Introduction to Parametrized Algorithms
Part 2

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  - Principle of **dynamic programming**.

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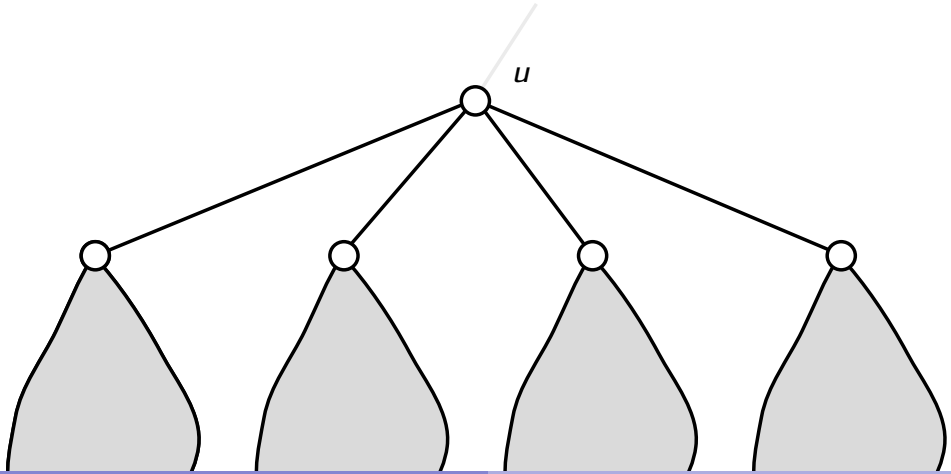

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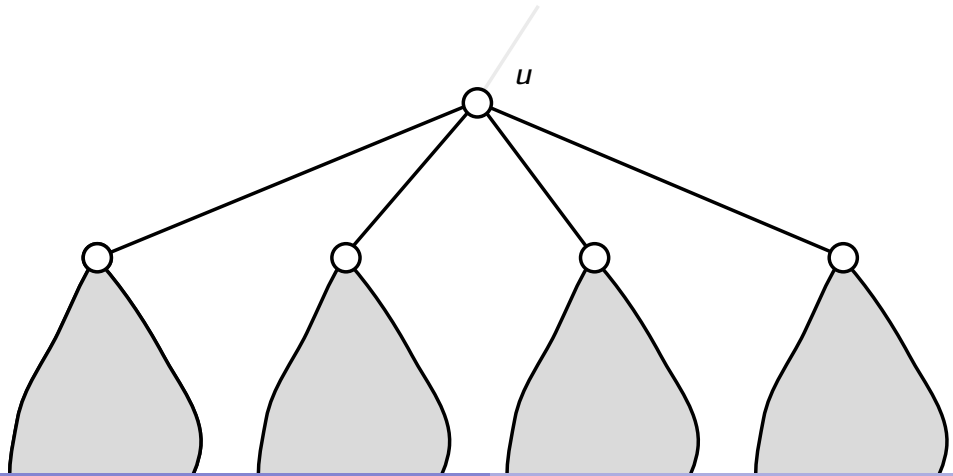

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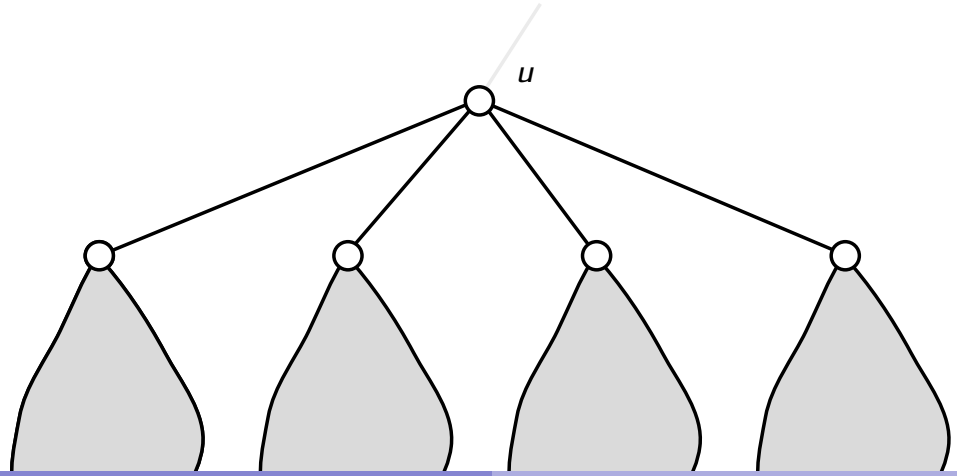

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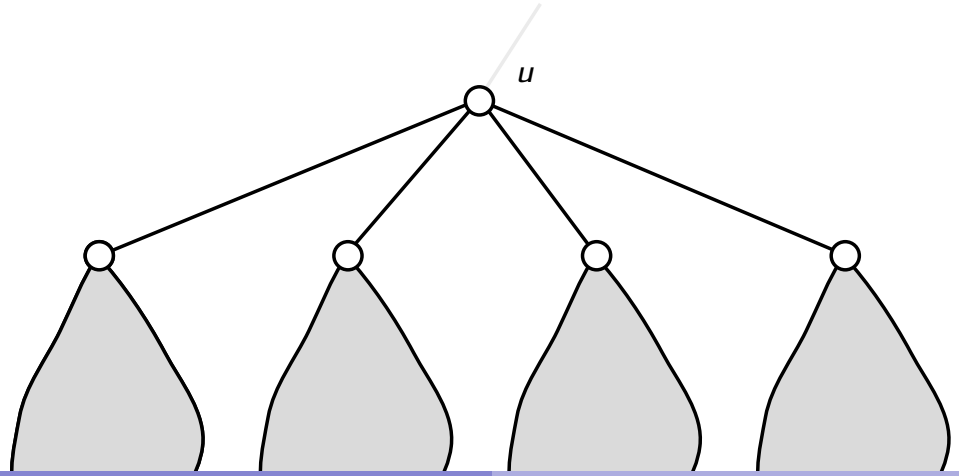

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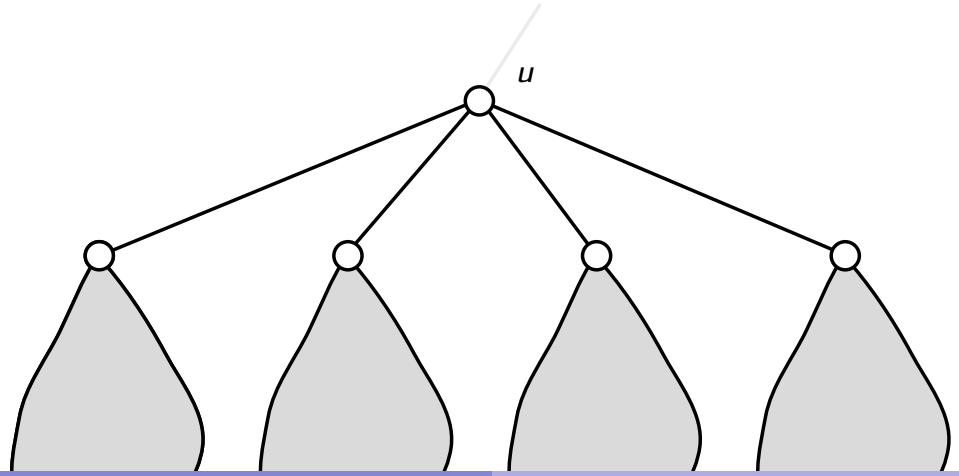

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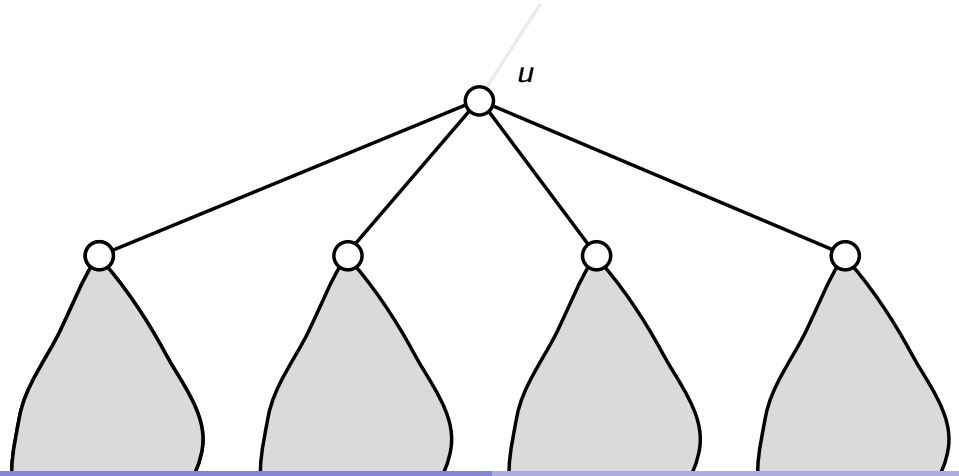

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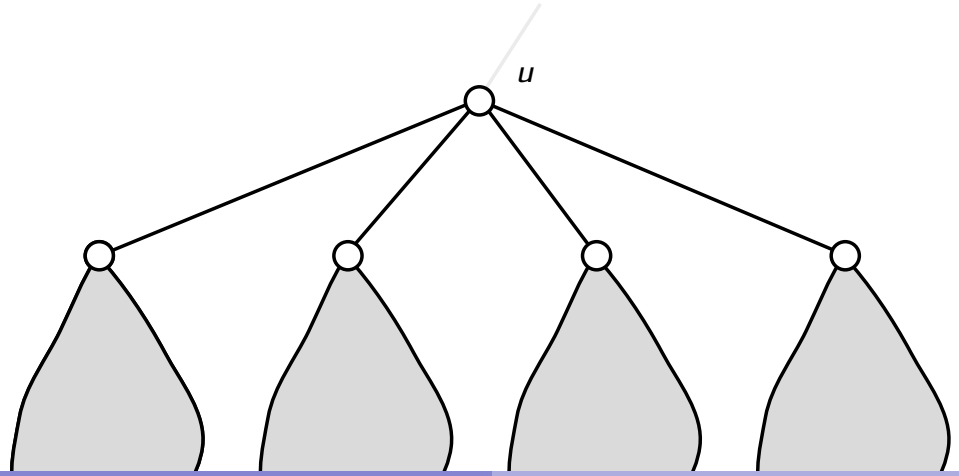

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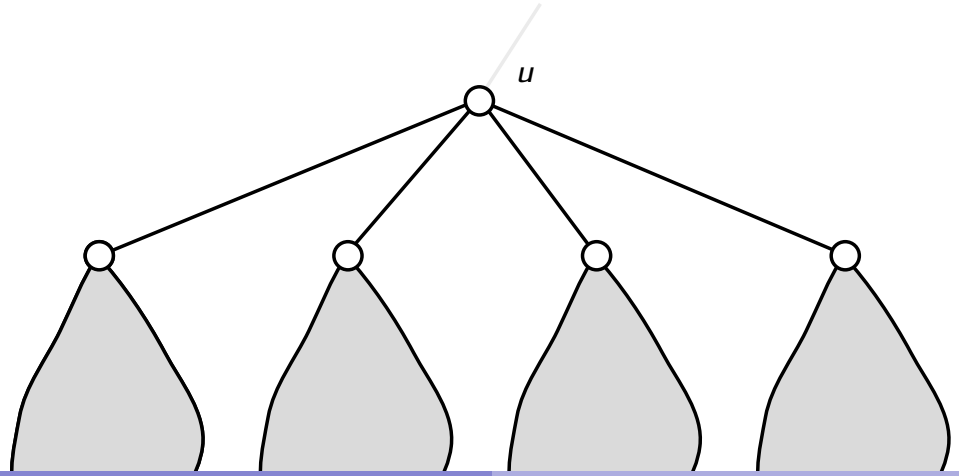

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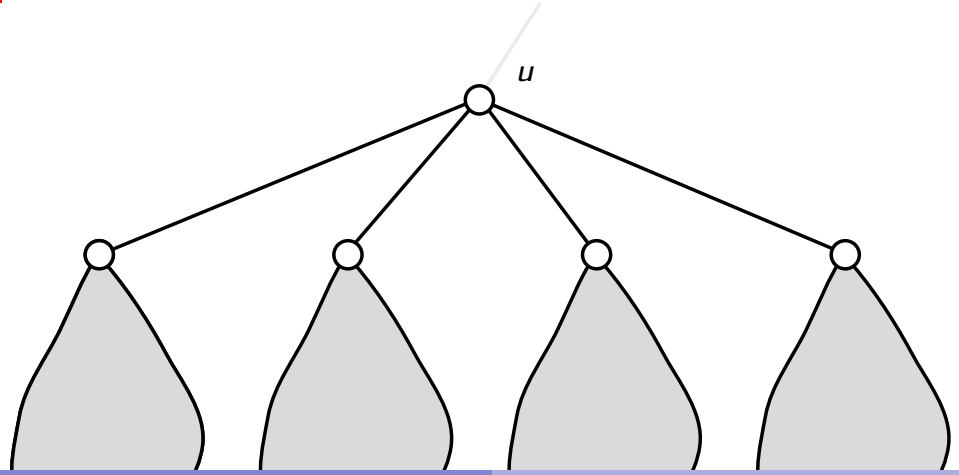

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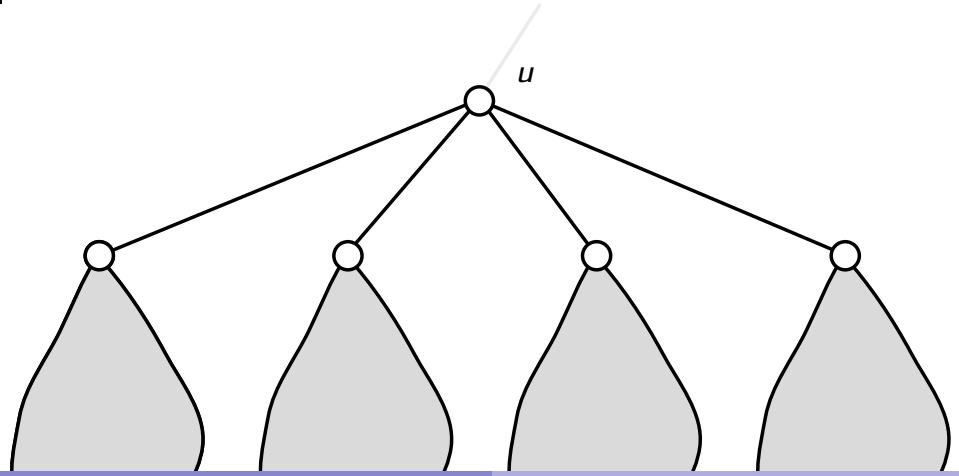
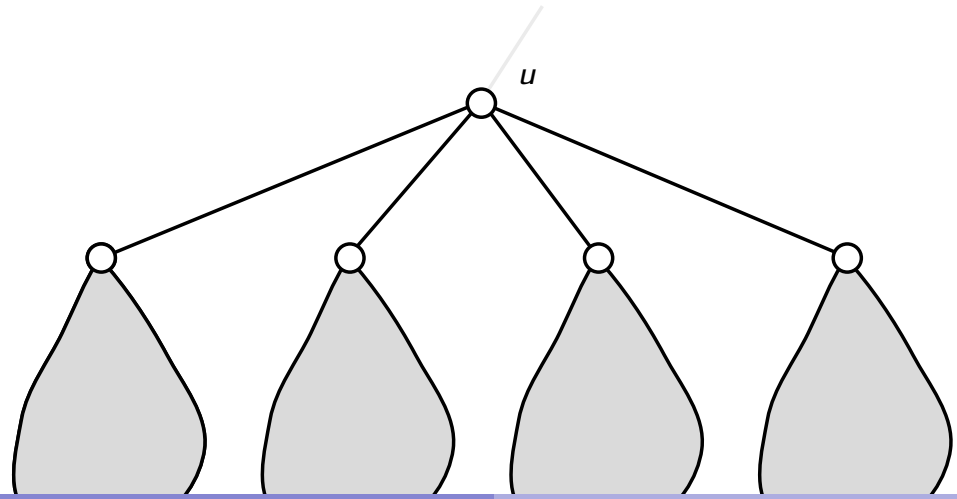
- **Answer**: $A[r]$.

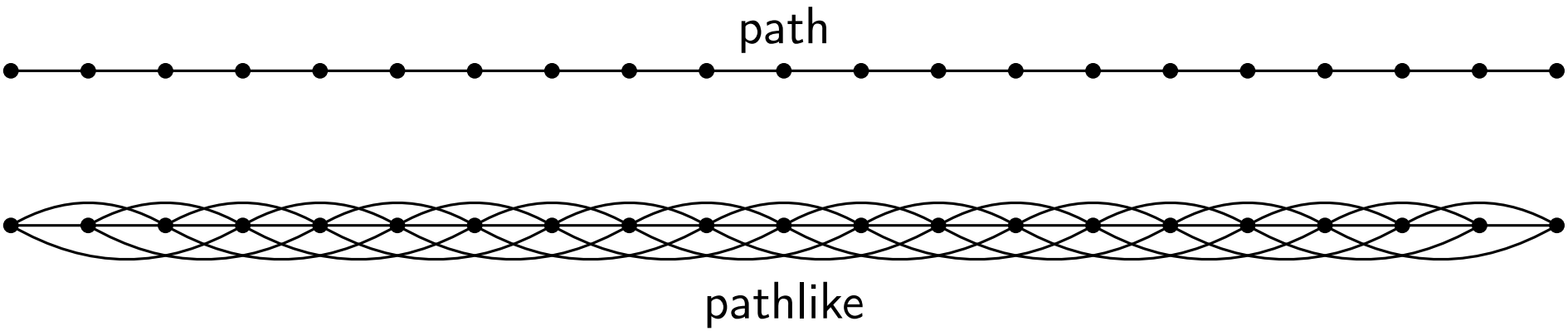


Independent Set on trees: Why does the DP work?

- Very limited dependence between vertices
- Essential information about subtree stored in a compact way.



Generalizations of paths and trees



Pathwidth

Graph of small pathwidth is path-like.
We hope that problems tractable on paths are also tractable on graphs of small pathwidth.

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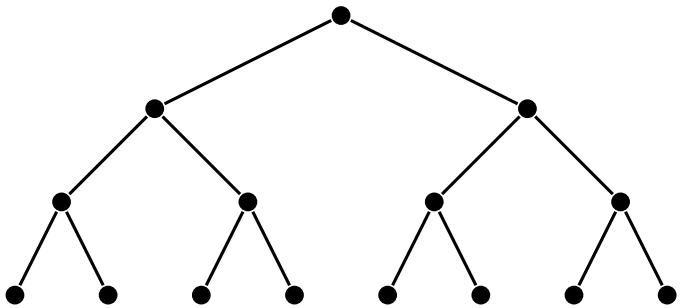

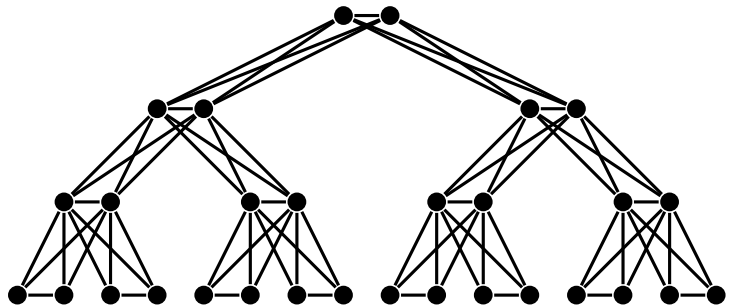

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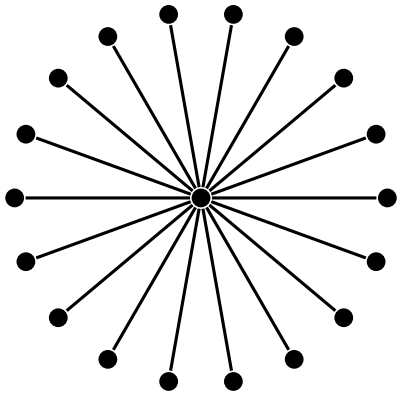
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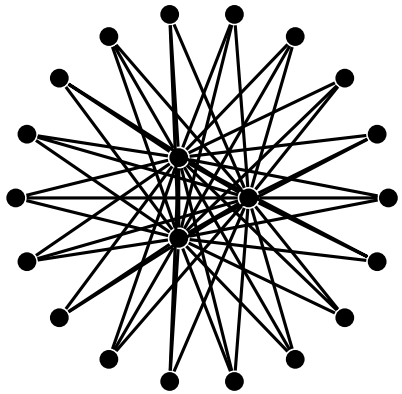
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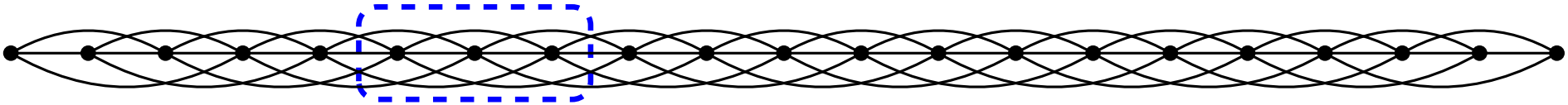

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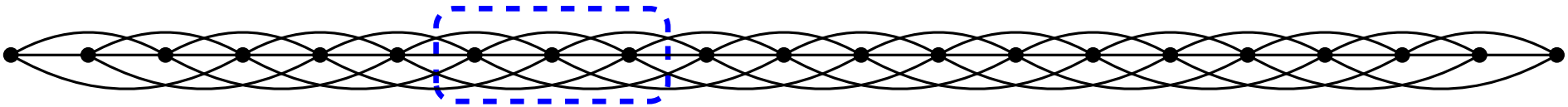

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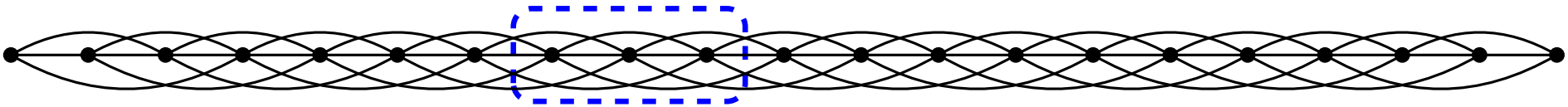

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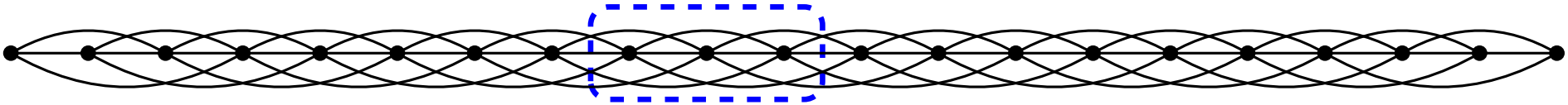

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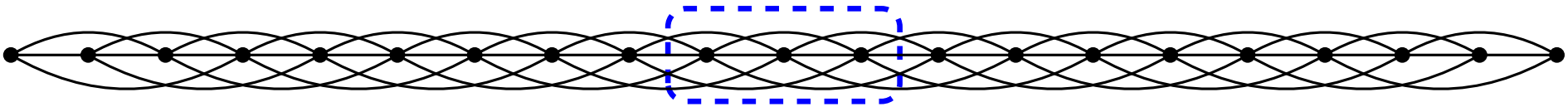

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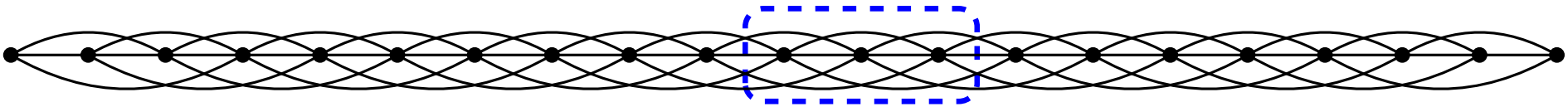

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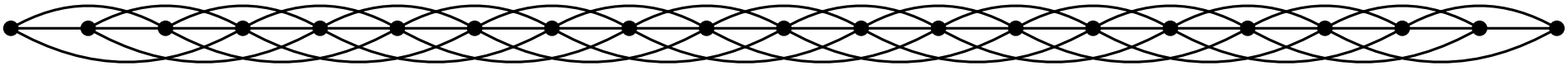

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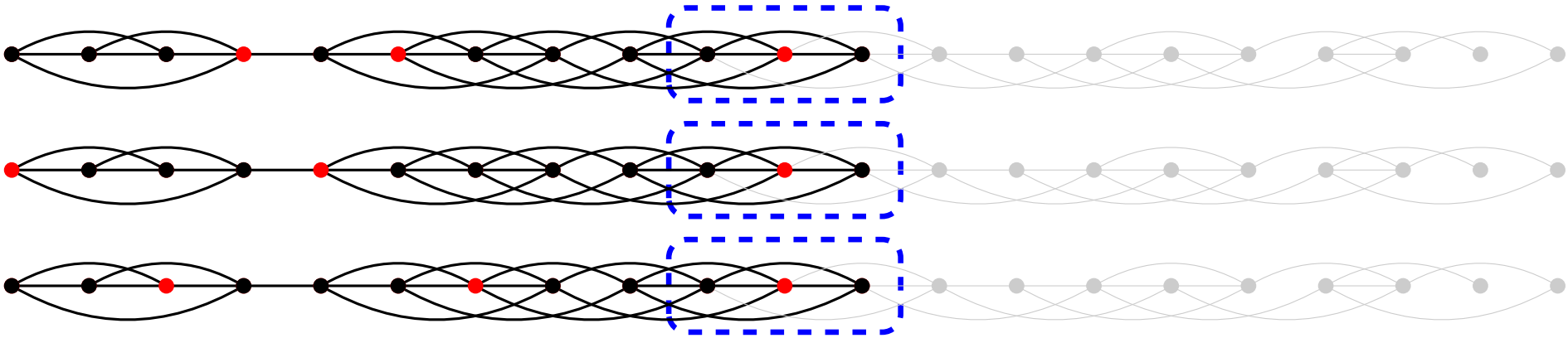

Definition (mapping to intervals)

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A graph G has *pathwidth* $\leq \mathrm{pw}$ if there exists a function $f : V \to \textit{intervals}$ such that:

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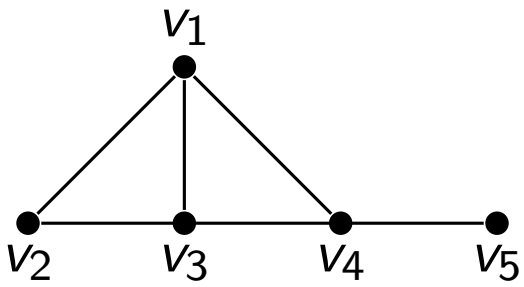

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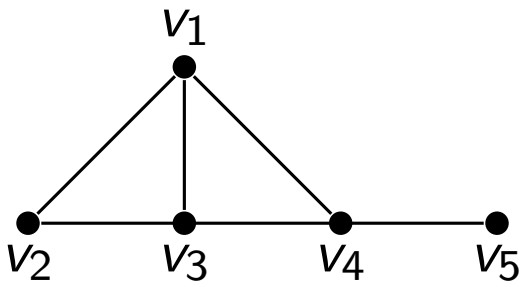

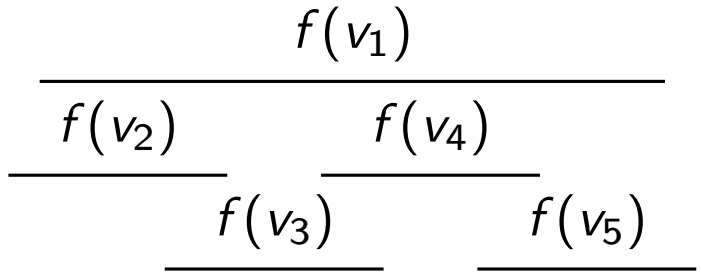

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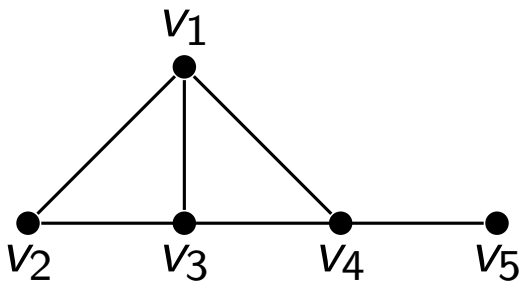
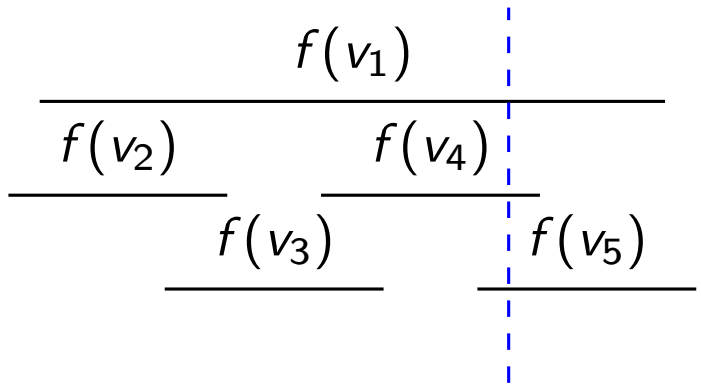
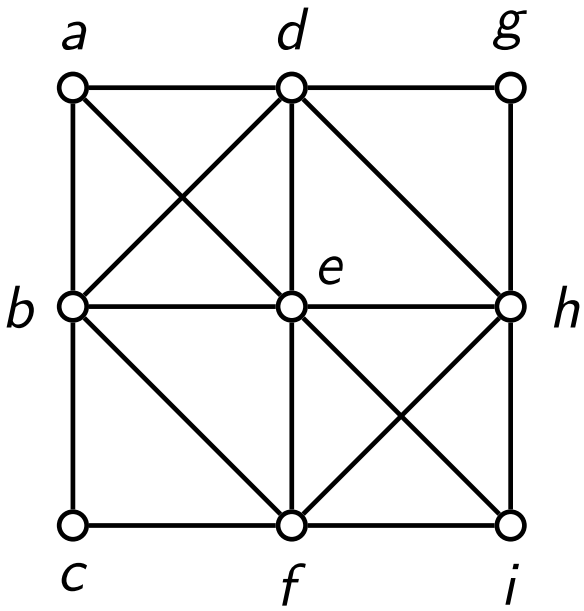




Path decompositions and pathwidth

Graph $G = (V, E)$



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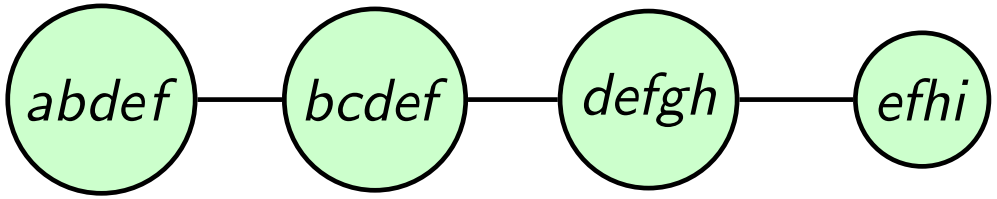

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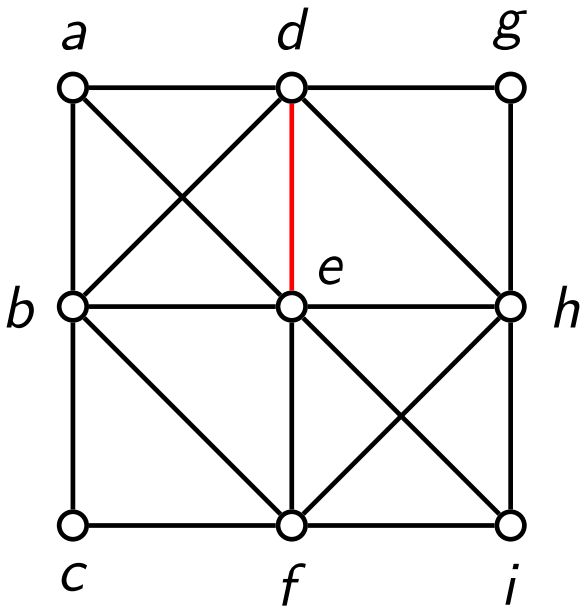

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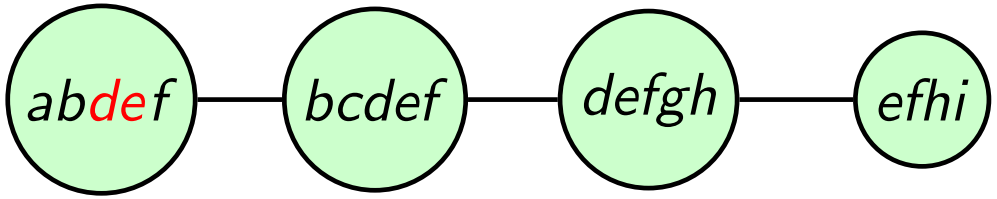

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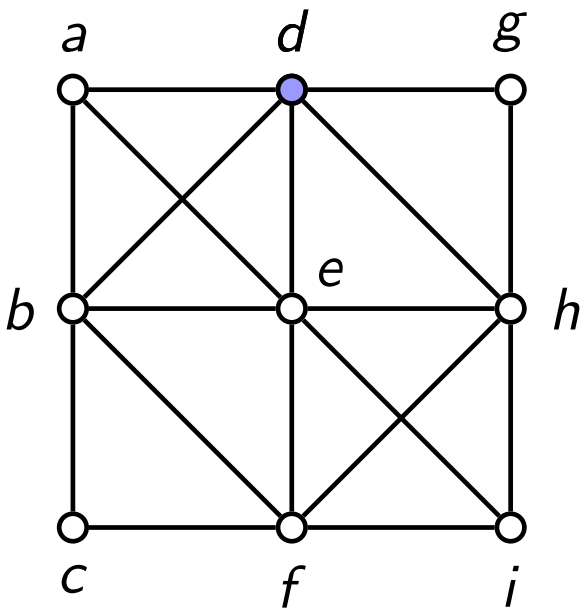

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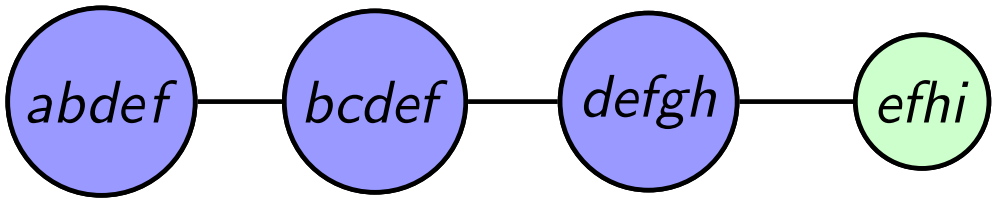

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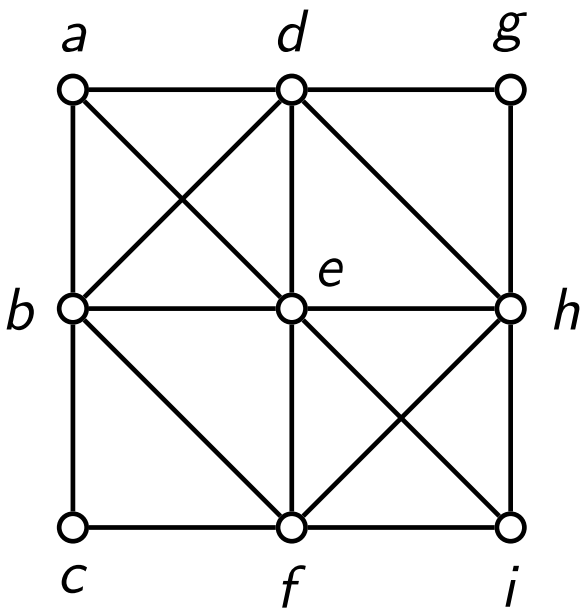

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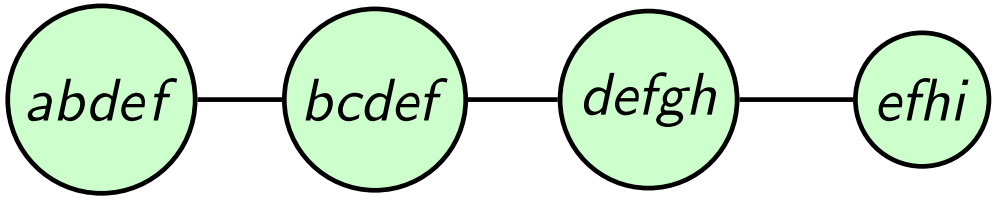

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Width of the decomposition: maximum bag size -1 (here: 4).

Pathwidth of G: minimum width of a decomposition of G .

Nice path decomposition

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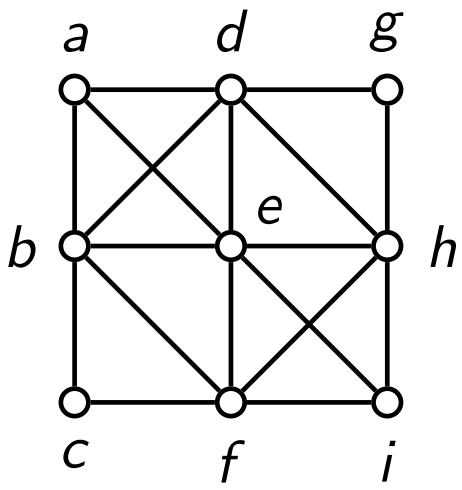

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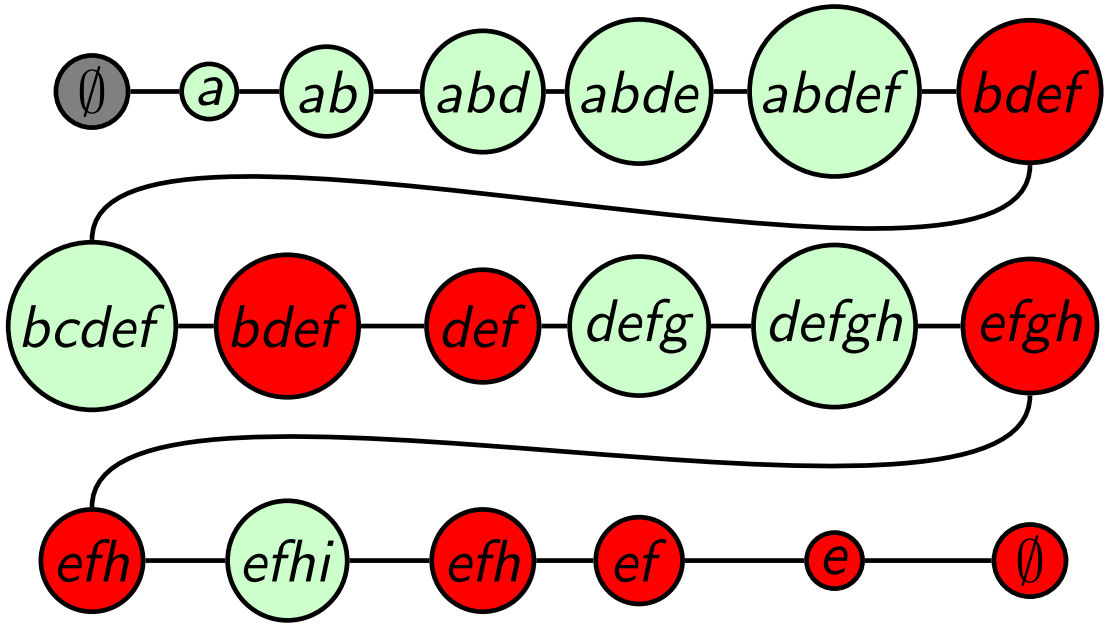

Nice path decomposition is a path decomposition $X_1, \ldots, X_r$ such that

- $X_1 = X_r = \emptyset$
- For $i = 2, \ldots, r$ bag X_i is of one of two types:
  1. Introduce bag: $X_i = X_{i-1} \cup \{v\}$ for some $v \in V \setminus X_{i-1}$,
  2. Forget bag: $X_i = X_{i-1} \setminus \{v\}$ for some $v \in X_{i-1}$.

(Easy) Theorem 1: Graph has a path decomposition of width w iff it has a nice path decomposition of width w (at most $2w$ times longer).

Nice path decomposition

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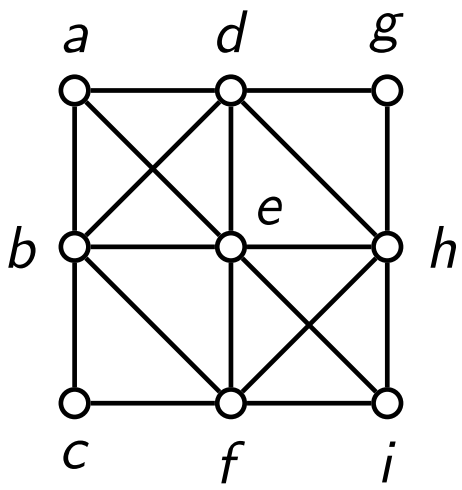
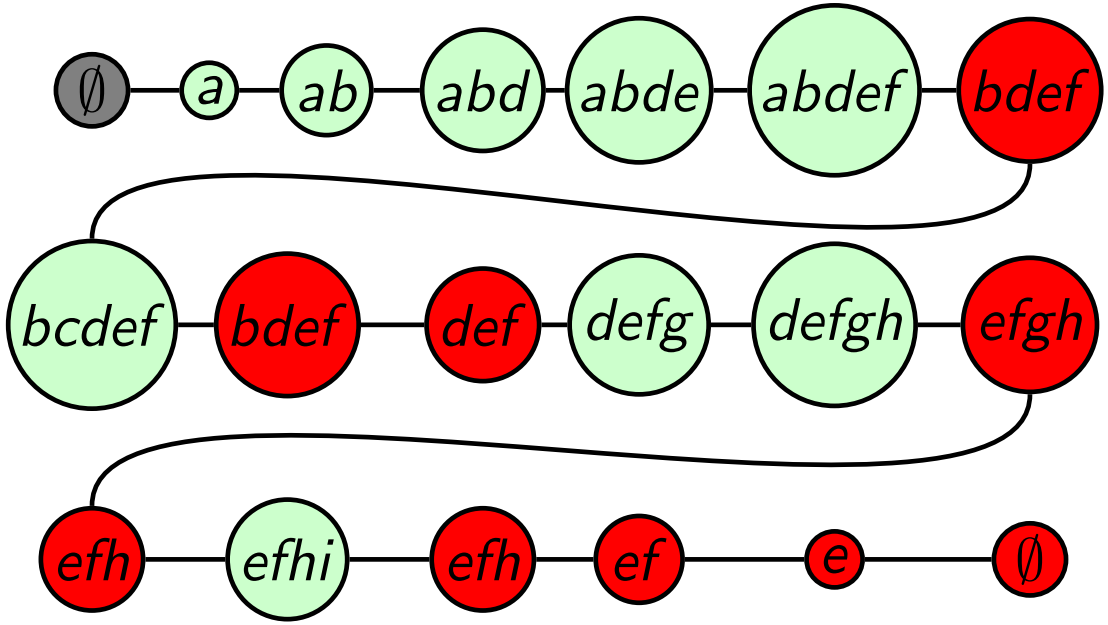


Path decomposition of G



Nice path decomposition is a path decomposition $X_1, \ldots, X_r$ such that

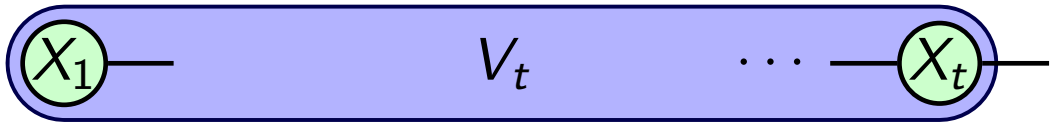
- $X_1 = X_r = \emptyset$
- For $i = 2, \ldots, r$ bag X_i is of one of two types:
  1. Introduce bag: $X_i = X_{i-1} \cup \{v\}$ for some $v \in V \setminus X_{i-1}$,
  2. Forget bag: $X_i = X_{i-1} \setminus \{v\}$ for some $v \in X_{i-1}$.

(Easy) Theorem 2: There is a mapping of G to p intervals iff G has nice path decomposition with bags of size p.

Dynamic programming for Independent Set

For every node t of a path decomposition of graph G:

- X_t = the bag at t,
- $V_t = \bigcup_{i \leq t} X_i$



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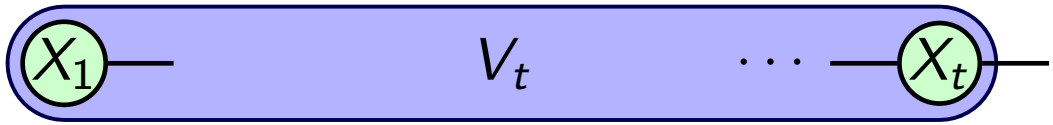

For every $f : X_i \to \{0, 1\}$ we want to compute the following.

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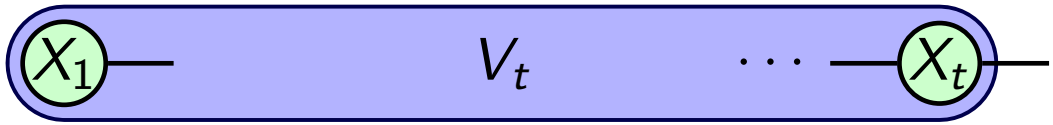

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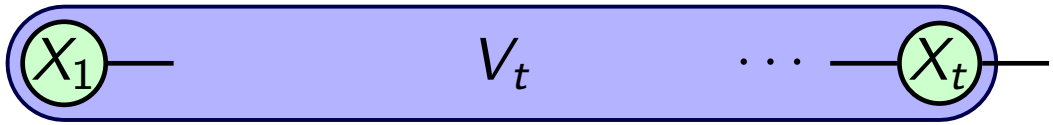

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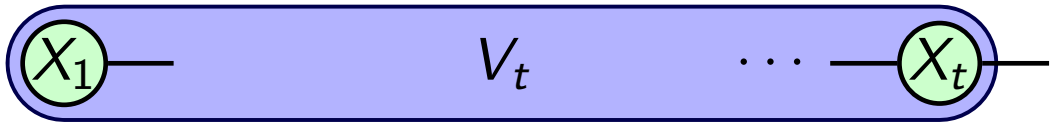

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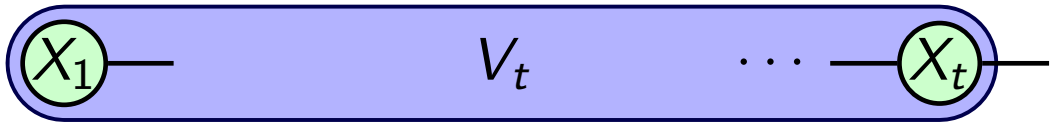

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- We now give recursive equations on $\Psi[\cdot, \cdot]$.
- $\Psi[i, \emptyset] = 0$ if $i = 1$.
- The answer is $\Psi[r, \emptyset]$.
- We follow convention that $\Psi[i, f] = -\infty$ if $f^{-1}(1)$ is already not independent.

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Introduce node i such that $X_i = X_{i-1} \cup \{v\}$.

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If $f(v) = 0$ then

$$\Psi[i, f] = \Psi[i - 1, f'].$$

If $f(v) = 1$ and there is $v' \in X_i$ s.t. $vv' \in E(G)$ and $f(v') = 1$, then

$$\Psi[i, f] = -\infty.$$

Otherwise, we have

$$\Psi[i, f] = 1 + \Psi[i - 1, f'].$$

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If there is $w' \in X_i$ such that $ww' \in E(G)$ and $f(w') = 1$, then

$$\Psi[i, f] = \Psi[i - 1, f[i \to 0]],$$

where

$$f[i \to \alpha](x) = \begin{cases} f(x) & x \neq v \\ \alpha & x = v. \end{cases}$$

Otherwise, we have

$$\Psi[i, f] = \max(\Psi[i - 1, f[w \to 0]], \Psi[i - 1, f[w \to 1]]).$$

DP for Independent Set: conclusion for pathwidth

Theorem

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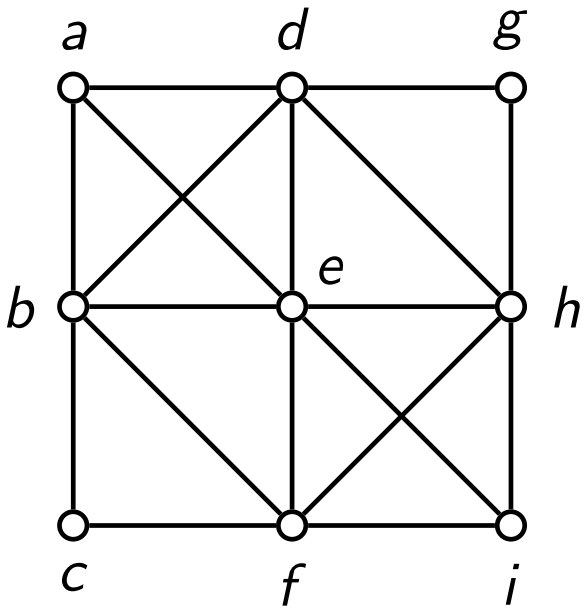
Tree decompositions and treewidth

Graph $G = (V, E)$



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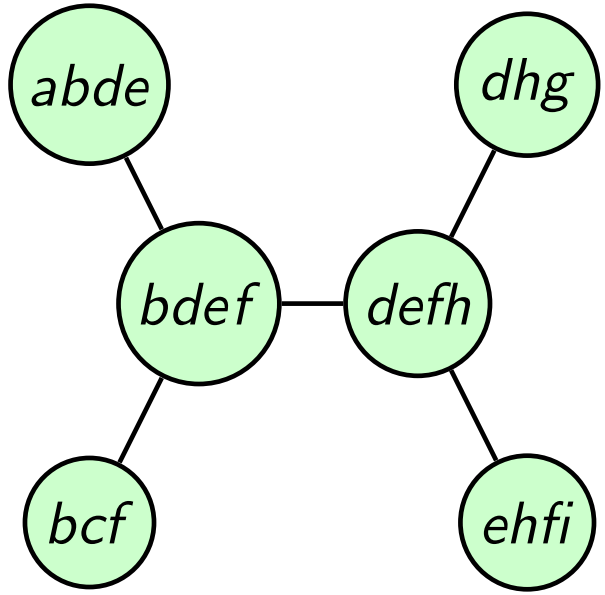

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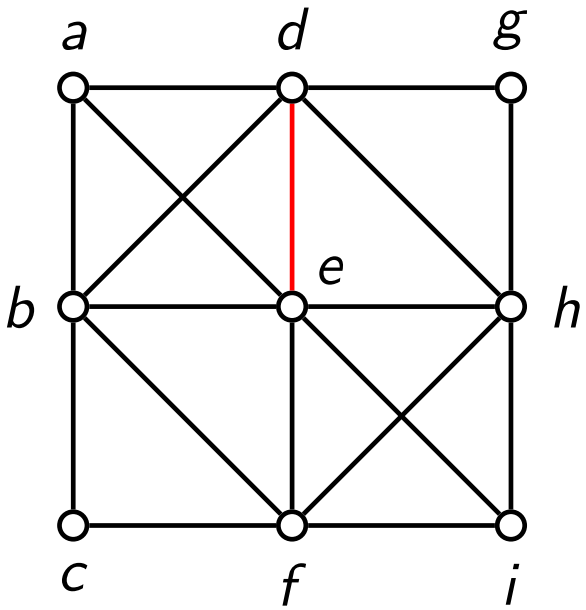

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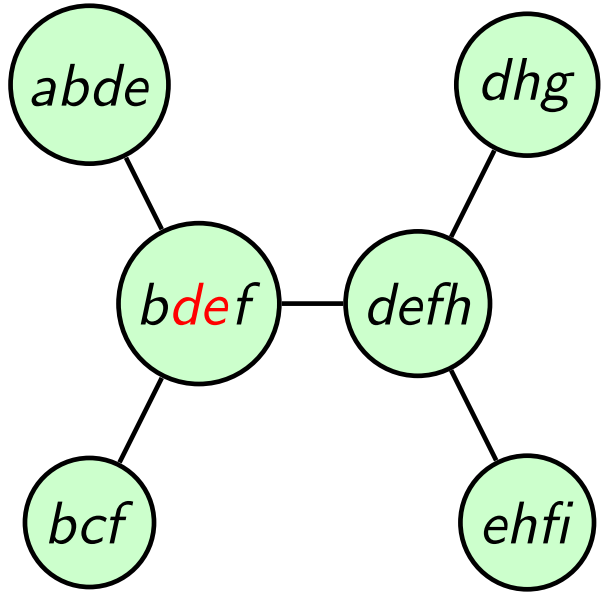

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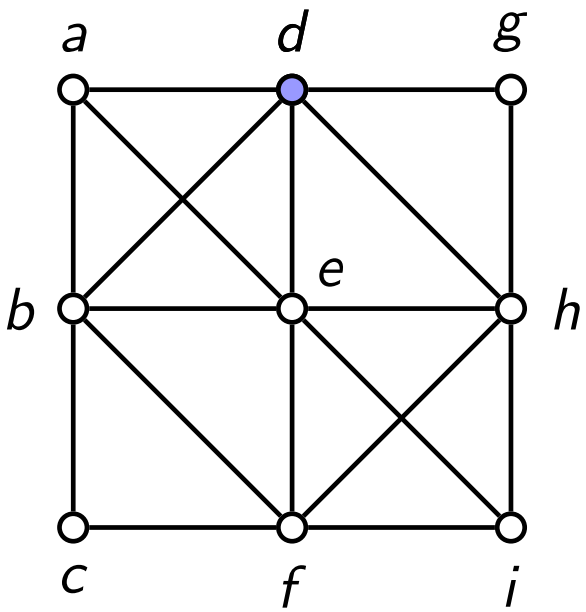

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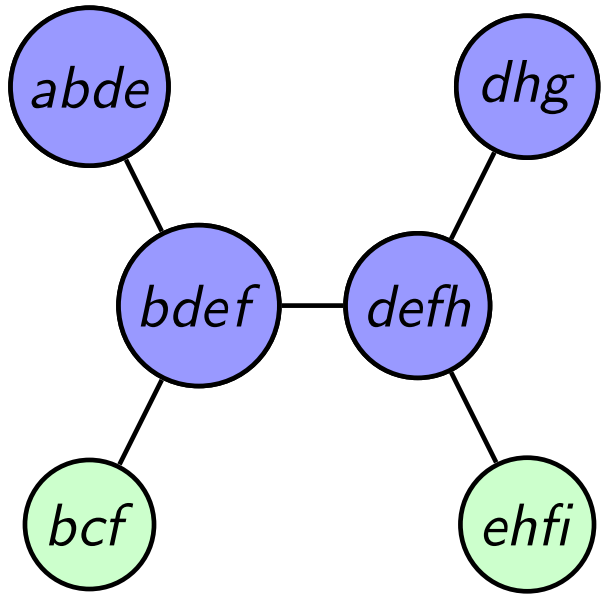

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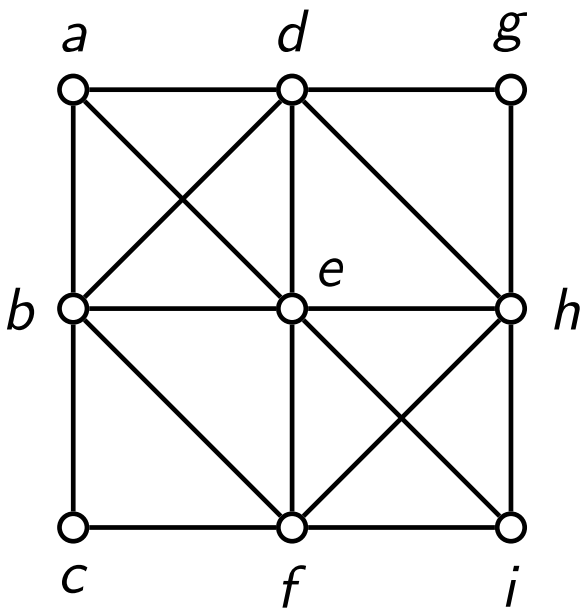

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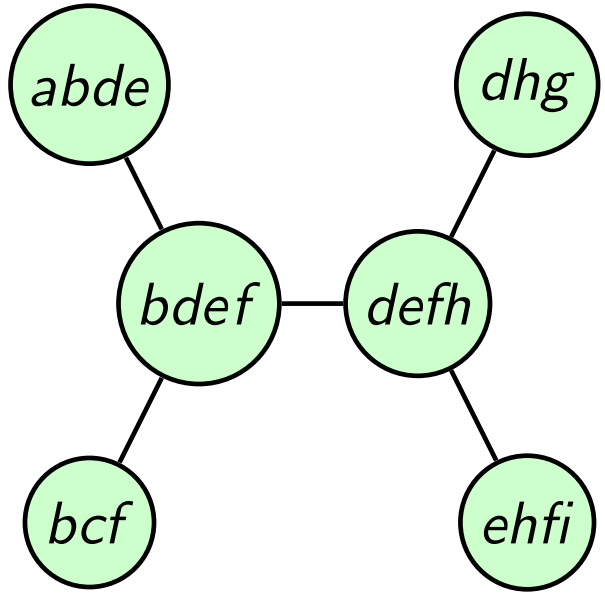

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Width of the decomposition: maximum bag size -1 (here: 3).

Treewidth of G: minimum width of a decomposition of G .

Nice tree decompositions

Nice tree decomposition is a rooted tree decomp. $(T, \{X_i\}_{i \in V(T)})$ s.t.

- $X_i = \emptyset$ for every leaf i,
- $X_r = \emptyset$ for the root r,
- Every non-leaf node $i \in V(T)$ is of one of three types:
  1. Introduce: $X_i = X_j \cup \{v\}$ for $v \in V \setminus X_j$, where j is the only child of i,
  2. Forget: $X_i = X_j \setminus \{v\}$ for $v \in X_j$, where j is the only child of i,
  3. Join: $X_i = X_j = X_k$ where j and k are the only two children of i.

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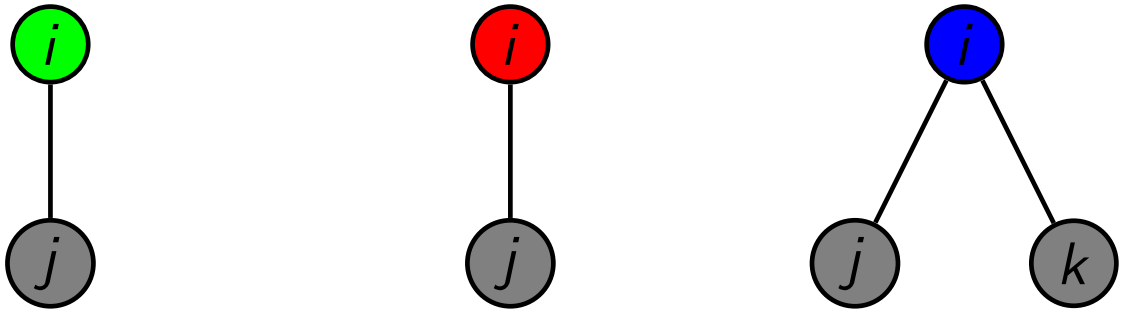
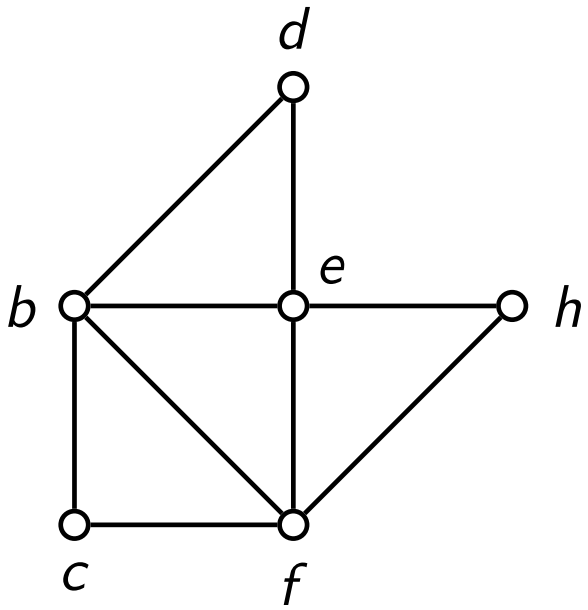
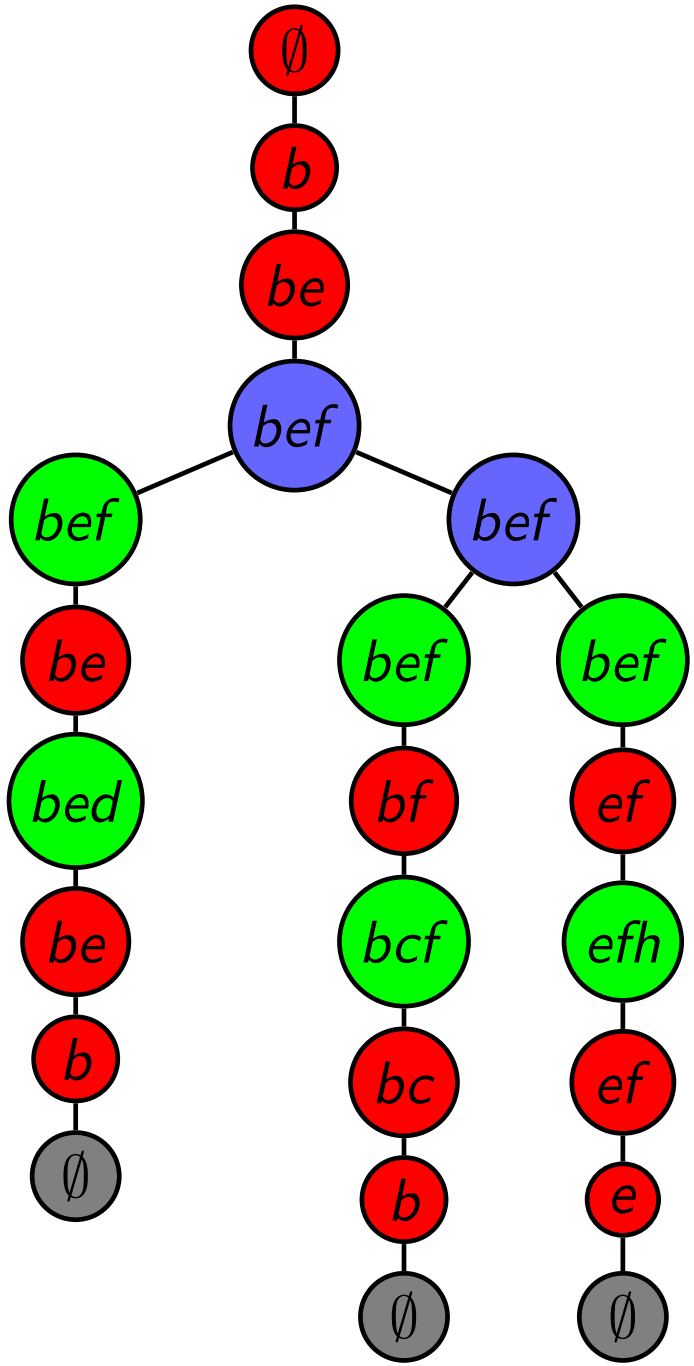


Introduce node Forget node Join node

Example





(Easy) Theorem 1: Graph has a tree decomposition of width w iff it has a nice tree decomposition of width w (at most $2w$ times larger).

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For every node t of a tree decomposition of graph G:

- X_t = the bag at t,
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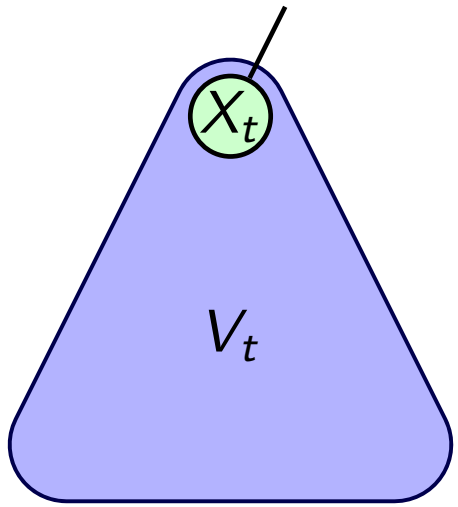

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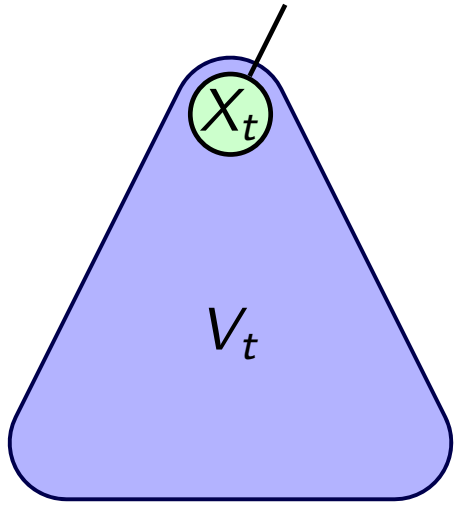

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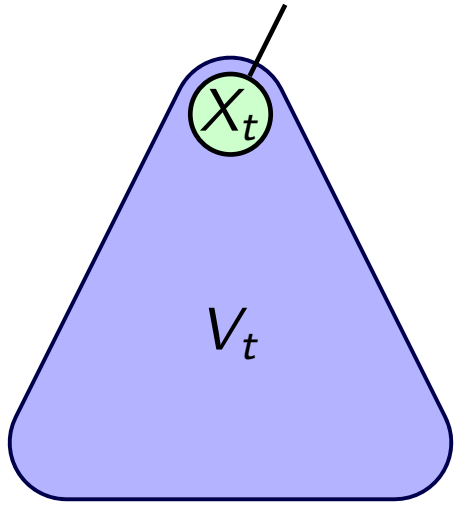

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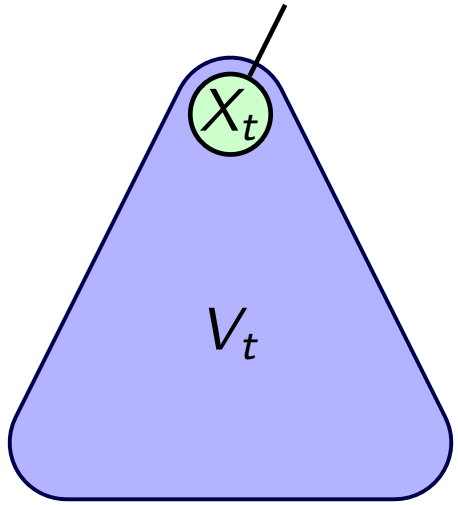

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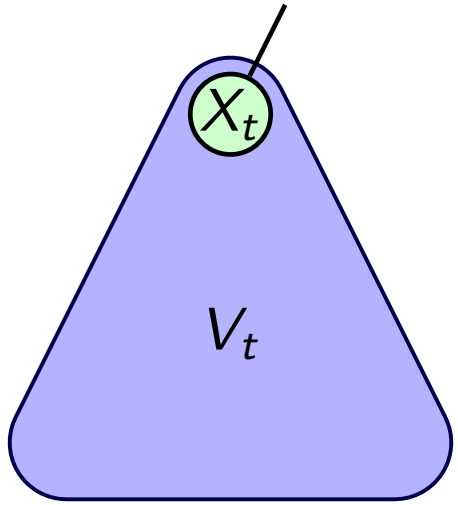

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$$\Psi[i, f] = \Psi[j, f] + \Psi[k, f] - |f^{-1}(1)|.$$

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  - Everybody discovers his or her own tricks after writing the first five DPs on treewidth.

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- **Overall**: the concept of bounded treewidth graphs presents a fundamental link between various fields of combinatorics and computer science.

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- Best known poly-time approximation has apx ratio $\mathcal{O}(\sqrt{\log OPT})$.

Computation of treewidth

- If we want to run a dynamic programming algorithm, we need to have a decomposition of small width.
- Computing treewidth optimally is NP-hard.
- Moreover, constant-factor approximation is not possible under the Small Set Expansion Conjecture.
- Best known poly-time approximation has apx ratio $\mathcal{O}(\sqrt{\log OPT})$.
- **Idea**: If we run an FPT algorithm anyways, we can afford FPT running time of computing treewidth.

Algorithms for treewidth

Input: Graph G and integer k

Robertson-Seymour 4-approximation (GM XIII, '95)

Output: Either conclusion that $\mathrm{tw}(G) > k$, or a tree decomposition of width $\leq 4k + 4$

Runtime: $\mathcal{O}(8^k k^2 \cdot n^2)$

Bodlaender's algorithm ('96)

Output: Either conclusion that $\mathrm{tw}(G) > k$, or a tree decomposition of width $\leq k$

Runtime: $k^{\mathcal{O}(k^3)} \cdot n$

BDDFLP algorithm ('13)

Output: Either conclusion that $\mathrm{tw}(G) > k$, or a tree decomposition of width $\leq 5k + 4$

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- Treewidth is one of the main technical tools.

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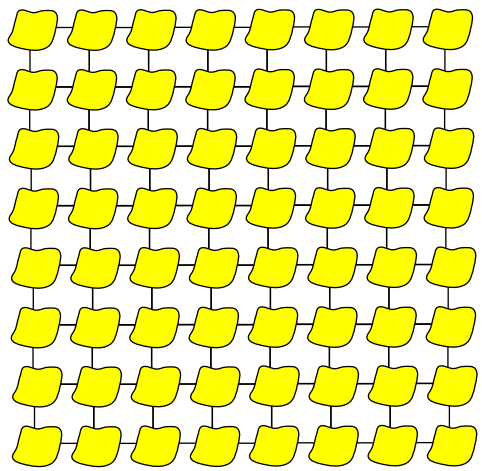
There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that if $\mathrm{tw}(G) > g(k)$, then G contains $k \times k$ grid as a minor.



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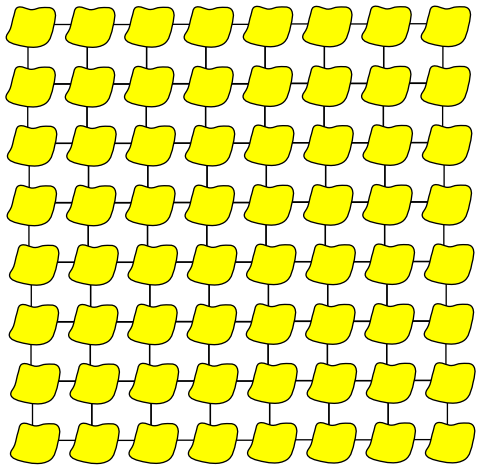

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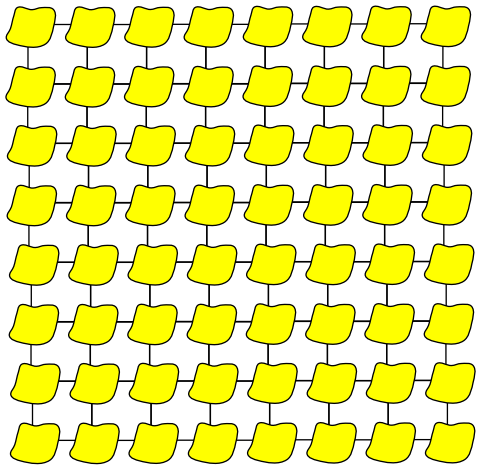

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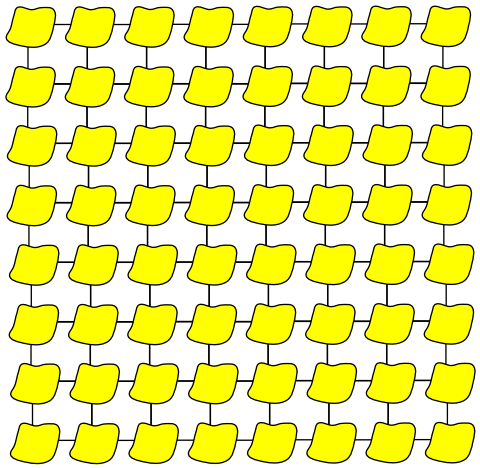

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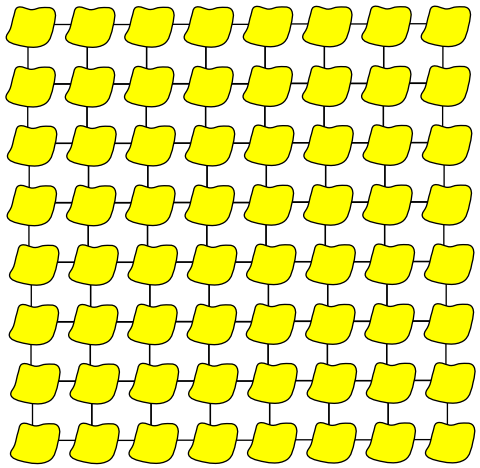
- **Important**: If G is planar (or H-minor-free), then $g(k)$ is **linear**.



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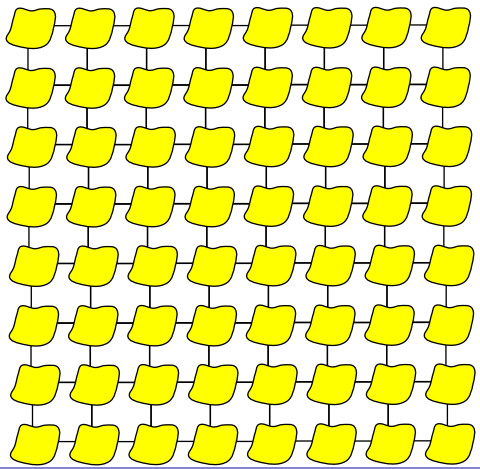

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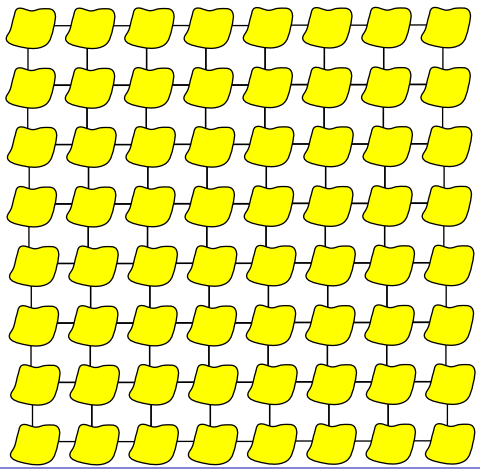

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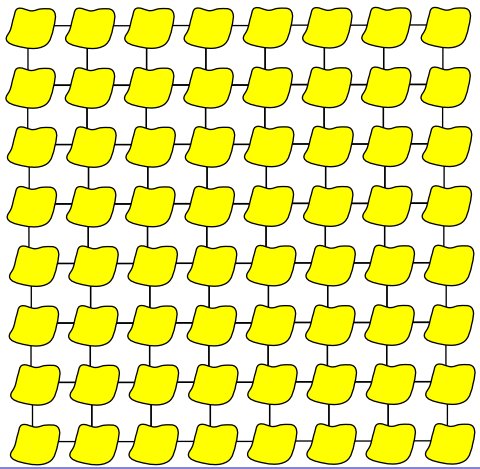

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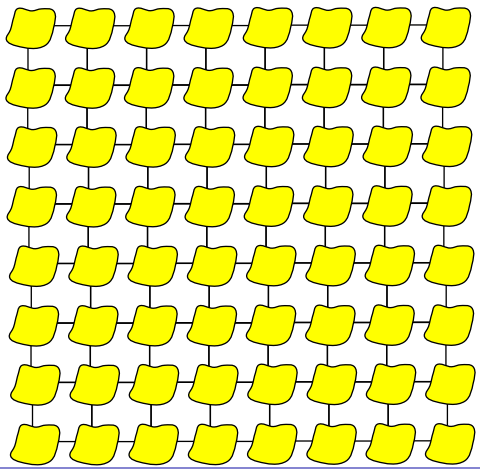

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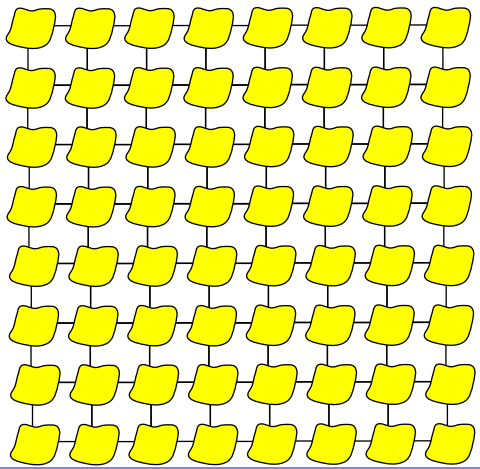

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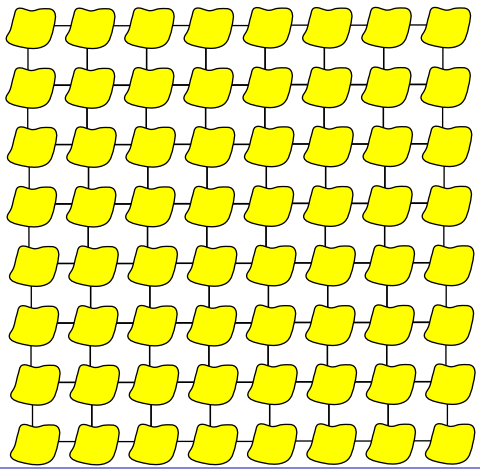

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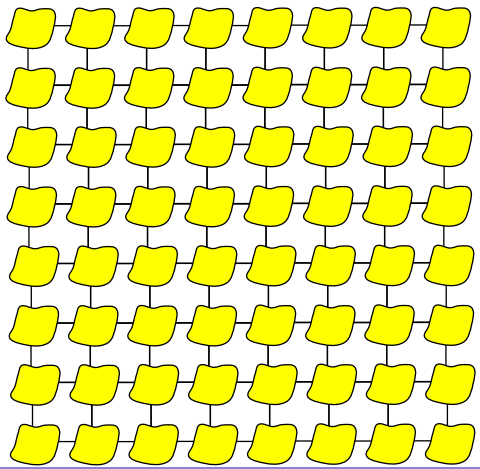

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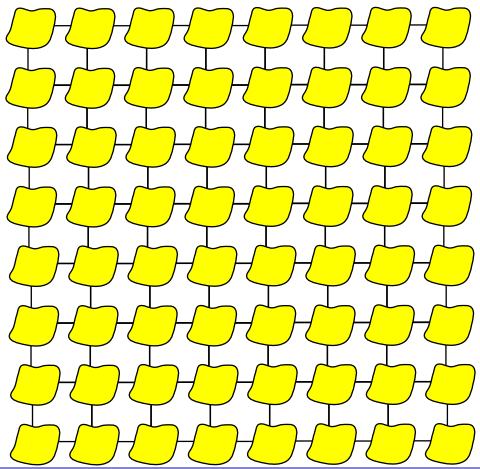

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