Universal Reconstruction of a String

Paweł Gawrychowski, Tomasz Kociumaka, Jakub Radoszewski,
Wojciech Rytter and Tomasz Waleń

University of Warsaw, Poland

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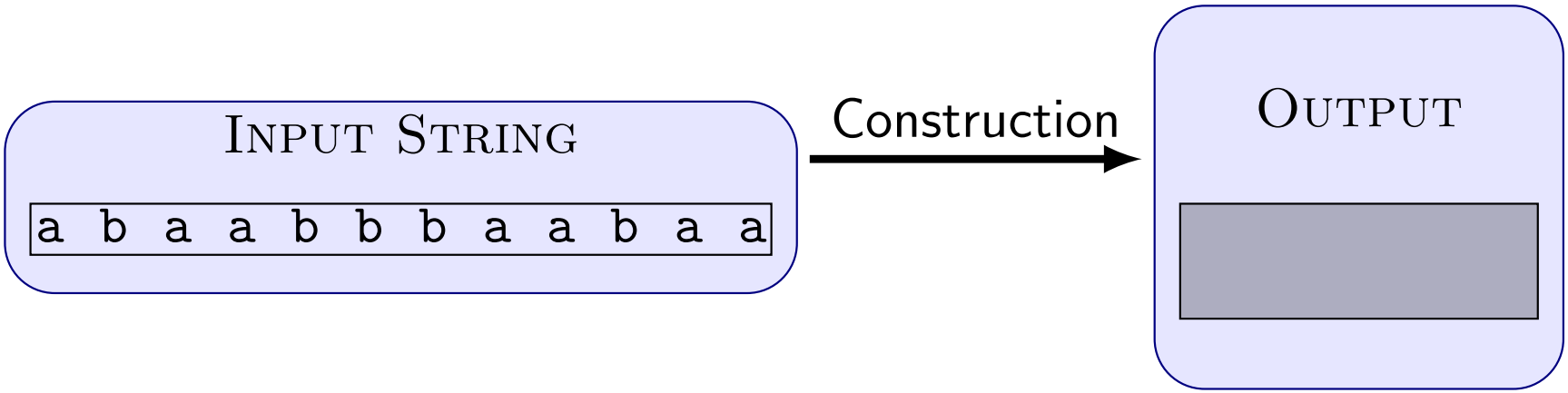
Reconstruction Problems



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Given a string w compute $\mathbf{F}(w)$.

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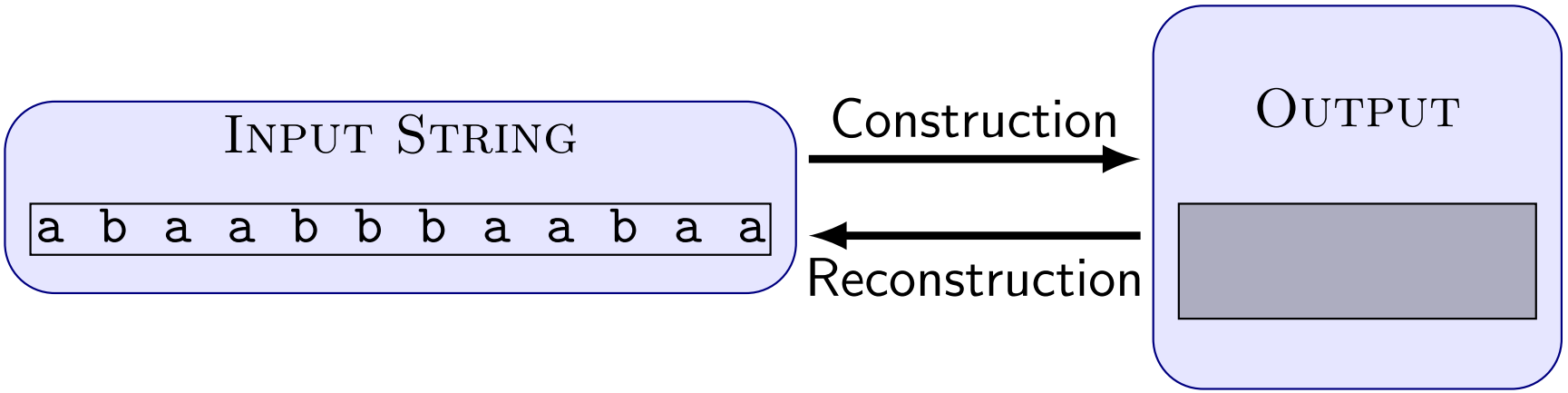

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Given F find a string w satisfying $F = \mathbf{F}(w)$.

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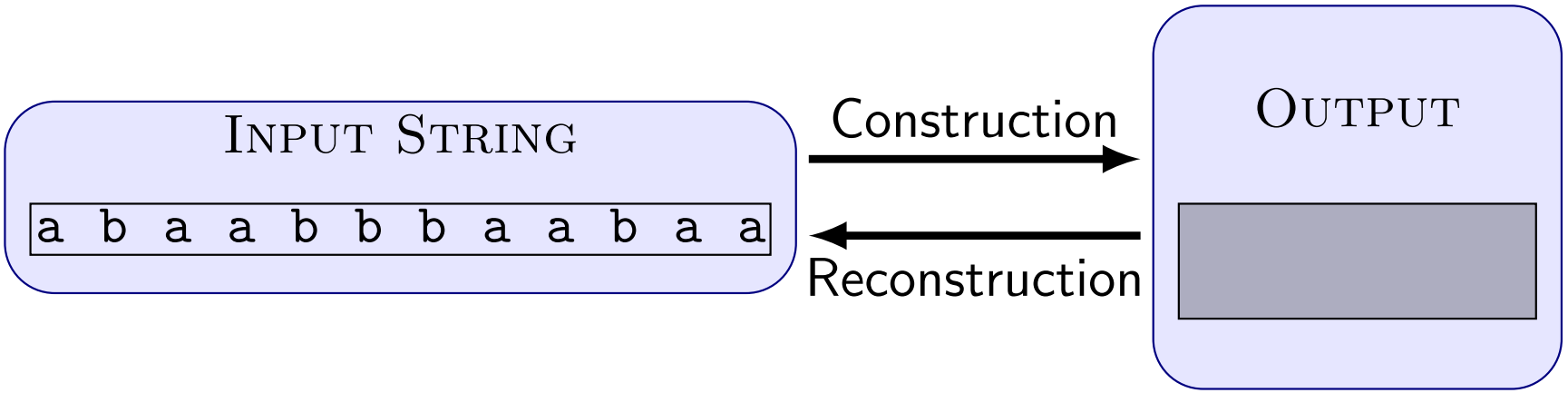

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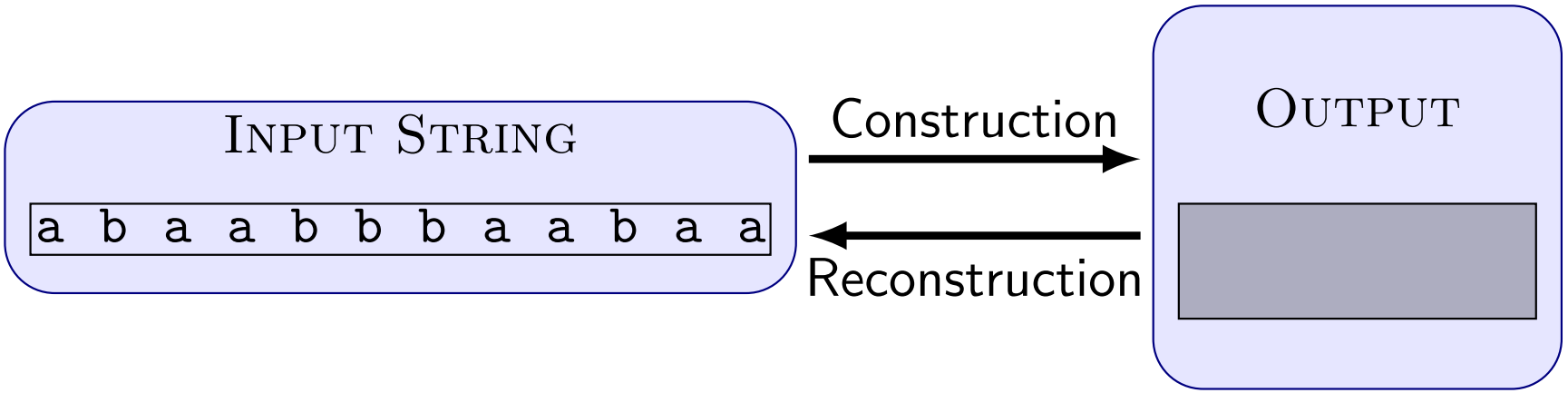

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Main motivation:

- develop a good understanding of $\mathbf{F}$,
- aid computer experiments for combinatorics on words.

Systems of Substring Equations

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$$E = \{ \qquad\qquad n = 9 \qquad\qquad \}$$

w :	?	?	?	?	?	?	?	?	?
	0	1	2	3	4	5	6	7	8

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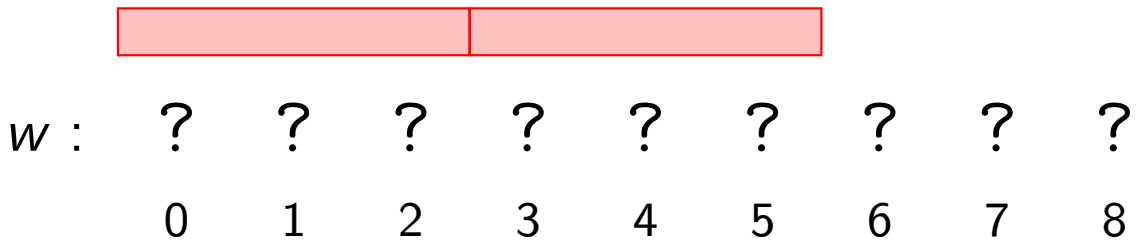

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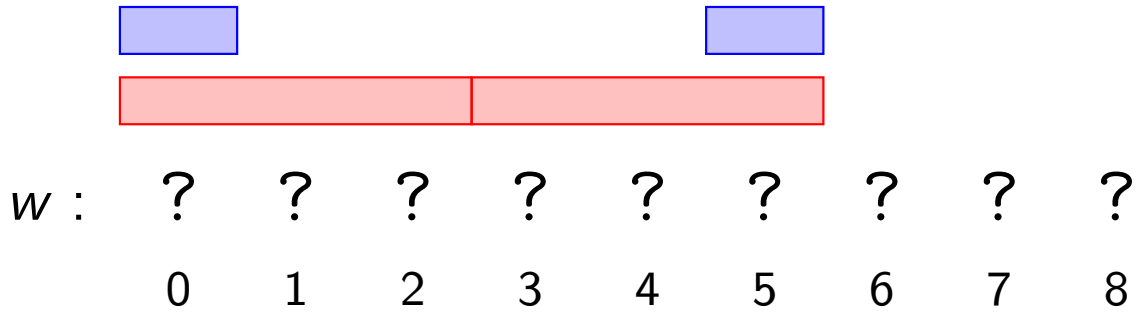

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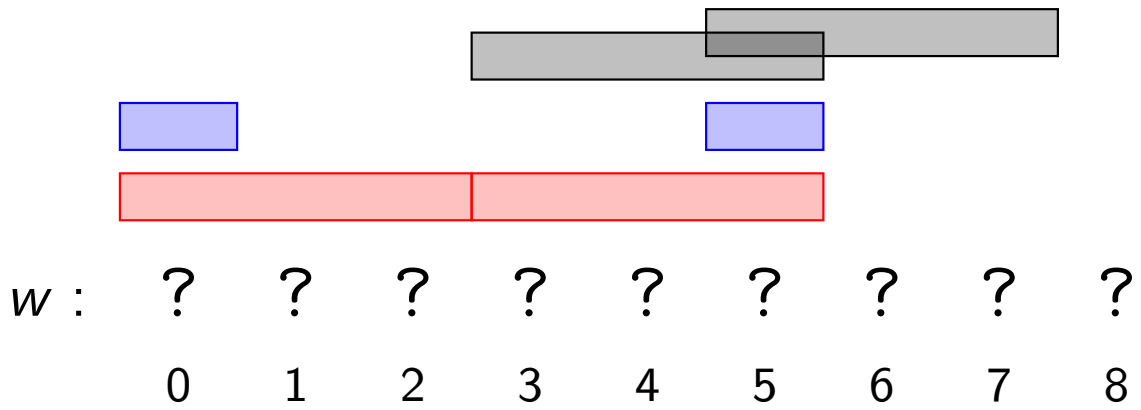

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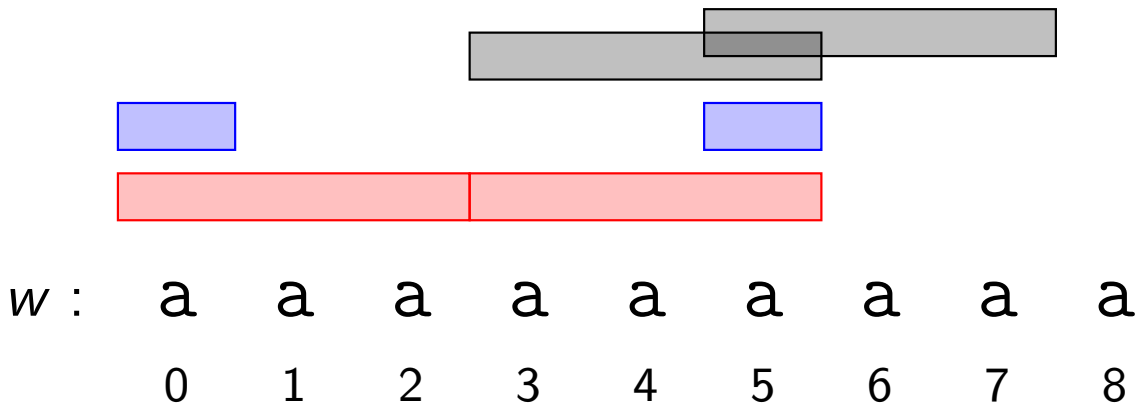

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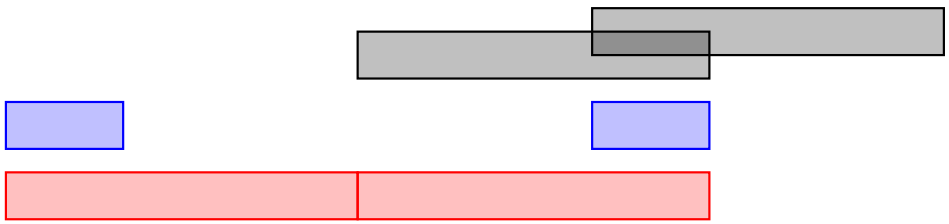

$\Phi(E) = w$:

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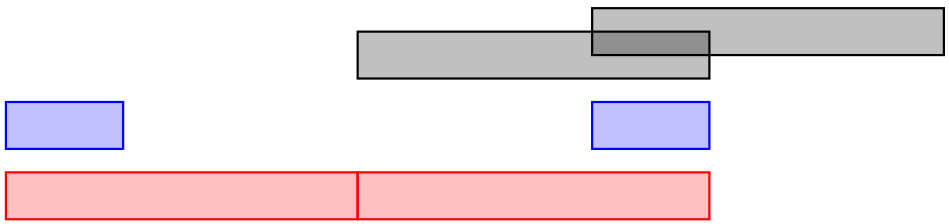

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Solutions of E = images of $\Phi(E)$ via letter-to-letter morphisms.

Our Results

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Several structures can be characterized using substring equations:

- border table (Morris-Pratt failure table),
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- maximal repetitions (runs),
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Details later in the talk.

Solving Systems of Substring Equations

Positions Graph

A substrings equation represents several letter equations.

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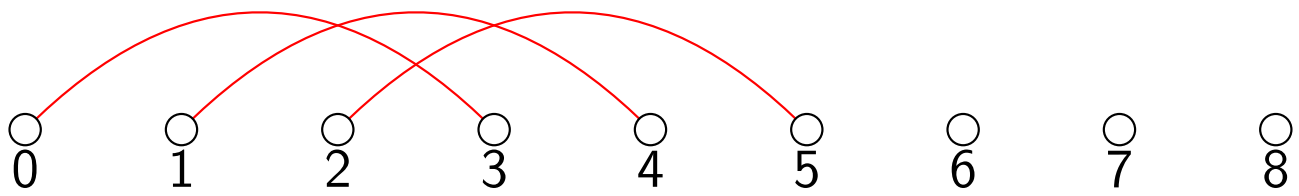

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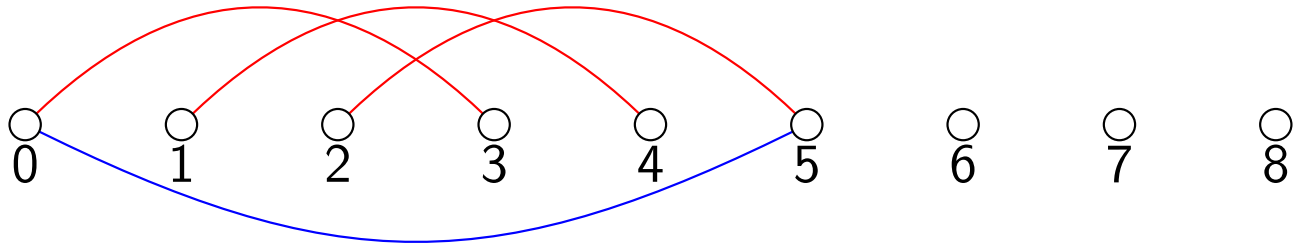

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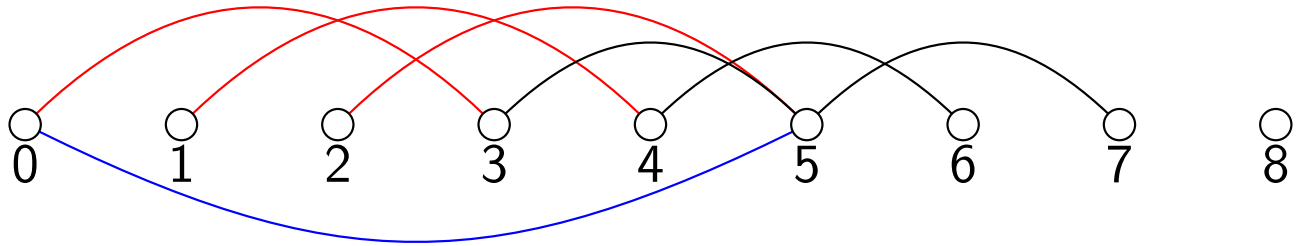

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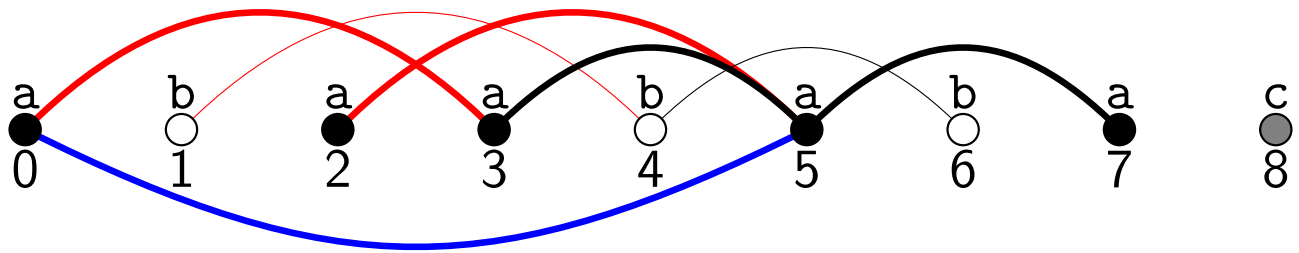

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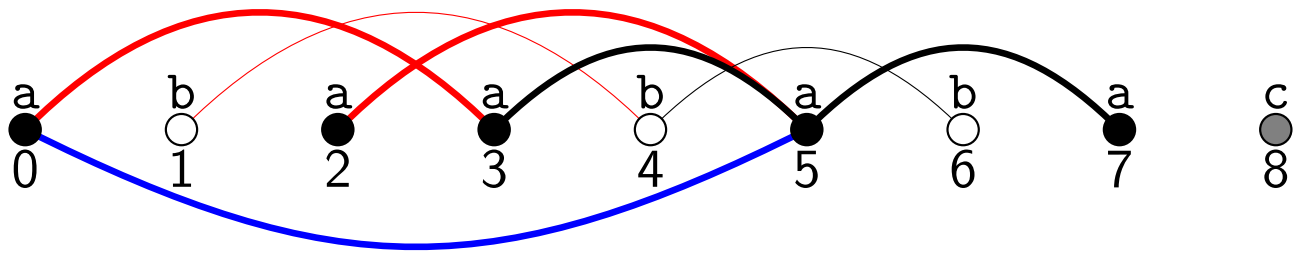

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Observation

A system can be solved in $\mathcal{O}(n + \|E\|)$ time ($\|E\|$ = total length).

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How to make equations shorter?

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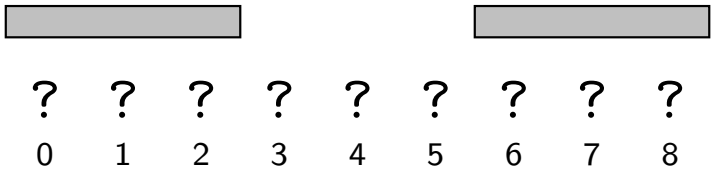

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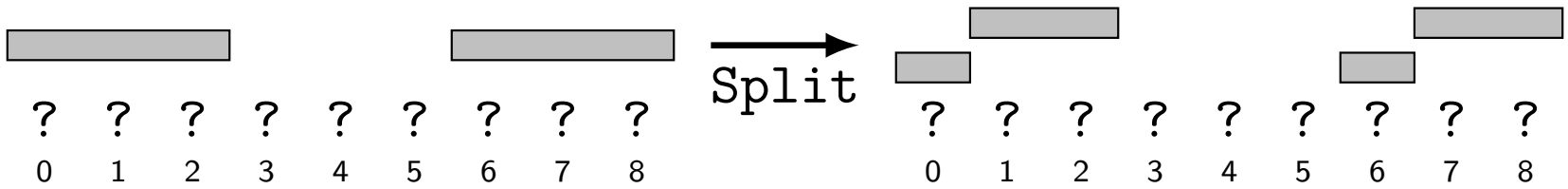

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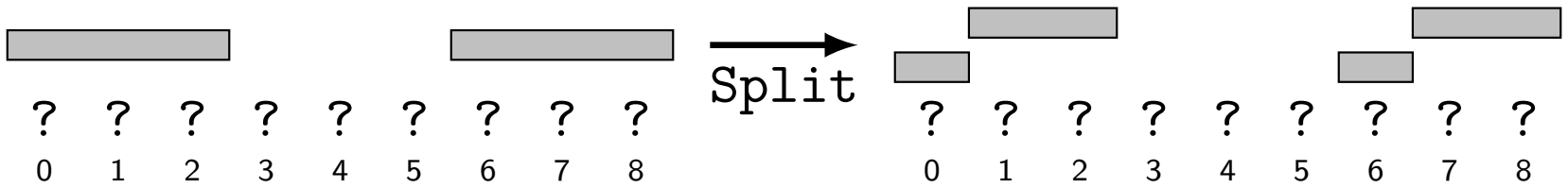

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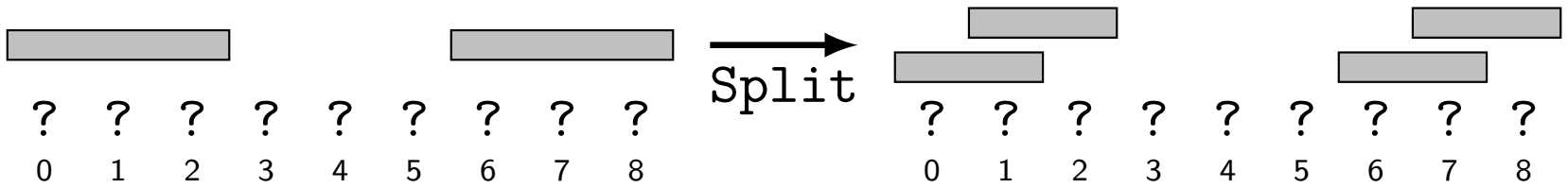

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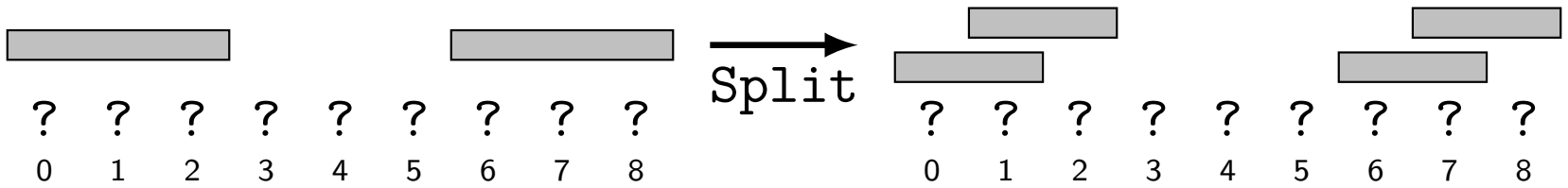

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Side effect:

- increase the number of equations.

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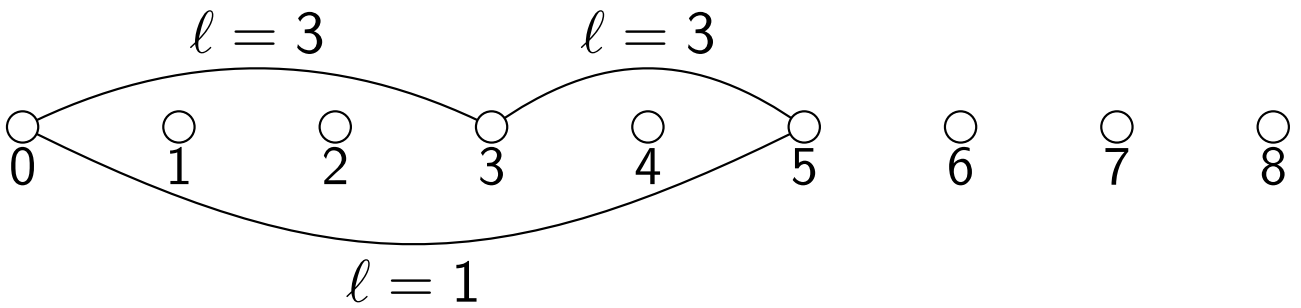
$G(E)$



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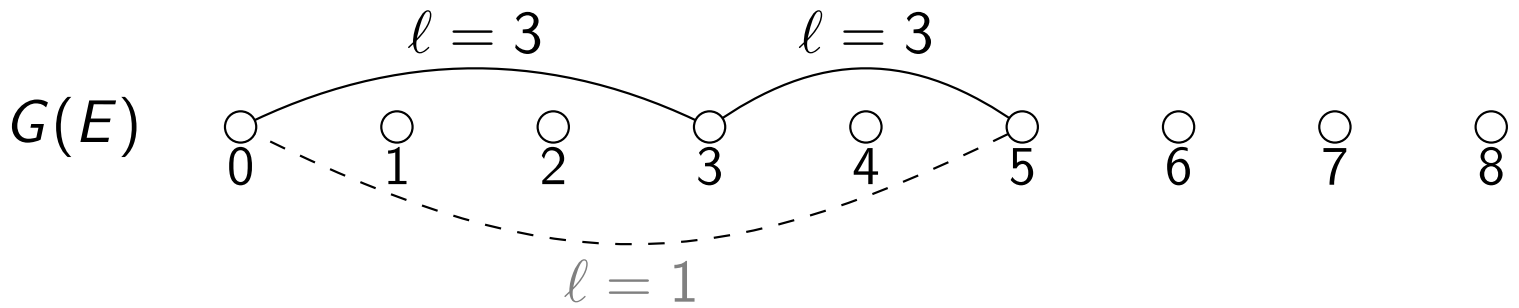

Lemma

$E \equiv MST(G(E))$ where MST is the max.-weight spanning forest.

$\mathcal{O}(n \log n + |E|)$-time Algorithm

Alternately apply

- `Split` to decrease the maximum length (by a factor of two),
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2. For $k := \lfloor \log n \rfloor$ downto 0
   1. $E_k := \{w[p..q] = w[p'..q'] \in E : 2^k \leq |w[p..q]| < 2^{k+1}\}$
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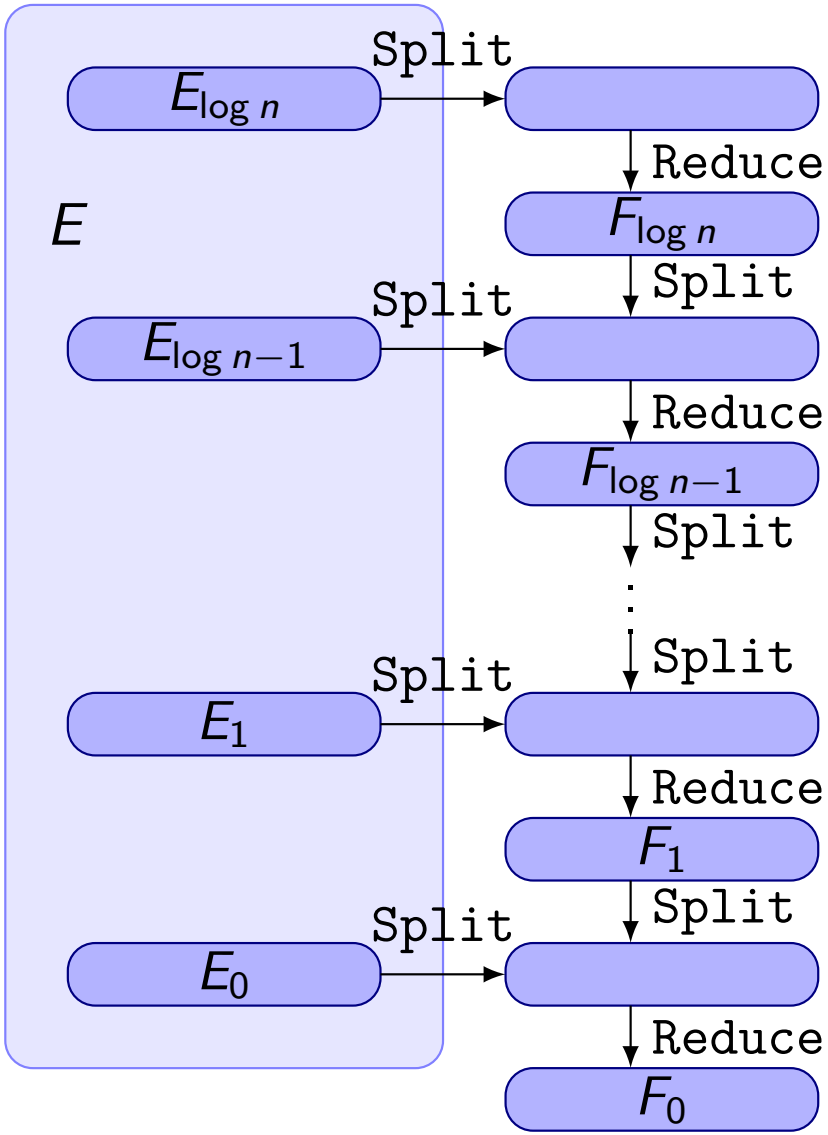
Speed-up Idea



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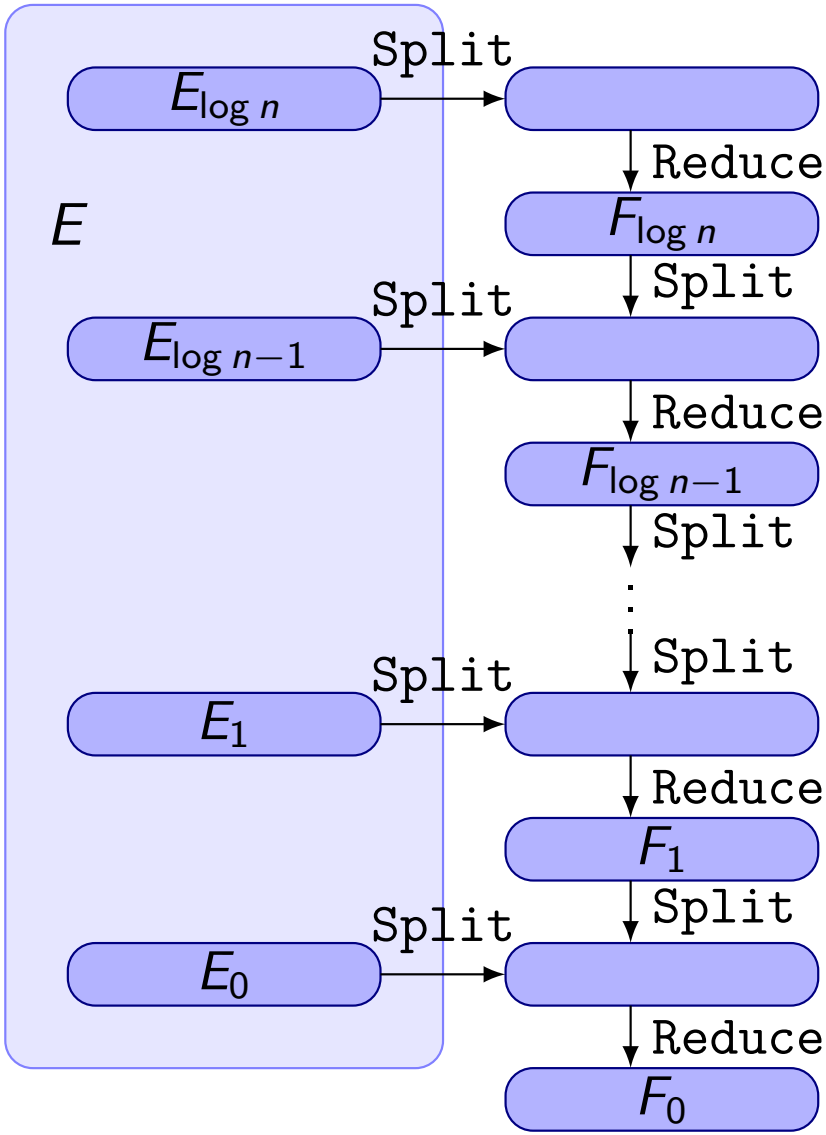

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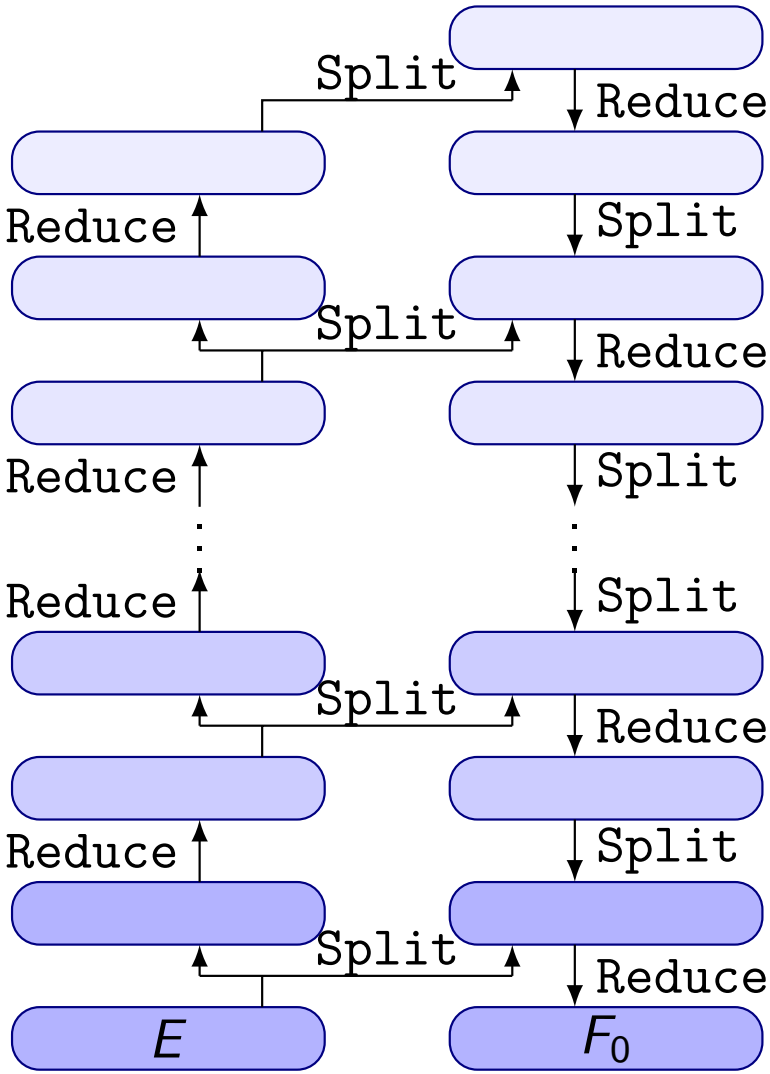

Idea: few positions of long equations.

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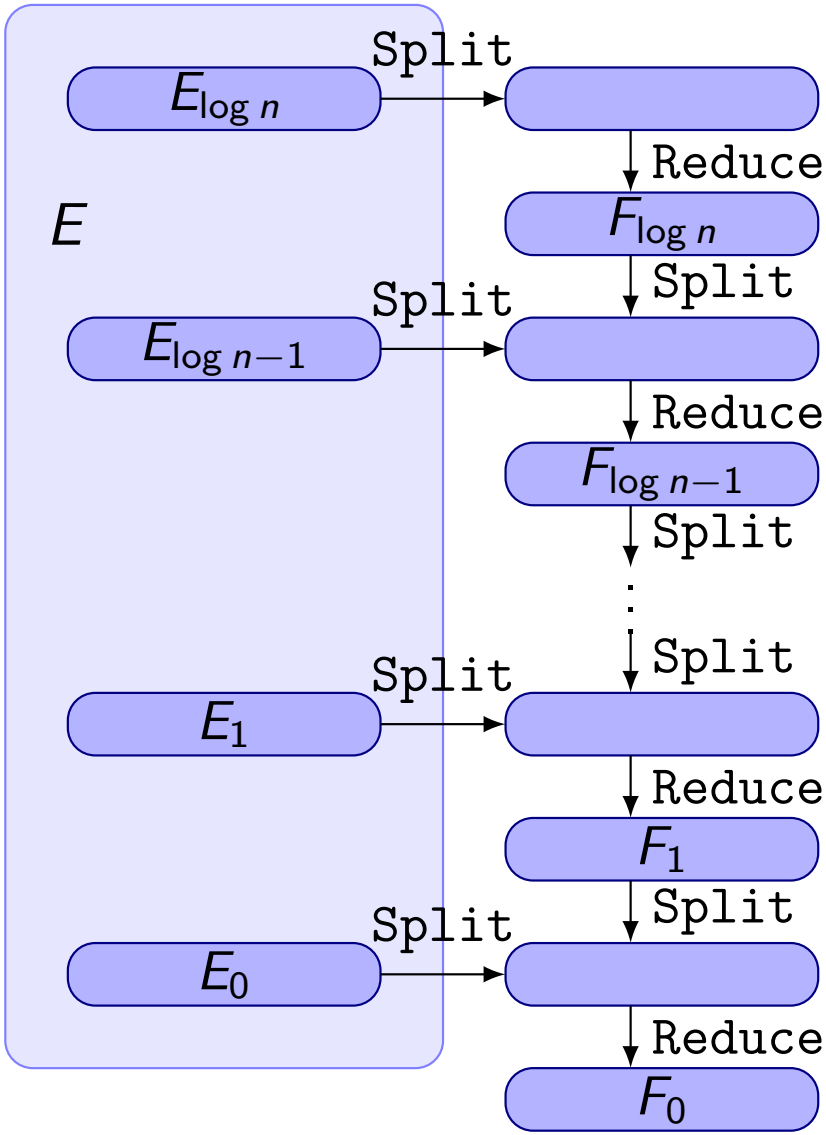

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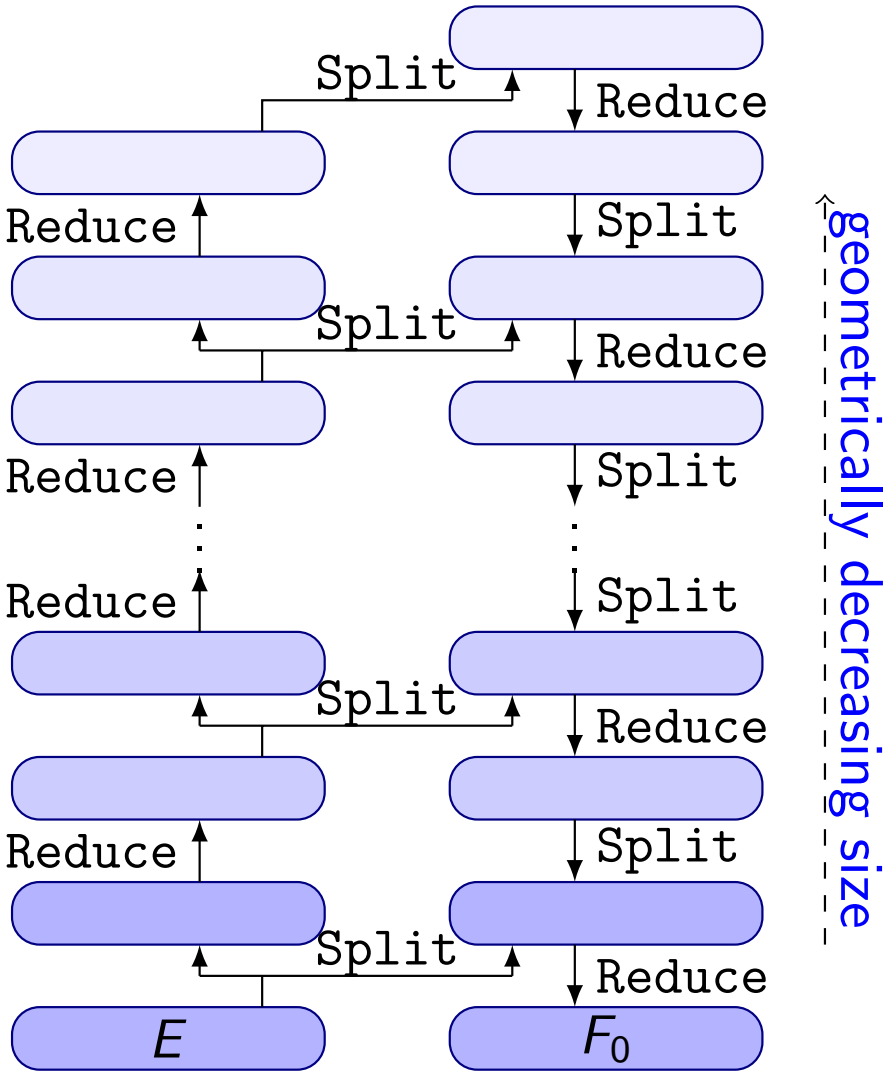

Idea: few positions of long equations.

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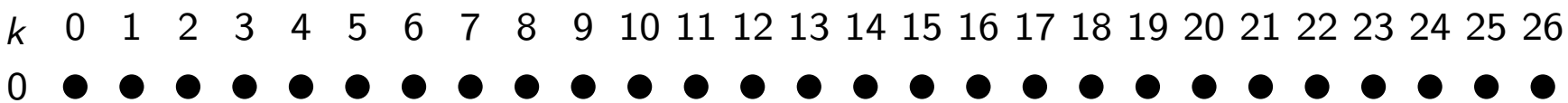

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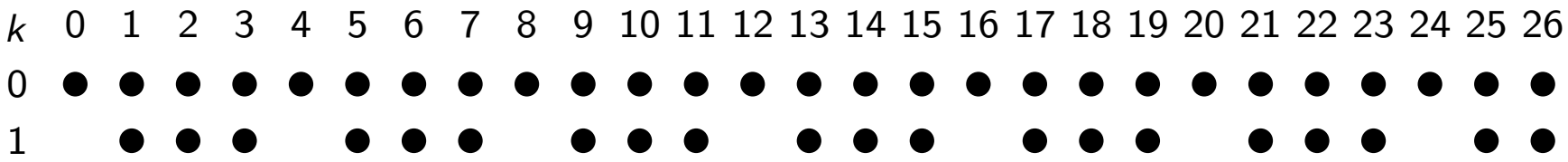

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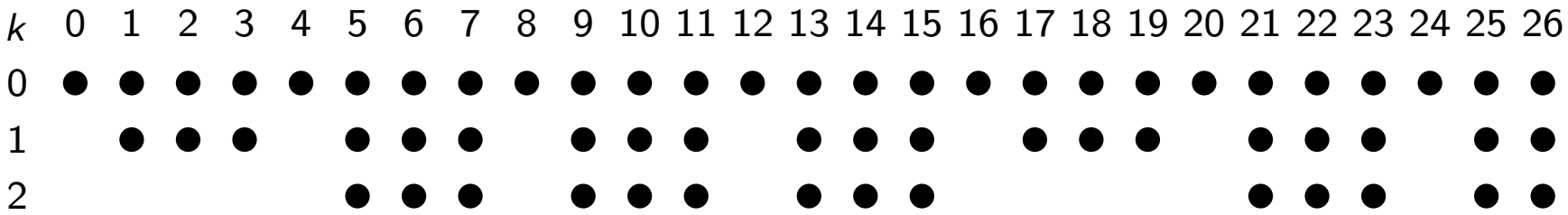

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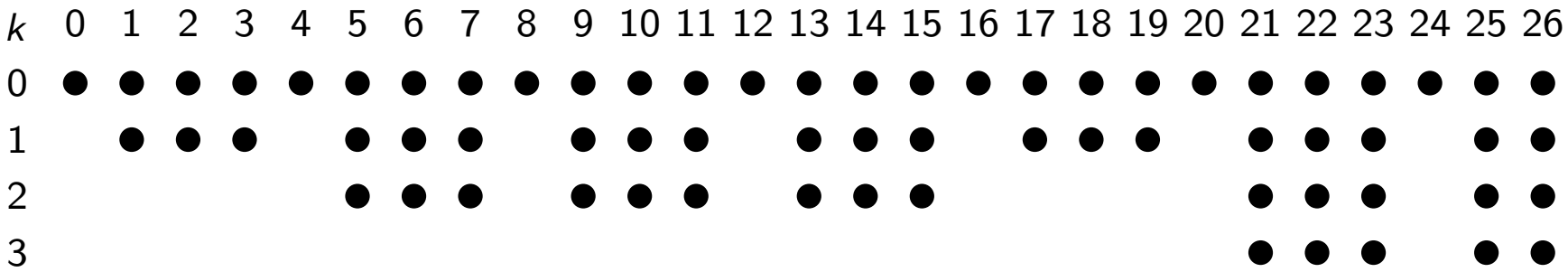

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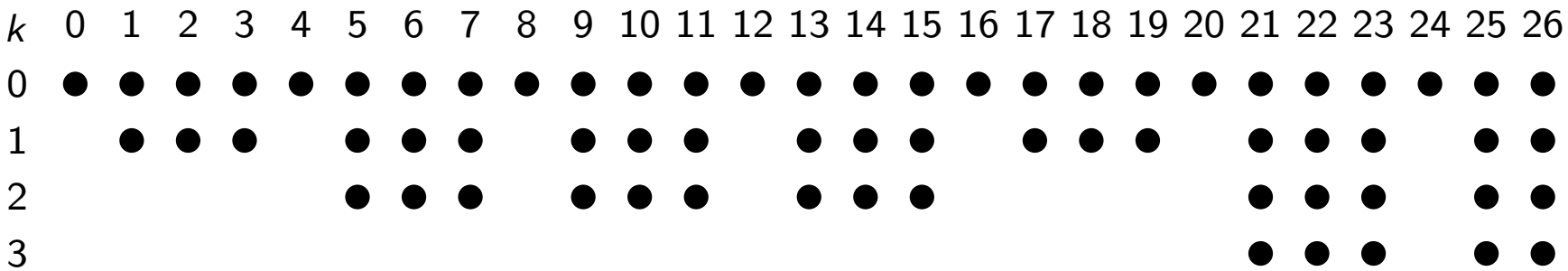

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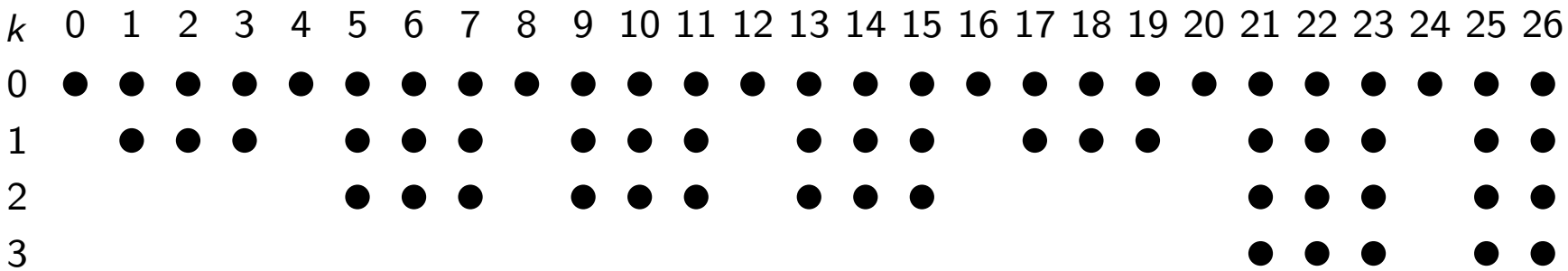

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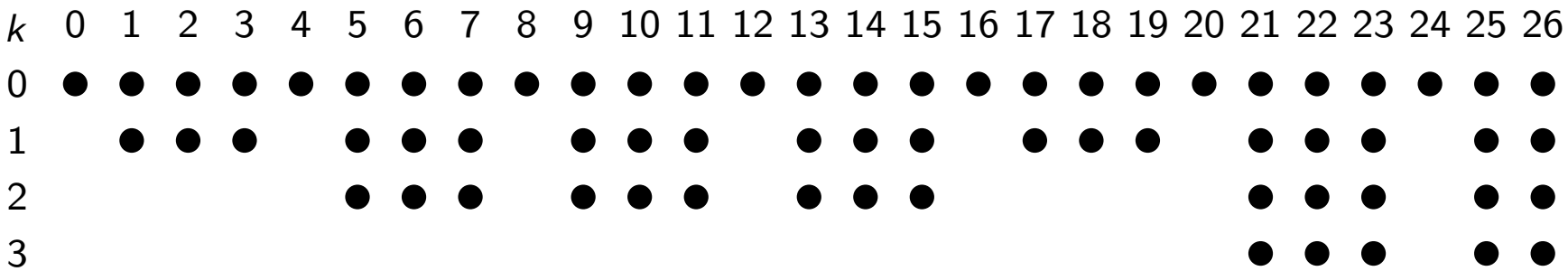

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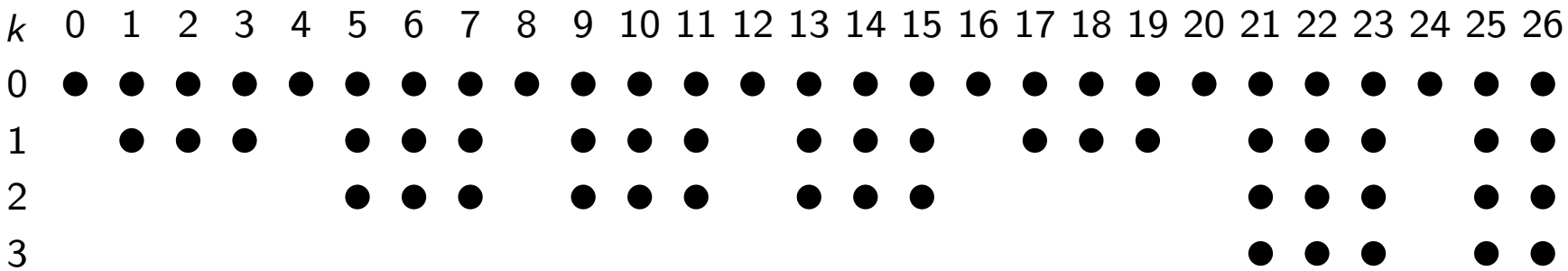

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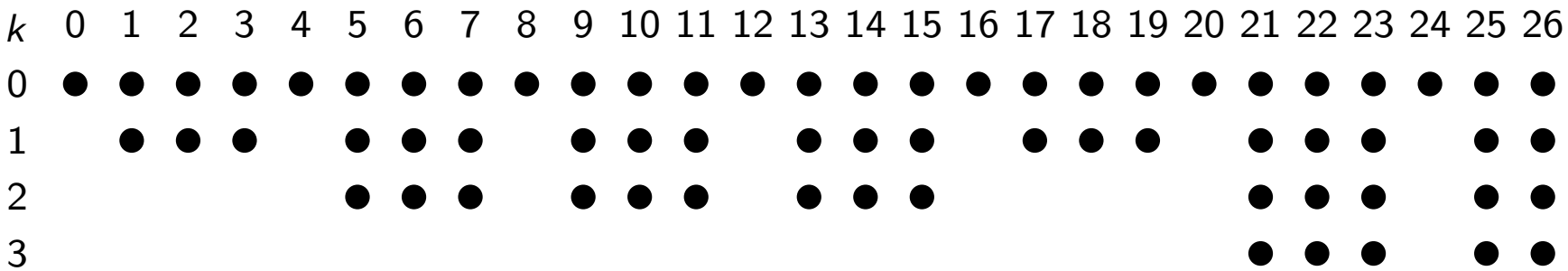

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Implementation:

1994: Fredman & Willard Linear-time MST for integer weights.

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Profit:

1. MST computation based on Dijkstra-Jarník-Prim algorithm:
   - priority queue for weights $2^0, \ldots, 2^{\log n}$,
     - constant-time operations (least significant set bit extraction).

Applications: Universal Reconstruction Framework

Border Table

Definition

A border of a string occurs both as its prefix and as its suffix.

a	b	a	a	b	c	b	a	a	b	a	a
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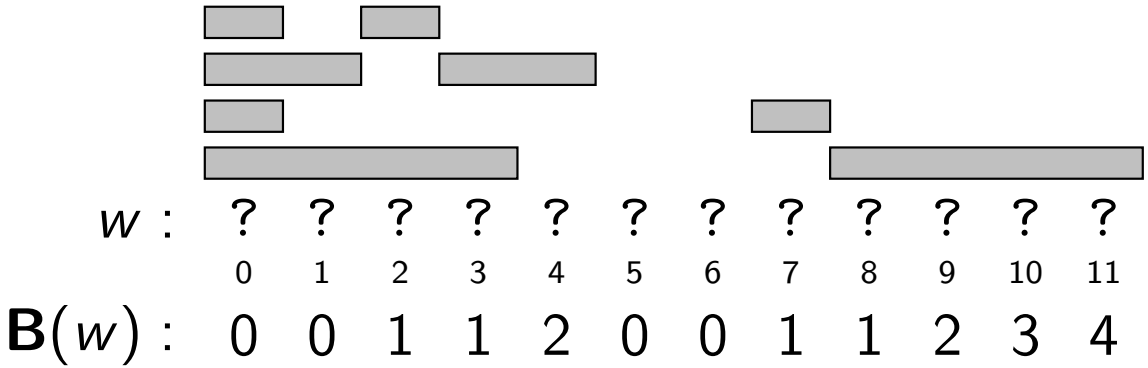

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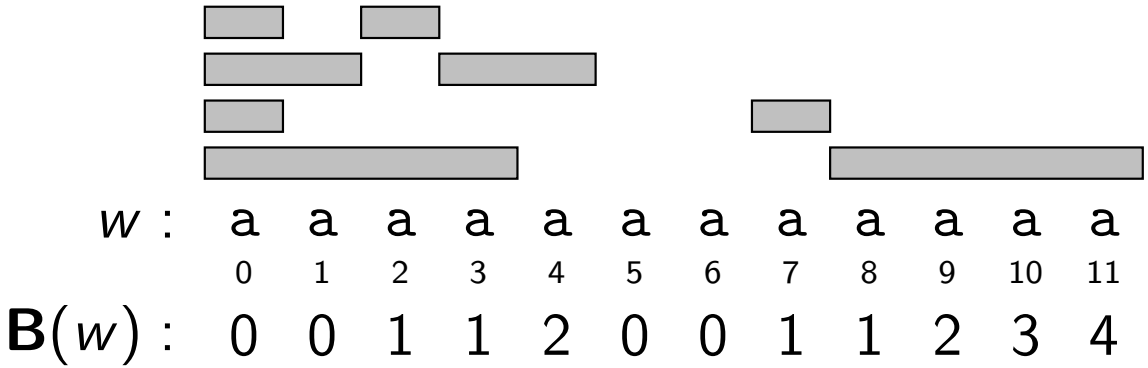

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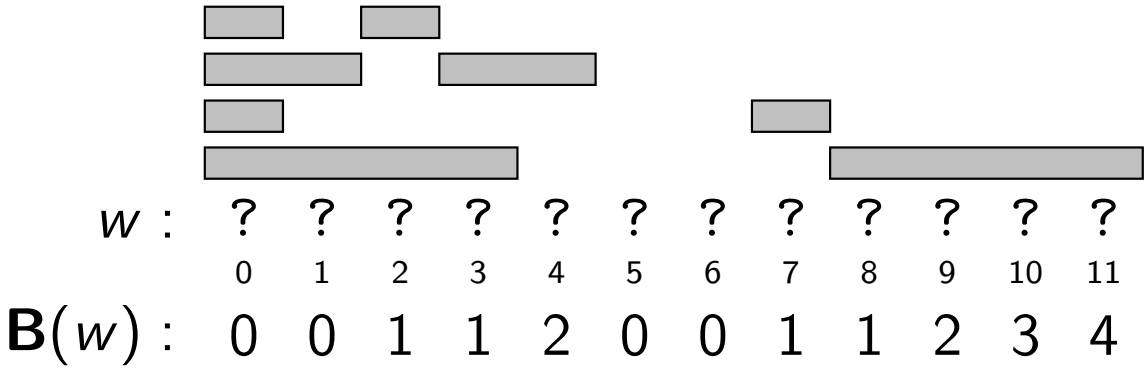

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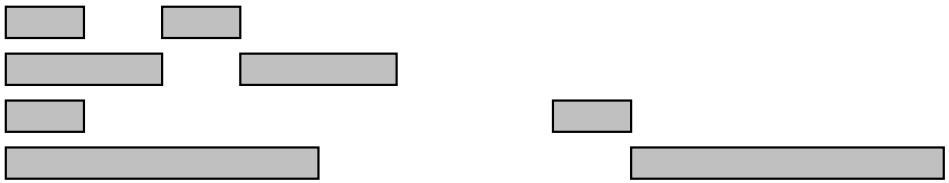

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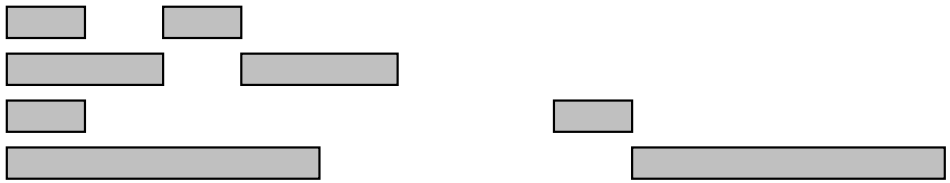

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A generic solution satisfies all satisfiable non-equations.

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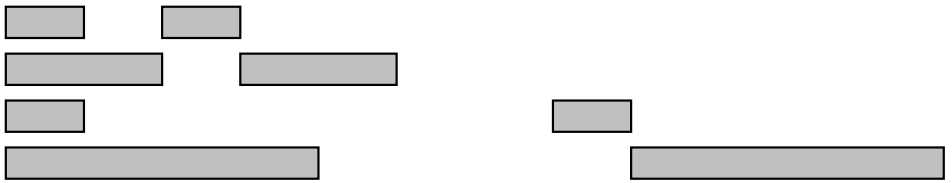

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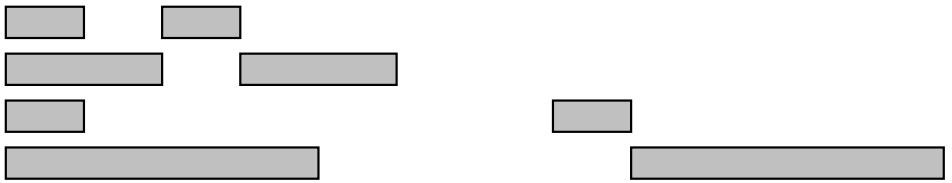

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Maximal Repetitions

Definition

A fragment $w[i..j]$ is a repetition of period p in a string w if $\mathrm{per}(w[i..j]) = p$ and $|w[i..j]| \geq 2p$. It is a maximal repetition (a run) if it cannot be extended preserving the period.

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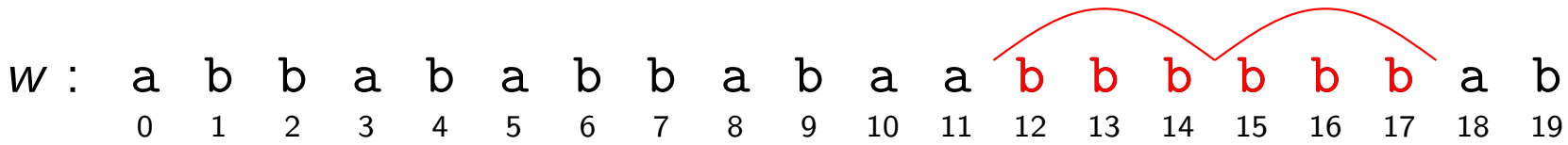

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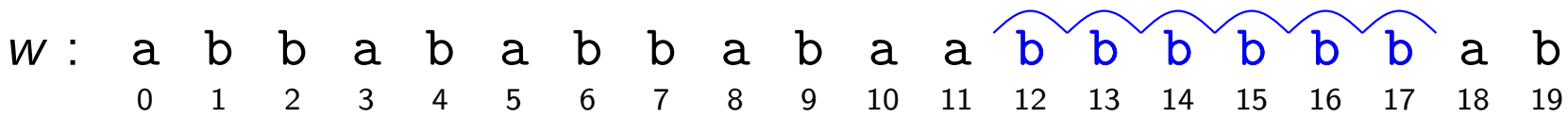

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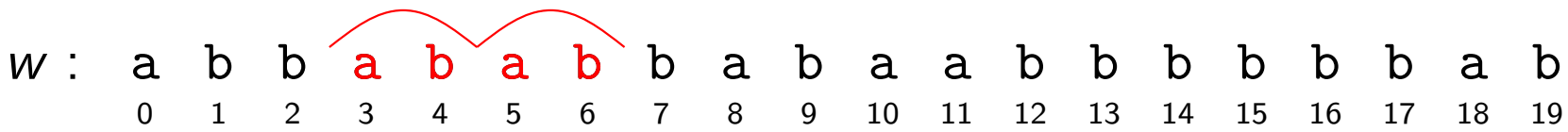

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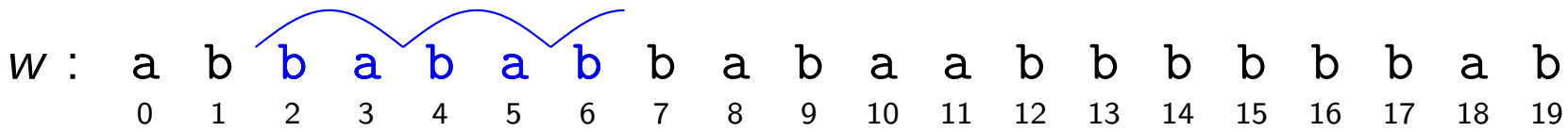

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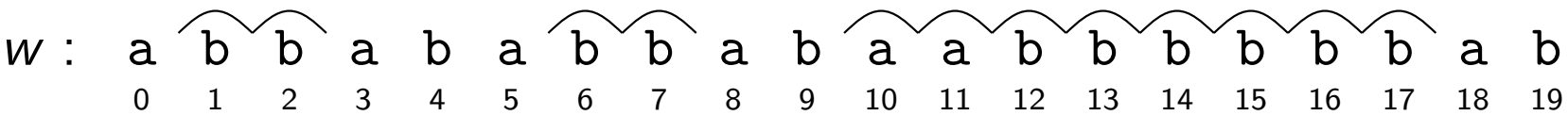

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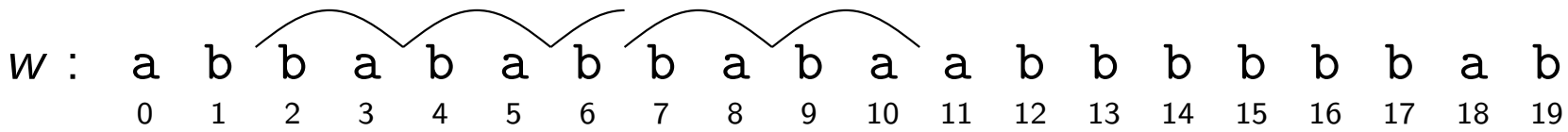

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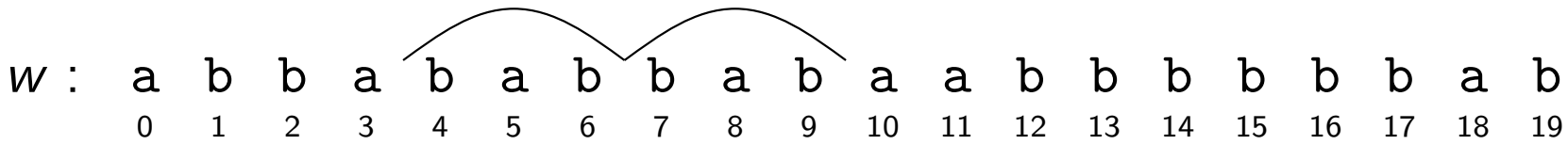

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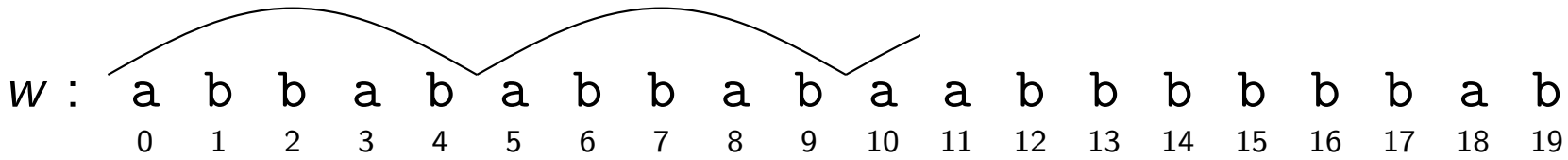

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Theorem (Bannai et al.; SODA 2015)

A string of length n has at most $n-1$ maximal repetitions.

Maximal Repetitions: Construction & Reconstruction

1996: Kolpakov & Kucherov Linear-time construction

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Example:

$$|w| = 20 \qquad \mathbf{R}(w) = \{(1,2,1),(6,7,1),(10,11,1),(12,17,1),(2,6,2),(7,10,2),(4,9,3),(0,10,5)\}$$

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2010: Matsubara et al. NP-hard to minimize the alphabet size

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   - construction algorithm for the structure.
2. Applies to structures characterized using a conjunction of:
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non-equations every square of length $2q$ can be extended to a maximal repetition with period p dividing q.

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Conclusions & Open Problems

Our contributions:

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- Formulation of general reconstruction setting.
- Several applications:
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Open problems:

- Further applications of substring equation systems?
- Generic algorithm for wider class of problems?
  - Include strong border table (KMP failure table).
- When alphabet minimization is possible?

Questions?

Thank you for your attention!