

Weighted Shortest Common Supersequence Problem Revisited*

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Abstract. A weighted string, also known as a position weight matrix, is a sequence of probability distributions over some alphabet. We revisit the Weighted Shortest Common Supersequence (WSCS) problem, introduced by Amir et al. [SPIRE 2011], that is, the SCS problem on weighted strings. In the WSCS problem, we are given two weighted strings W_1 and W_2 and a threshold $\frac{1}{z}$ on probability, and we are asked to compute the shortest (standard) string S such that both W_1 and W_2 match subsequences of S (not necessarily the same) with probability at least $\frac{1}{z}$. Amir et al. showed that this problem is NP-complete if the probabilities, including the threshold $\frac{1}{z}$, are represented by their logarithms (encoded in binary). We present an algorithm that solves the WSCS problem for two weighted strings of length n over a constant-sized alphabet in $\mathcal{O}(n^2\sqrt{z}\log z)$ time. Notably, our upper bound matches known conditional lower bounds stating that the WSCS problem cannot be solved in $\mathcal{O}(n^{2-\varepsilon})$ time or in $\mathcal{O}^*(z^{0.5-\varepsilon})$ time¹ unless there is a breakthrough improving upon long-standing upper bounds for fundamental NP-hard problems (CNF-SAT and SUBSET SUM, respectively).

We also discover a fundamental difference between the WSCS problem and the Weighted Longest Common Subsequence (WLCS) problem, introduced by Amir et al. [JDA 2010]. We show that the WLCS problem cannot be solved in $\mathcal{O}(n^{f(z)})$ time, for any function $f(z)$, unless $P = NP$.

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¹ The \mathcal{O}^* notation suppresses factors polynomial with respect to the instance size (with numeric values encoded in binary).

1 Introduction

Consider two strings X and Y . A common supersequence of X and Y is a string S such that X and Y are both subsequences of S . A shortest common supersequence (SCS) of X and Y is a common supersequence of X and Y of minimum length. The SHORTEST COMMON SUPERSEQUENCE problem (the SCS problem, in short) is to compute an SCS of X and Y . The SCS problem is a classic problem in theoretical computer science [23,25,18]. It is solvable in quadratic time using a standard dynamic-programming approach [13], which also allows computing a shortest common supersequence of any constant number of strings (rather than just two) in polynomial time. In case of an arbitrary number of input strings, the problem becomes NP-hard [23] even when the strings are binary [25].

A weighted string of length n over some alphabet Σ is a type of uncertain sequence. The uncertainty at any position of the sequence is modeled using a subset of the alphabet (instead of a single letter), with every element of this subset being associated with an occurrence probability; the probabilities are often represented in an $n \times |\Sigma|$ matrix. These kinds of data are common in various applications where: (i) imprecise data measurements are recorded; (ii) flexible sequence modeling, such as binding profiles of molecular sequences, is required; (iii) observations are private and thus sequences of observations may have artificial uncertainty introduced deliberately [2]. For instance, in computational biology they are known as position weight matrices or position probability matrices [26].

In this paper, we study the WEIGHTED SHORTEST COMMON SUPERSEQUENCE problem (the WSCS problem, in short) introduced by Amir et al. [5], which is a generalization of the SCS problem for weighted strings. In the WSCS problem, we are given two weighted strings W_1 and W_2 and a probability threshold $\frac{1}{z}$, and the task is to compute the shortest (standard) string such that both W_1 and W_2 match subsequences of S (not necessarily the same) with probability at least $\frac{1}{z}$. In this work, we show the first efficient algorithm for the WSCS problem.

A related problem is the WEIGHTED LONGEST COMMON SUBSEQUENCE problem (the WLCS problem, in short). It was introduced by Amir et al. [4] and further studied in [14] and, very recently, in [20]. In the WLCS problem, we are also given two weighted strings W_1 and W_2 and a threshold $\frac{1}{z}$ on probability, but the task is to compute the longest (standard) string S such that S matches a subsequence of W_1 with probability at least $\frac{1}{z}$ and S matches a subsequence of W_2 with probability at least $\frac{1}{z}$. For standard strings S_1 and S_2 , the length of their shortest common supersequence $|\text{SCS}(S_1, S_2)|$ and the length of their longest common subsequence $|\text{LCS}(S_1, S_2)|$ satisfy the following folklore relation:

$$|\text{LCS}(S_1, S_2)| + |\text{SCS}(S_1, S_2)| = |S_1| + |S_2|. \quad (1)$$

However, an analogous relation does not connect the WLCS and WSCS problems, even though both problems are NP-complete because of similar reductions, which remain valid even in the case that both weighted strings have the same length [4,5]. In this work, we discover an important difference between the two problems.

Kociumaka et al. [21] introduced a problem called WEIGHTED CONSENSUS, which is a special case of the WSCS problem asking whether the WSCS of two

weighted strings of length n is of length n , and they showed that the WEIGHTED CONSENSUS problem is NP-complete yet admits an algorithm running in pseudo-polynomial time $\mathcal{O}(n + \sqrt{z} \log z)$ for constant-sized alphabets². Furthermore, it was shown in [21] that the WEIGHTED CONSENSUS problem cannot be solved in $\mathcal{O}^*(z^{0.5-\varepsilon})$ time for any $\varepsilon > 0$ unless there is an $\mathcal{O}^*(2^{(0.5-\varepsilon)n})$ -time algorithm for the SUBSET SUM problem. Let us recall that the SUBSET SUM problem, for a set of n integers, asks whether there is a subset summing up to a given integer. Moreover, the $\mathcal{O}^*(2^{n/2})$ running time for the SUBSET SUM problem, achieved by a classic meet-in-the-middle approach of Horowitz and Sahni [15], has not been improved yet despite much effort; see e.g. [6].

Abboud et al. [1] showed that the LONGEST COMMON SUBSEQUENCE problem over constant-sized alphabets cannot be solved in $\mathcal{O}(n^{2-\varepsilon})$ time for $\varepsilon > 0$ unless the Strong Exponential Time Hypothesis [16,17,22] fails. By (1), the same conditional lower bound applies to the SCS problem, and since standard strings are a special case of weighted strings (having one letter occurring with probability equal to 1 at each position), it also applies to the WSCS problem.

The following theorem summarizes the above conditional lower bounds on the WSCS problem.

Theorem 1 (Conditional hardness of the WSCS problem; see [1,21]).

Even in the case of constant-sized alphabets, the WEIGHTED SHORTEST COMMON SUPERSEQUENCE problem is NP-complete, and for any $\varepsilon > 0$ it cannot be solved:

1. *in $\mathcal{O}(n^{2-\varepsilon})$ time unless the Strong Exponential Time Hypothesis fails;*
2. *in $\mathcal{O}^*(z^{0.5-\varepsilon})$ time unless there is an $\mathcal{O}^*(2^{(0.5-\varepsilon)n})$ -time algorithm for the SUBSET SUM problem.*

Our Results We give an algorithm for the WSCS problem with pseudo-polynomial running time that depends polynomially on n and z . Note that such algorithms have already been proposed for several problems on weighted strings: pattern matching [9,12,21,24], indexing [3,8,7,11], and finding regularities [10]. In contrast, we show that no such algorithm is likely to exist for the WLCS problem.

Specifically, we develop an $\mathcal{O}(n^2 \sqrt{z} \log z)$ -time algorithm for the WSCS problem in the case of a constant-sized alphabet³. This upper bound matches the conditional lower bounds of Theorem 1. We then show that unless $P = NP$, the WLCS problem cannot be solved in $\mathcal{O}(n^{f(z)})$ time for any function $f(\cdot)$.

Model of Computations We assume the word RAM model with word size $w = \Omega(\log n + \log z)$. We consider the log-probability representation of weighted sequences, that is, we assume that the non-zero probabilities in the weighted sequences and the threshold probability $\frac{1}{z}$ are all of the form $c \frac{p}{z^{d w}}$, where c and d are constants and p is an integer that fits in $\mathcal{O}(1)$ machine words.

² Note that in general $z \notin \mathcal{O}^*(1)$ unless z is encoded in unary.

³ We consider the case of $|\Sigma| = \mathcal{O}(1)$ just for simplicity. For a general alphabet, our algorithm can be modified to work in $\mathcal{O}(n^2 |\Sigma| \sqrt{z} \log z)$ time.

2 Preliminaries

A *weighted string* $W = W[1] \cdots W[n]$ of length $|W| = n$ over alphabet Σ is a sequence of sets of the form

$$W[i] = \{(c, \pi_i^{(W)}(c)) : c \in \Sigma\}.$$

Here, $\pi_i^{(W)}(c)$ is the occurrence probability of the letter c at the position $i \in [1 \dots n]$.⁴ These values are non-negative and sum up to 1 for a given index i .

By $W[i \dots j]$ we denote the weighted *substring* $W[i] \cdots W[j]$; it is called a prefix if $i = 1$ and a suffix if $j = |W|$.

The *probability of matching* of a string S with a weighted string W , with $|S| = |W| = n$, is

$$\mathcal{P}(S, W) = \prod_{i=1}^n \pi_i^{(W)}(S[i]) = \prod_{i=1}^n \mathcal{P}(S[i] = W[i]).$$

We say that a (standard) string S *matches a weighted string W with probability at least $\frac{1}{z}$* , denoted by $S \approx_z W$, if $\mathcal{P}(S, W) \geq \frac{1}{z}$. We also denote

$$\text{Matched}_z(W) = \{S \in \Sigma^n : \mathcal{P}(S, W) \geq \frac{1}{z}\}.$$

For a string S we write $W \subseteq_z S$ if $S' \approx_z W$ for some subsequence S' of S . Similarly we write $S \subseteq_z W$ if $S \approx_z W'$ for some subsequence W' of W .

Our main problem can be stated as follows.

WEIGHTED SHORTEST COMMON SUPERSEQUENCE (WSCS(W_1, W_2, z))
Input: Weighted strings W_1 and W_2 of length up to n and a threshold $\frac{1}{z}$.
Output: A shortest standard string S such that $W_1 \subseteq_z S$ and $W_2 \subseteq_z S$.

Example 2. If the alphabet is $\Sigma = \{\mathbf{a}, \mathbf{b}\}$, then we write the weighted string as $W = [p_1, p_2, \dots, p_n]$, where $p_i = \pi_i^{(W)}(\mathbf{a})$; in other words, p_i is the probability that the i th letter $W[i]$ is \mathbf{a} . For

$$W_1 = [1, 0.2, 0.5], \quad W_2 = [0.2, 0.5, 1], \quad \text{and } z = \frac{5}{2},$$

we have $\text{WSCS}(W_1, W_2, z) = \mathbf{baba}$ since $W_1 \subseteq_z \mathbf{baba}$, $W_2 \subseteq_z \mathbf{baba}$ (the witness subsequences are underlined), and \mathbf{baba} is a shortest string with this property.

We first show a simple solution to WSCS based on the following facts.

Observation 3 (Amir et al. [3]). *Every weighted string W matches at most z standard strings with probability at least $\frac{1}{z}$, i.e., $|\text{Matched}_z(W)| \leq z$.*

Lemma 4. *The set $\text{Matched}_z(W)$ can be computed in $\mathcal{O}(nz)$ time if $|\Sigma| = \mathcal{O}(1)$.*

⁴ For any two integers $\ell \leq r$, we use $[\ell \dots r]$ to denote the integer range $\{\ell, \dots, r\}$.

Proof. If $S \in \text{Matched}_z(W)$, then $S[1..i] \in \text{Matched}_z(W[1..i])$ for every index i . Hence, the algorithm computes the sets Matched_z for subsequent prefixes of W . Each string $S \in \text{Matched}_z(W[1..i])$ is represented as a triple (c, p, S') , where $c = S[i]$ is the last letter of S , $p = \mathcal{P}(S, W[1..i])$, and $S' = S[1..i-1]$ points to an element of $\text{Matched}_z(W[1..i-1])$. Such a triple is represented in $\mathcal{O}(1)$ space.

Assume that $\text{Matched}_z(W[1..i-1])$ has already been computed. Then, for every $S' = (c', p', S'') \in \text{Matched}_z(W[1..i-1])$ and every $c \in \Sigma$, if $p := p' \cdot \pi_i^{(W)}(c) \geq \frac{1}{z}$, then the algorithm adds (c, p, S') to $\text{Matched}_z(W[1..i])$.

By Observation 3, $|\text{Matched}_z(W[1..i-1])| \leq z$ and $|\text{Matched}_z(W[1..i])| \leq z$. Hence, the $\mathcal{O}(nz)$ time complexity follows. \square

Proposition 5. *The WSCS problem can be solved in $\mathcal{O}(n^2z^2)$ time if $|\Sigma| = \mathcal{O}(1)$.*

Proof. The algorithm builds $\text{Matched}_z(W_1)$ and $\text{Matched}_z(W_2)$ using Lemma 4. These sets have size at most z by Observation 3. The result is the shortest string in

$$\{\text{SCS}(S_1, S_2) : S_1 \in \text{Matched}_z(W_1), S_2 \in \text{Matched}_z(W_2)\}.$$

Recall that the SCS of two strings can be computed in $\mathcal{O}(n^2)$ time using a standard dynamic programming algorithm [13]. \square

We substantially improve upon this upper bound in Sections 3 and 4.

2.1 Meet-in-the-middle Technique

In the decision version of the KNAPSACK problem, we are given n items with weights w_i and values v_i , and we seek for a subset of items with total weight up to W and total value at least V . In the classic meet-in-the-middle solution to the KNAPSACK problem by Horowitz and Sahni [15], the items are divided into two sets S_1 and S_2 of sizes roughly $\frac{1}{2}n$. Initially, the total value and the total weight is computed for every subset of elements of each set S_i . This results in two sets A, B , each with $\mathcal{O}(2^{n/2})$ pairs of numbers. The algorithm needs to pick a pair from each set such that the first components of the pairs sum up to at most W and the second components sum up to at least V . This problem can be solved in linear time w.r.t. the set sizes provided that the pairs in both sets A and B are sorted by the first component.

Let us introduce a modified version this problem.

MERGE(A, B, w)

Input: Two sets A and B of points in 2 dimensions and a threshold w .

Output: Do there exist $(x_1, y_1) \in A, (x_2, y_2) \in B$ such that $x_1x_2, y_1y_2 \geq w$?

A linear-time solution to this problem is the same as for the problem in the meet-in-the-middle solution for KNAPSACK. However, for completeness we prove the following lemma (see also [21, Lemma 5.6]):

Lemma 6 (Horowitz and Sahni [15]). *The MERGE problem can be solved in linear time assuming that the points in A and B are sorted by the first component.*

Proof. A pair (x, y) is *irrelevant* if there is another pair (x', y') in the same set such that $x' \geq x$ and $y' \geq y$. Observe that removing an irrelevant point from A or B leads to an equivalent instance of the MERGE problem.

Since the points in A and B are sorted by the first component, a single scan through these pairs suffices to remove all irrelevant elements. Next, for each $(x, y) \in A$, the algorithm computes $(x', y') \in B$ such that $x' \geq w/x$ and additionally x' is smallest possible. As the irrelevant elements have been removed from B , this point also maximizes y' among all pairs satisfying $x' \geq w/x$. If the elements (x, y) are processed by non-decreasing values x , the values x' do not increase, and thus the points (x', y') can be computed in $\mathcal{O}(|A| + |B|)$ time in total. \square

3 Dynamic Programming Algorithm for WSCS

Our algorithm is based on dynamic programming. We start with a less efficient procedure and then improve it in the next section. Henceforth, we only consider computing the length of the WSCS; an actual common supersequence of this length can be recovered from the dynamic programming using a standard approach (storing the parent of each state).

For a weighted string W , we introduce a data structure that stores, for every index i , the set $\{\mathcal{P}(S, W[1..i]) : S \in \text{Matched}_z(W[1..i])\}$ represented as an array of size at most z (by Observation 3) with entries in the increasing order. This data structure is further denoted as $\text{Freq}_i(W, z)$. Moreover, for each element $p \in \text{Freq}_{i+1}(W, z)$ and each letter $c \in \Sigma$, a pointer to $p' = p / \pi_{i+1}^{(W)}(c)$ in $\text{Freq}_i(W, z)$ is stored provided that $p' \in \text{Freq}_i(W, z)$. A proof of the next lemma is essentially the same as of Lemma 4.

Lemma 7. *For a weighted string W of length n , the arrays $\text{Freq}_i(W, z)$, with $i \in [1..n]$, can be constructed in $\mathcal{O}(nz)$ total time if $|\Sigma| = \mathcal{O}(1)$.*

Proof. Assume that $\text{Freq}_i(W, z)$ is computed. For every $c \in \Sigma$, we create a list

$$L_c = \{p \cdot \pi_{i+1}^{(W)}(c) : p \in \text{Freq}_i(W, z), p \cdot \pi_{i+1}^{(W)}(c) \geq \frac{1}{z}\}.$$

The lists are sorted since $\text{Freq}_i(W, z)$ was sorted. Then $\text{Freq}_{i+1}(W, z)$ can be computed by merging all the lists L_c (removing duplicates). This can be done in $\mathcal{O}(z)$ time since $\sigma = \mathcal{O}(1)$. The desired pointers can be computed within the same time complexity. \square

Let us extend the WSCS problem in the following way:

WSCS'(W₁, W₂, ℓ, p, q):

Input: Weighted strings W₁, W₂, an integer ℓ, and probabilities p, q.

Output: Is there a string S of length ℓ with subsequences S₁ and S₂ such that $\mathcal{P}(S_1, W_1) = p$ and $\mathcal{P}(S_2, W_2) = q$?

In the following, a *state* in the dynamic programming denotes a quadruple (i, j, ℓ, p) , where $i \in [0..|W_1|]$, $j \in [0..|W_2|]$, $\ell \in [0..|W_1| + |W_2|]$, and $p \in \text{Freq}_i(W_1, z)$.

Observation 8. *There are $\mathcal{O}(n^3z)$ states.*

In the dynamic programming, for all states (i, j, ℓ, p) , we compute

$$\mathbf{DP}[i, j, \ell, p] = \max\{q : \text{WSCS}'(W_1[1..i], W_2[1..j], \ell, p, q) = \mathbf{true}\}. \quad (2)$$

Let us denote $\pi_i^k(c) = \pi_i^{(W_k)}(c)$. Initially, the array \mathbf{DP} is filled with zeroes, except that the values $\mathbf{DP}[0, 0, \ell, 1]$ for $\ell \in [0..|W_1| + |W_2|]$ are set to 1. In order to cover corner cases, we assume that $\pi_0^1(c) = \pi_0^2(c) = 1$ for any $c \in \Sigma$ and that $\mathbf{DP}[i, j, \ell, p] = 0$ if (i, j, ℓ, p) is not a state. The procedure **Compute** implementing the dynamic-programming algorithm is shown as Algorithm 1.

Algorithm 1: $\text{Compute}(W_1, W_2, z)$

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for  $\ell = 0$  to  $|W_1| + |W_2|$  do
   $\mathbf{DP}[0, 0, \ell, 1] := 1;$ 
foreach state  $(i, j, \ell, p)$  in lexicographic order do
  foreach  $c \in \Sigma$  do
     $x := \pi_i^1(c); y := \pi_j^2(c);$ 
     $\mathbf{DP}[i, j, \ell, p] := \max\{$ 
       $\mathbf{DP}[i, j, \ell, p],$ 
       $\mathbf{DP}[i - 1, j, \ell - 1, \frac{p}{x}],$ 
       $y \cdot \mathbf{DP}[i, j - 1, \ell - 1, p],$ 
       $y \cdot \mathbf{DP}[i - 1, j - 1, \ell - 1, \frac{p}{x}]$ 
     $\};$ 
return  $\min\{\ell : \mathbf{DP}[|W_1|, |W_2|, \ell, p] \geq \frac{1}{z} \text{ for some } p \in \text{Freq}_{|W_1|}(W_1, z)\};$ 

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The correctness of the algorithm is implied by the following lemma:

Lemma 9 (Correctness of Algorithm 1). *The array \mathbf{DP} satisfies (2). In particular, $\text{Compute}(W_1, W_2, z) = \text{WSCS}(W_1, W_2, z)$.*

Proof. The proof that \mathbf{DP} satisfies (2) goes by induction on $i + j$. The base case of $i + j = 0$ holds trivially. It is simple to verify the cases that $i = 0$ or $j = 0$. Let us henceforth assume that $i > 0$ and $j > 0$.

We first show that

$$\mathbf{DP}[i, j, \ell, p] \leq \max\{q : \text{WSCS}'(W_1[1..i], W_2[1..j], \ell, p, q) = \mathbf{true}\}.$$

The value $q = \mathbf{DP}[i, j, \ell, p]$ was derived from $\mathbf{DP}[i - 1, j, \ell - 1, p/x] = q$, or $\mathbf{DP}[i, j - 1, \ell - 1, p] = q/y$, or $\mathbf{DP}[i - 1, j - 1, \ell - 1, p/x] = q/y$, where $x = \pi_i^1(c)$ and $y = \pi_j^2(c)$ for some $c \in \Sigma$. In the first case, by the inductive hypothesis, there exists a string T that is a solution to $\text{WSCS}'(W_1[1..i-1], W_2[1..j], \ell - 1, p/x, q)$. That is, T has subsequences T_1 and T_2 such that

$$\mathcal{P}(T_1, W_1[1..i-1]) = p/x \quad \text{and} \quad \mathcal{P}(T_2, W_2[1..j]) = q.$$

Then, for $S = Tc$, $S_1 = T_1c$, and $S_2 = T_2$, we indeed have

$$\mathcal{P}(S_1, W_1[1..i]) = p \quad \text{and} \quad \mathcal{P}(S_2, W_2[1..j]) = q$$

The two remaining cases are analogous.

Let us now show that

$$\mathbf{DP}[i, j, \ell, p] \geq \max\{q : \text{WSCS}'(W_1[1..i], W_2[1..j], \ell, p, q) = \mathbf{true}\}.$$

Assume a that string S is a solution to $\text{WSCS}'(W_1[1..i], W_2[1..j], \ell, p, q)$. Let S_1 and S_2 be the subsequences of S such that $\mathcal{P}(S_1, W_1) = p$ and $\mathcal{P}(S_2, W_2) = q$.

Let us first consider the case that $S_1[i] = S[\ell] \neq S_2[j]$. Then $T_1 = S_1[1..i-1]$ and $T_2 = S_2$ are subsequences of $T = S[1..\ell-1]$. We then have

$$p' := \mathcal{P}(T_1, W_1[1..i-1]) = p/\pi_i^1(S_1[i])$$

By the inductive hypothesis, $\mathbf{DP}[i - 1, j, \ell - 1, p'] \geq q$. Hence, $\mathbf{DP}[i, j, \ell, p] \geq q$ because $\mathbf{DP}[i - 1, j, \ell - 1, p']$ is present as the second argument of the maximum in the dynamic programming algorithm for $c = S[\ell]$.

The cases that $S_1[i] \neq S[\ell] = S_2[j]$ and that $S_1[i] = S[\ell] = S_2[j]$ rely on the values $\mathbf{DP}[i, j - 1, \ell - 1, p] \geq q/y$ and $\mathbf{DP}[i - 1, j - 1, \ell - 1, p/x] \geq q/y$, respectively.

Finally, the case that $S_1[i] \neq S[\ell] \neq S_2[j]$ is reduced to one of the previous cases by changing $S[\ell]$ to $S_1[i]$ so that S is still a supersequence of S_1 and S_2 and a solution to $\text{WSCS}'(W_1[1..i], W_2[1..j], \ell, p, q)$. \square

Proposition 10. *The WSCS problem can be solved in $\mathcal{O}(n^3z)$ time if $|\Sigma| = \mathcal{O}(1)$.*

Proof. The correctness follows from Lemma 9. As noted in Observation 8, the dynamic programming has $\mathcal{O}(n^3z)$ states. The number of transitions from a single state is constant provided that $|\Sigma| = \mathcal{O}(1)$.

Before running the dynamic programming algorithm of Proposition 10, we construct the data structures $\text{Freq}_i(W_1, z)$ for all $i \in [1..n]$ using Lemma 7. The last dimension in the $\mathbf{DP}[i, j, \ell, p]$ array can then be stored as a position in $\text{Freq}_i(W_1, z)$. The pointers in the arrays Freq_i are used to follow transitions. \square

4 Improvements

4.1 First Improvement: Bounds on ℓ

Our approach here is to reduce the number of states (i, j, ℓ, p) in Algorithm 1 from $\mathcal{O}(n^3 z)$ to $\mathcal{O}(n^2 z \log z)$. This is done by limiting the number of values of ℓ considered for each pair of indices i, j from $\mathcal{O}(n)$ to $\mathcal{O}(\log z)$.

For a weighted string W , we define $\mathcal{H}(W)$ as a standard string generated by taking the most probable letter at each position, breaking ties arbitrarily. The string $\mathcal{H}(W)$ is also called the *heavy* string of W . By $d_H(S, T)$ we denote the Hamming distance of strings S and T . Let us recall an observation from [21].

Observation 11 ([21, Observation 4.3]). *If $S \approx_z W$ for a string S and a weighted string W , then $d_H(S, \mathcal{H}(W)) \leq \log_2 z$.*

The lemma below follows from Observation 11.

Lemma 12. *If strings S_1 and S_2 satisfy $S_1 \approx_z W_1$ and $S_2 \approx_z W_2$, then*

$$|\text{SCS}(S_1, S_2) - \text{SCS}(\mathcal{H}(W_1), \mathcal{H}(W_2))| \leq 2 \log_2 z.$$

Proof. By Observation 11,

$$d_H(S_1, \mathcal{H}(W_1)) \leq \log_2 z \quad \text{and} \quad d_H(S_2, \mathcal{H}(W_2)) \leq \log_2 z.$$

Due to the relation (1) between LCS and SCS, it suffices to show the following.

Claim. Let S_1, H_1, S_2, H_2 be strings such that $|S_1| = |H_1|$ and $|S_2| = |H_2|$. If $d_H(S_1, H_1) \leq d$ and $d_H(S_2, H_2) \leq d$, then $|\text{LCS}(S_1, S_2) - \text{LCS}(H_1, H_2)| \leq 2d$.

Proof. Notice that if S'_1, S'_2 are strings resulting from S_1, S_2 by removing up to d letters from each of them, then $\text{LCS}(S'_1, S'_2) \geq \text{LCS}(S_1, S_2) - 2d$.

We now create strings S'_k for $k = 1, 2$, by removing from S_k letters at positions i such that $S_k[i] \neq H_k[i]$. Then, according to the observation above, we have

$$\text{LCS}(S'_1, S'_2) \geq \text{LCS}(S_1, S_2) - 2d.$$

Any common subsequence of S'_1 and S'_2 is also a common subsequence of H_1 and H_2 since S'_1 and S'_2 are subsequences of H_1 and H_2 , respectively. Consequently,

$$\text{LCS}(H_1, H_2) \geq \text{LCS}(S_1, S_2) - 2d.$$

In a symmetric way, we can show that $\text{LCS}(S_1, S_2) \geq \text{LCS}(H_1, H_2) - 2d$. This completes the proof of the claim. \square

We apply the claim for $H_1 = \mathcal{H}(W_1)$, $H_2 = \mathcal{H}(W_2)$, and $d = \log_2 z$. \square

Let us make the following simple observation.

Observation 13. *If $S = \text{WSCS}(W_1, W_2, z)$, then $S = \text{SCS}(S_1, S_2)$ for some strings S_1 and S_2 such that $W_1 \subseteq_z S_1$ and $W_2 \subseteq_z S_2$.*

Algorithm 2: Improved1(W_1, W_2, z)

In the beginning, we apply the classic $\mathcal{O}(n^2)$ -time dynamic-programming solution to the standard SCS problem on $H_1 = \mathcal{H}(W_1)$ and $H_2 = \mathcal{H}(W_2)$. It computes a 2D array T such that

$$T[i, j] = \text{SCS}(H_1[1..i], H_2[1..j]).$$

Let us denote an interval

$$L[i, j] = [T[i, j] - \lfloor 2 \log_2 z \rfloor \dots T[i, j] + \lfloor 2 \log_2 z \rfloor].$$

We run the dynamic programming algorithm `Compute` restricted to states (i, j, ℓ, p) with $\ell \in L[i, j]$.

Let \mathbf{DP}' denote the resulting array, restricted to states satisfying $\ell \in L[i, j]$. We return $\min \{ \ell : \mathbf{DP}'[|W_1|, |W_2|, \ell, p] \geq \frac{1}{z} \text{ for some } p \in \text{Freq}_{|W_1|}(W_1, z) \}$.

Using Lemma 12, we refine the previous algorithm as shown in Algorithm 2.

Lemma 14 (Correctness of Algorithm 2). *For every state (i, j, ℓ, p) , an inequality $\mathbf{DP}'[i, j, \ell, p] \leq \mathbf{DP}[i, j, \ell, p]$ holds. Moreover, if $S = \text{SCS}(S_1, S_2)$, $|S| = \ell$, $\mathcal{P}(S_1, W_1[1..i]) = p \geq \frac{1}{z}$ and $\mathcal{P}(S_2, W_2[1..j]) = q \geq \frac{1}{z}$, then $\mathbf{DP}'[i, j, \ell, p] \geq q$. Consequently, $\text{Improved1}(W_1, W_2, z) = \text{WSCS}(W_1, W_2, z)$.*

Proof. A simple induction on $i + j$ shows that the array \mathbf{DP}' is lower bounded by \mathbf{DP} . This is because Algorithm 2 is restricted to a subset of states considered by Algorithm 1, and because $\mathbf{DP}'[i, j, \ell, p]$ is assumed to be 0 while $\mathbf{DP}[i, j, \ell, p] \geq 0$ for states (i, j, ℓ, p) ignored in Algorithm 2.

We prove the second part of the statement also by induction on $i + j$. The base cases satisfying $i = 0$ or $j = 0$ can be verified easily, so let us henceforth assume that $i > 0$ and $j > 0$.

First, consider the case that $S_1[i] = S[\ell] \neq S_2[j]$. Let $T = S[1.. \ell - 1]$ and $T_1 = S_1[1..i - 1]$. We then have

$$p' := \mathcal{P}(T_1, W_1[1..i - 1]) = p / \pi_i^1(S_1[i])$$

Claim. If $S_1[i] = S[\ell] \neq S_2[j]$, then $T = \text{SCS}(T_1, S_2)$.

Proof. Let us first show that T is a common supersequence of T_1 and S_2 . Indeed, if T_1 was not a subsequence of T , then $T_1 S_1[i] = S_1$ would not be a subsequence of $T S_1[i] = S$, and if S_2 was not a subsequence of T , then it would not be a subsequence of $T S_1[i] = S$ since $S_1[i] \neq S_2[j]$.

Finally, if T_1 and S_2 had a common supersequence T' shorter than T , then $T' S_1[i]$ would be a common supersequence of S_1 and S_2 shorter than S . \square

By the claim and the inductive hypothesis, $\mathbf{DP}'[i - 1, j, \ell - 1, p'] \geq q$. Hence, $\mathbf{DP}'[i, j, \ell, p] \geq q$ due to the presence of the second argument of the maximum in

the dynamic programming algorithm for $c = S[\ell]$. Note that (i, j, ℓ, p) is a state in Algorithm 2 since $\ell \in L[i, j]$ follows from Lemma 12.

The cases that $S_1[i] \neq S[\ell] = S_2[j]$ and that $S_1[i] = S[\ell] = S_2[j]$ use the values $\mathbf{DP}'[i, j - 1, \ell - 1, p] \geq q/y$ and $\mathbf{DP}'[i - 1, j - 1, \ell - 1, p/x] \geq q/y$, respectively. Finally, the case that $S_1[i] \neq S[\ell] \neq S_2[j]$ is impossible as $S = \text{SCS}(S_1, S_2)$. \square

Example 15. Let $W_1 = [1, 0]$, $W_2 = [0]$ (using the notation from Example 2), and $z \geq 1$. The only strings that match W_1 and W_2 are $S_1 = \mathbf{ab}$ and $S_2 = \mathbf{b}$, respectively. We have $\mathbf{DP}[2, 1, 3, 1] = 1$ which corresponds, in particular, to a solution $S = \mathbf{abb}$ which is not an SCS of S_1 and S_2 . However, $\mathbf{DP}[2, 1, 2, 1] = \mathbf{DP}'[2, 1, 2, 1] = 1$ which corresponds to $S = \mathbf{ab} = \text{SCS}(S_1, S_2)$.

Proposition 16. *The WSCS problem can be solved in $\mathcal{O}(n^2 z \log z)$ time if $|\Sigma| = \mathcal{O}(1)$.*

Proof. The correctness of the algorithm follows from Lemma 14. The number of states is now $\mathcal{O}(n^2 z \log z)$ and thus so is the number of considered transitions. \square

4.2 Second Improvement: Meet in the Middle

The second improvement is to apply a meet-in-the-middle approach, which is possible due to following observation resembling Observation 6.6 in [21].

Observation 17. *If $S \approx_z W$ for a string S and weighted string W of length n , then there exists a position $i \in [1..n]$ such that*

$$S[1..i-1] \approx_{\sqrt{z}} W[1..i-1] \quad \text{and} \quad S[i+1..n] \approx_{\sqrt{z}} W[i+1..n].$$

Proof. Select i as the maximum index with $S[1..i-1] \approx_{\sqrt{z}} W[1..i-1]$. \square

We first use dynamic programming to compute two arrays, $\overrightarrow{\mathbf{DP}}$ and $\overleftarrow{\mathbf{DP}}$. The array $\overrightarrow{\mathbf{DP}}$ contains a subset of states from \mathbf{DP}' ; namely the ones that satisfy $p \geq \frac{1}{\sqrt{z}}$. The array $\overleftarrow{\mathbf{DP}}$ is an analogous array defined for suffixes of W_1 and W_2 . Formally, we compute $\overrightarrow{\mathbf{DP}}$ for the reversals of W_1 and W_2 , denoted as $\overrightarrow{\mathbf{DP}}^R$, and set $\overleftarrow{\mathbf{DP}}[i, j, \ell, p] = \overrightarrow{\mathbf{DP}}^R[|W_1| + 1 - i, |W_2| + 1 - j, \ell, p]$. Proposition 16 yields

Observation 18. *Arrays $\overrightarrow{\mathbf{DP}}$ and $\overleftarrow{\mathbf{DP}}$ can be computed in $\mathcal{O}(n^2 \sqrt{z} \log z)$ time.*

Henceforth, we consider only a simpler case in which there exists a solution S to $\text{WSCS}(W_1, W_2, z)$ with a decomposition $S = S_L \cdot S_R$ such that

$$W_1[1..i] \subseteq_{\sqrt{z}} S_L \quad \text{and} \quad W_1[i+1..|W_1|] \subseteq_{\sqrt{z}} S_R \quad (3)$$

holds for some $i \in [0..|W_1|]$.

In the pseudocode, we use the array $L[i, j]$ from the first improvement, denoted here as $\overrightarrow{L}[i, j]$, and a symmetric array \overleftarrow{L} from right to left, i.e.:

$$\begin{aligned} \overleftarrow{T}[i, j] &= \text{SCS}(\mathcal{H}(W_1)[i..|W_1|], \mathcal{H}(W_2)[j..|W_2|]), \\ \overleftarrow{L}[i, j] &= [\overleftarrow{T}[i, j] - \lfloor 2 \log_2 z \rfloor \dots \overleftarrow{T}[i, j] + \lfloor 2 \log_2 z \rfloor]. \end{aligned}$$

Algorithm 3 is applied for every $i \in [0..|W_1|]$ and $j \in [0..|W_2|]$.

Algorithm 3: Improved2(W_1, W_2, z, i, j)

```

res := ∞;
foreach  $\ell_L \in \overrightarrow{L}[i, j], \ell_R \in \overleftarrow{L}[i + 1, j + 1]$  do
  A := {(p, q) :  $\overrightarrow{DP}[i, j, \ell_L, p] = q$ };
  B := {(p, q) :  $\overleftarrow{DP}[i + 1, j + 1, \ell_R, p] = q$ };
  if MERGE(A, B, z) then
    res := min(res,  $\ell_L + \ell_R$ );
return res;

```

Lemma 19 (Correctness of Algorithm 3). *Assuming that there is a solution S to $\text{WSCS}(W_1, W_2, z)$ that satisfies (3), we have*

$$\text{WSCS}(W_1, W_2, z) = \min_{i,j}(\text{Improved2}(W_1, W_2, z, i, j)).$$

Proof. Assume that $\text{WSCS}(W_1, W_2, z)$ has a solution $S = S_L \cdot S_R$ that satisfies (3) for some $i \in [0..|W_1|]$ and denote $\ell_L = |S_L|$, $\ell_R = |S_R|$. Let S'_L and S'_R be subsequences of S_L and S_R such that

$$p_L := \mathcal{P}(S'_L, W_1[1..i]) \geq \frac{1}{\sqrt{z}} \quad \text{and} \quad p_R := \mathcal{P}(S'_R, W_1[i+1..|W_1|]) \geq \frac{1}{\sqrt{z}}.$$

Let S''_L and S''_R be subsequences of S_L and S_R such that

$$\mathcal{P}(S''_L, W_2[1..j]) = q_L \quad \text{and} \quad \mathcal{P}(S''_R, W_2[j+1..|W_2|]) = q_R$$

for some j and $q_L q_R \geq \frac{1}{z}$.

By Lemma 14, $\overrightarrow{DP}[i, j, \ell_L, p_L] \geq q_L$ and $\overleftarrow{DP}[i+1, j+1, \ell_R, p_R] \geq q_R$. Hence, the set A will contain a pair (p_L, q'_L) such that $q'_L \geq q_L$ and the set B will contain a pair (p_R, q'_R) such that $q'_R \geq q_R$. Consequently, $\text{MERGE}(A, B, z)$ will return a positive answer.

Similarly, if $\text{MERGE}(A, B, z)$ returns a positive answer for given i, j, ℓ_L and ℓ_R , then

$$\overrightarrow{DP}[i, j, \ell_L, p_L] \geq q_L \quad \text{and} \quad \overleftarrow{DP}[i+1, j+1, \ell_R, p_R] \geq q_R$$

for some $p_L p_R, q_L q_R \geq \frac{1}{z}$. By Lemma 14, this implies that

$$\text{WSCS}'(W_1[1..i], W_2[1..j], \ell_L, p_L, q_L)$$

and

$$\text{WSCS}'(W_1[i+1..|W_1|], W_2[j+1..|W_2|], \ell_R, p_R, q_R)$$

have a positive answer, so

$$\text{WSCS}'(W_1, W_2, \ell_L + \ell_R, p_L p_R, q_L q_R)$$

has a positive answer too. Due to $p_L p_R, q_L q_R \geq \frac{1}{z}$, this completes the proof. \square

Proposition 20. *The WSCS problem can be solved in $\mathcal{O}(n^2\sqrt{z}\log^2 z)$ time if $|\Sigma| = \mathcal{O}(1)$.*

Proof. We use the algorithm Improved2, whose correctness follows from Lemma 19 in case (3) is satisfied. The general case of Observation 17 requires only a minor technical change to the algorithm. Namely, the computation of $\overrightarrow{\mathbf{DP}}$ then additionally includes all states (i, j, ℓ, p) such that $\ell \in \overrightarrow{L}[i, j]$, $p \geq \frac{1}{z}$, and $p = \pi_i^1(c)p'$ for some $c \in \Sigma$ and $p' \in \text{Freq}_{i-1}(W_1, \sqrt{z})$. Due to $|\Sigma| = \mathcal{O}(1)$, the number of such states is still $\mathcal{O}(n^2\sqrt{z}\log z)$.

For every i and j , the algorithm solves $\mathcal{O}(\log^2 z)$ instances of MERGE, each of size $\mathcal{O}(\sqrt{z})$. This results in the total running time of $\mathcal{O}(n^2\sqrt{z}\log^2 z)$. \square

4.3 Third Improvement: Removing one log z Factor

The final improvement is obtained by a structural transformation after which we only need to consider $\mathcal{O}(\log z)$ pairs (ℓ_L, ℓ_R) .

For this to be possible, we compute prefix maxima on the ℓ -dimension of the $\overrightarrow{\mathbf{DP}}$ and $\overleftarrow{\mathbf{DP}}$ arrays in order to guarantee monotonicity. That is, if $\text{MERGE}(A, B, z)$ returns true for ℓ_L and ℓ_R , then we make sure that it would also return true if any of these two lengths increased (within the corresponding intervals).

This lets us compute, for every $\ell_L \in \overrightarrow{L}[i, j]$ the smallest $\ell_R \in \overleftarrow{L}[i, j]$ such that $\text{MERGE}(A, B, z)$ returns true using $\mathcal{O}(\log z)$ iterations because the sought ℓ_R may only decrease as ℓ_L increases. The pseudocode is given in Algorithm 4.

Algorithm 4: Improved3(W_1, W_2, z, i, j)

```

foreach state  $(i, j, \ell, p)$  of  $\overrightarrow{\mathbf{DP}}$  in lexicographic order do
     $\overrightarrow{\mathbf{DP}}[i, j, \ell, p] := \max(\overrightarrow{\mathbf{DP}}[i, j, \ell, p], \overrightarrow{\mathbf{DP}}[i, j, \ell - 1, p]);$ 
foreach state  $(i, j, \ell, p)$  of  $\overleftarrow{\mathbf{DP}}$  in lexicographic order do
     $\overleftarrow{\mathbf{DP}}[i, j, \ell, p] := \max(\overleftarrow{\mathbf{DP}}[i, j, \ell, p], \overleftarrow{\mathbf{DP}}[i, j, \ell - 1, p]);$ 
 $[a..b] := \overrightarrow{L}[i, j]; [a'..b'] := \overleftarrow{L}[i + 1, j + 1];$ 
 $\ell_L := a; \ell_R := b' + 1; res := \infty;$ 
while  $\ell_L \leq b$  and  $\ell_R \geq a'$  do
     $A := \{(p, q) : \overrightarrow{\mathbf{DP}}[i, j, \ell_L, p] = q\};$ 
     $B := \{(p, q) : \overleftarrow{\mathbf{DP}}[i + 1, j + 1, \ell_R - 1, p] = q\};$ 
    if  $\text{MERGE}(A, B, z)$  then  $\triangleright \ell_R$  is too large for the current  $\ell_L$ 
         $\ell_R := \ell_R - 1;$ 
    else  $\triangleright \ell_R$  reached the target value for the current  $\ell_L$ 
        if  $\ell_R \leq b'$  then  $res := \min(res, \ell_L + \ell_R);$ 
         $\ell_L := \ell_L + 1;$ 
return  $res;$ 

```

Theorem 21. *The WSCS problem can be solved in $\mathcal{O}(n^2\sqrt{z}\log z)$ time if $|\Sigma| = \mathcal{O}(1)$.*

Proof. Let us fix indices i and j . Let us denote $\text{Freq}_i(W, z)$ by $\overrightarrow{\text{Freq}}_i(W, z)$ and introduce a symmetric array

$$\overleftarrow{\text{Freq}}_i(W, z) = \{\mathcal{P}(S, W[i..|W|]) : S \in \text{Matched}_z(W[i..|W|])\}.$$

In the first loop of prefix maxima computation, we consider all $\ell \in \overrightarrow{L}[i, j]$ and $p \in \overrightarrow{\text{Freq}}_i(W_1, \sqrt{z})$, and in the second loop, all $\ell \in \overleftarrow{L}[i, j]$ and $p \in \overleftarrow{\text{Freq}}_i(W_1, \sqrt{z})$. Hence, prefix maxima take $\mathcal{O}(\sqrt{z}\log z)$ time to compute.

Each step of the while-loop in **Improved3** increases ℓ_L or decreases ℓ_R . Hence, the algorithm produces only $\mathcal{O}(\log z)$ instances of **MERGE**, each of size $\mathcal{O}(\sqrt{z})$. The time complexity follows. \square

5 Lower Bound for WLCS

Let us first define the WLCS problem as it was stated in [4,14].

WEIGHTED LONGEST COMMON SUBSEQUENCE (WLCS(W_1, W_2, z))
Input: Weighted strings W_1 and W_2 of length up to n and a threshold $\frac{1}{z}$.
Output: A longest standard string S such that $S \subseteq_z W_1$ and $S \subseteq_z W_2$.

We consider the following well-known NP-complete problem [19]:

SUBSET SUM
Input: A set S of positive integers and a positive integer t .
Output: Is there a subset of S whose elements sum up to t ?

Theorem 22. *The WLCS problem cannot be solved in $\mathcal{O}(n^{f(z)})$ time if $\text{P} \neq \text{NP}$.*

Proof. We show the hardness result by reducing the NP-complete SUBSET SUM problem to the WLCS problem with a constant value of z .

For a set $S = \{s_1, s_2, \dots, s_n\}$ of n positive integers, a positive integer t , and an additional parameter $p \in [2..n]$, we construct two weighted strings W_1 and W_2 over the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}\}$, each of length n^2 .

Let $q_i = \frac{s_i}{t}$. At positions $i \cdot n$, for all $i \in [1..n]$, the weighted string W_1 contains letter \mathbf{a} with probability 2^{-q_i} and \mathbf{b} otherwise, while W_2 contains \mathbf{a} with probability $2^{\frac{1}{p-1}(q_i-1)}$ and \mathbf{b} otherwise. All the other positions contain letter \mathbf{b} with probability 1. We set $z = 2$.

We assume that S contains only elements smaller than t (we can ignore the larger ones and if there is an element equal to t , then there is no need for a reduction). All the weights of \mathbf{a} are then in the interval $(\frac{1}{2}, 1)$ since $-q_i \in (-1, 0)$ and $\frac{1}{p-1}(q_i - 1) \in (-1, 0)$. Thus, since $z = 2$, letter \mathbf{b} originating from a position

$i \cdot n$ can never occur in a subsequence of W_1 or in a subsequence of W_2 . Hence, every common subsequence of W_1 and W_2 is a subsequence of $(\mathbf{b}^{n-1}\mathbf{a})^n$.

For $I \subseteq [1 \dots n]$, we have

$$\prod_{i \in I} \pi_{i \cdot n}^{(W_1)}(\mathbf{a}) = \prod_{i \in I} 2^{-s_i/t} \geq 2^{-1} = \frac{1}{2} \iff \sum_{i \in I} s_i \leq t$$

and

$$\begin{aligned} \prod_{i \in I} \pi_{i \cdot n}^{(W_2)}(\mathbf{a}) &= \prod_{i \in I} 2^{\frac{1}{p-1}(s_i/t-1)} \geq 2^{-1} = \frac{1}{2} \iff \\ &\frac{1}{t(p-1)} \left(\sum_{i \in I} s_i \right) - \frac{|I|}{p-1} \geq -1 \iff \sum_{i \in I} s_i \geq t(1-p+|I|). \end{aligned}$$

If I is a solution to the instance of the SUBSET SUM problem, then for $p = |I|$ there is a weighted common subsequence of length $n(n-1) + p$ obtained by choosing all the letters \mathbf{b} and the letters \mathbf{a} that correspond to the elements of I .

Conversely, suppose that the constructed WLCS instance with a parameter $p \in [2 \dots n]$ has a solution of length at least $n(n-1) + p$. Notice that \mathbf{a} at position $i \cdot n$ in W_1 may be matched against \mathbf{a} at position $i' \cdot n$ in W_2 only if $i = i'$. (Otherwise, the length of the subsequence would be at most $(n - |i - i'|)n \leq (n-1)n < n(n-1) + p$.) Consequently, the solution yields a subset $I \subseteq [1 \dots n]$ of at least p indices i such that \mathbf{a} at position $i \cdot n$ in W_1 is matched against \mathbf{a} at position $i \cdot n$ in W_2 . By the relations above, we have (a) $|I| \geq p$, (b) $\sum_{i \in I} s_i \leq t$, and (c) $\sum_{i \in I} s_i \geq t(1-p+|I|)$. Combining these three inequalities, we obtain $\sum_{i \in I} s_i = t$ and conclude that the SUBSET SUM instance has a solution.

Hence, the SUBSET SUM instance has a solution if and only if there exists $p \in [2 \dots n]$ such that the constructed WLCS instance with p has a solution of length at least $n(n-1) + p$. This concludes that an $\mathcal{O}(n^{f(z)})$ -time algorithm for the WLCS problem implies the existence of an $\mathcal{O}(n^{2f(2)+1}) = \mathcal{O}(n^{\mathcal{O}(1)})$ -time algorithm for the SUBSET SUM problem. The latter would yield $P = NP$. \square

Example 23. For $S = \{3, 7, 11, 15, 21\}$ and $t = 25 = 3 + 7 + 15$, both weighted strings W_1 and W_2 are of the form:

$$\mathbf{b}^4 * \mathbf{b}^4 * \mathbf{b}^4 * \mathbf{b}^4 * \mathbf{b}^4 *,$$

where each $*$ is equal to either \mathbf{a} or \mathbf{b} with different probabilities.

The probabilities of choosing \mathbf{a} 's for W_1 are equal respectively to

$$\left(2^{-\frac{3}{25}}, 2^{-\frac{7}{25}}, 2^{-\frac{11}{25}}, 2^{-\frac{15}{25}}, 2^{-\frac{21}{25}} \right),$$

while for W_2 they depend on the value of p , and are equal respectively to

$$\left(2^{-\frac{22}{25(p-1)}}, 2^{-\frac{18}{25(p-1)}}, 2^{-\frac{14}{25(p-1)}}, 2^{-\frac{10}{25(p-1)}}, 2^{-\frac{4}{25(p-1)}} \right).$$

For $p = 3$, we have: $\text{WLCS}(W_1, W_2, 2) = \mathbf{b}^4 \mathbf{a} \mathbf{b}^4 \mathbf{a} \mathbf{b}^4 \mathbf{a} \mathbf{b}^4$, which corresponds to taking the first, the second, and the fourth \mathbf{a} . The length of this string is equal to $23 = n(n-1) + p$, and its probability of matching is $\frac{1}{2} = 2^{-\frac{22}{50}} \cdot 2^{-\frac{18}{50}} \cdot 2^{-\frac{10}{50}}$. Thus, the subset $\{3, 7, 15\}$ of S consisting of its first, second, and fourth element is a solution to the SUBSET SUM problem.

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