamma)^2 \frac{K-(K-1)\gamma}{(K+1)(1-\gamma)} \right\} w(M). \tag{6}$$

Now consider a minimum weight Hamiltonian cycle H^* in G and let $\text{OPT} = w(H^*)$. Let $H^*_{V\setminus X_0}$ be a minimum weight Hamiltonian cycle in $G[V \setminus X_0]$. Let C be a Hamiltonian cycle in $G[V \setminus X_0]$ which visitis vertices of $V \setminus X_0$ in the order they appear in H^*. Then,

$$w(H^*_{V\setminus X_0}) \leq w(C) \leq w(H^*) = \text{OPT}, \tag{7}$$

where the second inequality follows from the γ-parameterized triangle inequality. Since $w(M) \leq 2w(H^*_{V\setminus X_0})$ by Theorem 4, we get

$$w(\mathcal{H}) \leq \max\left\{ \frac{2-\gamma}{3(1-\gamma)}, \frac{1}{2}(1+\gamma)^2 \frac{K-(K-1)\gamma}{(K+1)(1-\gamma)} \right\} \text{OPT}. \tag{8}$$

Now it suffices to bound the weight of the edges added during the patching phase. We use the following lemma (proof deferred to the journal version).

Lemma 6. *For any fixed* $\gamma \in [\frac{1}{2}, 1)$,

$$\sum_{x \in X_0} D(x) = O(\tfrac{K}{n}\text{OPT})$$

Note that in our patching phase we use only edges leaving vertices of X_0. Hence using the above lemma we bound their weight by $O(\frac{K}{n}\text{OPT})$. This, together with (8) gives the following theorem.

Theorem 6. *For any integer $K \geq 1$ there is a polynomial time algorithm which finds a Hamiltonian cycle of weight at most*

$$\left[\max\left\{ \frac{2-\gamma}{3(1-\gamma)}, \frac{1}{2}(1+\gamma)^2 \frac{K-(K-1)\gamma}{(K+1)(1-\gamma)} \right\} + O(\tfrac{K}{n}) \right] \text{OPT}.$$

Corollary 6. *There is a constant* γ_0, $\gamma_0 \approx 0.7003$ *such that*

(i) For any $\gamma \in (\gamma_0, 1)$ *there is a polynomial time algorithm for ATSP with* γ*-parameterized triangle inequality, which has approximation ratio of* $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$.

(ii) For any $\gamma \in [\frac{1}{2}, \gamma_0]$*, and for any* $\epsilon > 0$ *there is a polynomial time algorithm for ATSP with* γ*-parameterized triangle inequality, which has approximation ratio of* $\frac{1}{2}(1+\gamma)^2 + \epsilon$.

Proof. One can easily check that the inequality in the variable x

$$\frac{1}{2}(1+\gamma)^2 \frac{x-(x-1)\gamma}{(x+1)(1-\gamma)} \leq \frac{2-\gamma}{3(1-\gamma)}$$

has the set of solutions of the form $x \geq f(\gamma)$ for some function f, if γ satisfies the inequality

$$\frac{2-\gamma}{3} - \frac{1}{2}(1-\gamma)(1+\gamma)^2 > 0, \tag{9}$$

and has no solutions otherwise. Moreover,the set of solutions of (9) that belong to $[\frac{1}{2}, 1)$ is of the form $\gamma \in (\gamma_0, 1)$, where γ_0 is an irrational number, $\gamma_0 \approx 0.7003$. It follows that for any $\gamma \in (\gamma_0, 1)$ we can put $K = \lceil f(\gamma) \rceil$ in Theorem 6 and we obtain an algorithm with approximation ratio of $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$.

For the second claim we know that whenever $\gamma \in [\frac{1}{2}, \gamma_0]$, the algortihm from Thorem 6 has an approximation ratio of $\frac{1}{2}(1+\gamma)^2 \frac{K-(K-1)\gamma}{(K+1)(1-\gamma)}$. It is easy to check that $\frac{1}{2}(1+\gamma)^2 + \epsilon$ upperbounds this for

$$K \geq \frac{\frac{1}{2}(1+\gamma)^2(2\gamma-1)}{(1-\gamma)\epsilon} - 1.$$

□

3.4 Tightness

As we mentioned earlier, the algorithm from Section 2 is optimal with respect to the cycle cover relaxation. Recall that for $\gamma \in (\gamma_0, 1)$ the algorithm from Section 2 has approximation ratio of $\frac{2-\gamma}{3(1-\gamma)} + O(\frac{1}{n})$. In what follows we show that this ratio is nearly optimal w.r.t. the double cycle cover relaxation of Kaplan et al. It is also nearly optimal w.r.t. the minimum cycle cover with no 2-cycles. Let $\mathrm{DC}(G)$ be the half of the minimum weight of a pair of cycle covers of G described in Theorem 4 and let $\mathrm{AP}_3(G)$ be the minimum weight of a cycle cover of G with no 2-cycles. The proof of the following theorem is deferred to the journal version.

Theorem 7. *For every* $\gamma \in [\frac{1}{2}, 1)$*, there exists an infinite family of graphs* $\mathcal{G}$ *such that for every* $G \in \mathcal{G}$,

$$\frac{\mathrm{ATSP}(G)}{\mathrm{DC}(G)} = \frac{\mathrm{ATSP}(G)}{\mathrm{AP}_3(G)} = \frac{2-\gamma}{3(1-\gamma)}.$$

It is an interesting open problem whether the approximation ratio of $\frac{2-\gamma}{3(1-\gamma)}$ can be achieved for all $\gamma \in [\frac{1}{2}, 1)$.

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A Proofs omitted in the paper

Proof (of Theorem 3). We are going to break the cycles one by one and pick the corresponding connecting edges. Let P_i and e_i be defined as in the proof of Theorem 2.

We start by breaking the cycle C_k. It follows from Corollary 1 that we can do it in such a way, that

$$E[w(P_k) + w(e_k)] \leq \frac{1}{2(1-\gamma)} w(C_k),$$

where the expectation is taken over random choices made for other cycles. In fact, the above is also true when only C_1 is broken randomly and the other cycles are broken arbitrarily. We can choose P_k so that

$$w(P_k) + E[w(e_k)] \leq \frac{1}{2(1-\gamma)} w(C_k)$$

by simply computing, for every possible P_k, the average weight of $w(e_k)$ over all choices for P_1.

Once C_k is broken, it follows from Corollary 3, that C_{k-1} can be broken so that

$$w(P_{k-1}) + w(e_{k-1}) \leq \frac{1}{2(1-\gamma)} w(C_{k-1}).$$

The same reasoning applies to all the remaining cycles, except for C_1. Here, special care needs to be taken, because by deciding how to break C_1, we are also selecting e_k. Notice however, that if we break C_1 randomly we have both

$$w(P_k) + E[w(e_k)] \leq \frac{1}{2(1-\gamma)} w(C_k)$$

and

$$E[w(P_1) + w(e_1)] \leq \frac{1}{2(1-\gamma)} w(C_1),$$

and so

$$w(P_k) + E[w(e_k) + w(P_1) + w(e_1)] \leq \frac{1}{2(1-\gamma)} (w(C_k) + w(C_1)).$$

Again, by running over all choices for P_1, we can choose one so that

$$w(P_k) + w(e_k) + w(P_1) + w(e_1) \leq \frac{1}{2(1-\gamma)} (w(C_k) + w(C_1)).$$

By summing the bounds for all $w(P_i) + w(e_i)$ we get the claim of the theorem. □

Proof (of Lemma 6). Let $H^* = v_0 v_1 \ldots v_{n-1} v_0$ be a minimum weight Hamiltonian cycle in G, and let H_* be the cycle obtained by reversing H^*. We are now going to bound $w(H_*)$. In what follows, indices are taken modulo n. By Lemma 3, for any $i = 0, \ldots, n-1$,

$$w(x_{i+1} x_i) \leq \gamma^{n-2} w(x_{i+1} x_{i+2}) + \sum_{j=2}^{n-1} \gamma^{n-j} w(x_{i+j} x_{i+j+1}).$$

Hence,

$$\begin{aligned} w(H_*) &= \sum_{i=0}^{n-1} w(x_{i+1}, x_i) \leq \sum_{i=0}^{n-1} \left(\gamma^{n-2} w(x_{i+1} x_{i+2}) + \sum_{j=2}^{n-1} \gamma^{n-j} w(x_{i+j} x_{i+j+1}) \right) \\ &= \sum_{\ell=0}^{n-1} \left(\gamma^{n-2} + \sum_{j=1}^{n-2} \gamma^j \right) w(x_\ell, x_{\ell+1}) < \left(\gamma^{n-2} + \frac{\gamma}{1-\gamma} \right) w(H^*). \end{aligned}$$

It follows that

$$w(H^*) + w(H_*) = O(\mathrm{OPT}). \tag{10}$$

Now consider all 2-cycles of the form $C_i = x_i x_{i+1} x_i$ for $i = 0, \dots, n-1$, and let $C_{i_1}, \dots, C_{i_K}$ be the K lightest cycles among them. By (10),

$$\sum_{j=1}^{K} w(C_{i_j}) = O(\tfrac{K}{n}\mathrm{OPT}). \tag{11}$$

Now, by Lemma 1, for any $i = 0, \dots, n-1$, $D(x_i) \leq \frac{\gamma}{1-\gamma} w(C_i)$. Hence,

$$\sum_{x \in X_0} D(x) \leq \sum_{j=1}^{K} D(x_{i_j}) \leq \sum_{j=1}^{K} \frac{\gamma}{1-\gamma} w(C_{i_j}) = O(\tfrac{K}{n}\mathrm{OPT}),$$

where the last equality follows from (11). □

Proof (of Theorem 7). Our argument is similar to that of Chandran and Ram [3], Theorem 9. Let $k \geq 2$ be an integer. We describe a complete graph G with $3k$ vertices $v_1, v_2, \dots, v_{3k}$. For $i = 1, \dots, k$, for any two distinct vertices $x, y \in \{v_{3i-2}, v_{3i-1}, v_{3i}\}$ we define $w(x, y) = 1$. For all the remaining pairs of distinct vertices in G the value of w is set to $\frac{\gamma}{1-\gamma}$. It is straightforward to verify that w satisfies the triangle inequality. Note that $\frac{\gamma}{1-\gamma} \geq 1$ in the assumed range of γ.

Now consider a cycle cover $\mathcal{C}$ in G consisting of cycles

$$\mathcal{C} = \{v_1 v_2 v_3 v_1, v_4 v_5 v_6 v_4, \dots, v_{3k-2} v_{3k-1} v_{3k} v_{3k-2}\}.$$

Note that two copies of $\mathcal{C}$ form a pair of cycle covers that satisfy the assumptions of Theorem 4. Moreover, $w(\mathcal{C}) = 3k$ and hence it is a minimum weight cycle cover since all the weights in G are at least 1. It follows that

$$\mathrm{DC}(G) = \mathrm{AP}_3(G) = 3k. \tag{12}$$

Finally observe that an arbitrary Hamiltonian cycle for any $i = 1, \dots, k$ contains at most two edges between vertices v_{3i-2}, v_{3i-1}, and v_{3i}. Hence $H = v_1 v_2 v_3 v_4 v_5 v_6 \cdots v_{3k} v_1$ is a minimum weight Hamiltonian cycle, so $\mathrm{ATSP}(G) = w(H) = (2 + \frac{\gamma}{1-\gamma})k$. Together with (12) we get $\frac{\mathrm{ATSP}(G)}{\mathrm{DC}(G)} = \frac{\mathrm{ATSP}(G)}{\mathrm{AP}_3(G)} = \frac{2-\gamma}{3(1-\gamma)}$, as required. □