Near-Optimal Computation of Runs over General Alphabet via Non-Crossing LCE Queries

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LCE Problem

For a given word w of length n, answer a sequence of $q = \mathcal{O}(n)$ queries $\mathrm{LCE}(i, j)$ in an on-line manner.

Algorithms for the LCE Problem

Integer alphabet

Letters can be sorted in $\mathcal{O}(n)$ time (e.g., integers $\{1, \ldots, n\}$).

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$\mathcal{O}(n \log^{2/3} n)$	Kosolobov; IPL 2016
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Runs (Maximal Repetitions)

Definition

A run is a maximal periodic fragment $w[i..j]$. For $p = \mathrm{per}(w[i..j])$,

- $2p \leq |w[i..j]|$,
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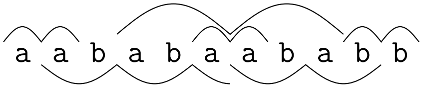

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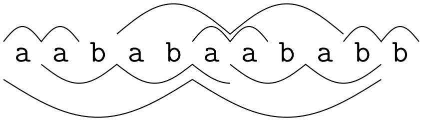

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period 5: $w[1..10]$

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Improvements via non-crossing LCE queries:

This work: $\mathcal{O}(n\alpha(n))$ time, $\mathcal{O}(n)$ comparisons.

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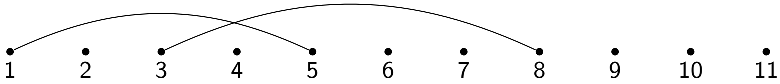

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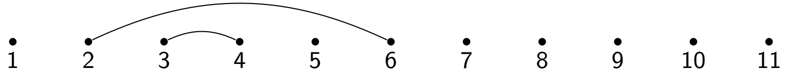

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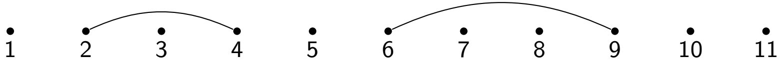

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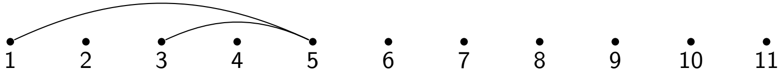

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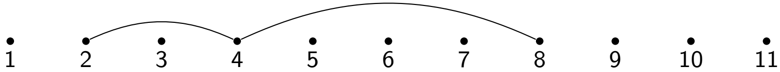

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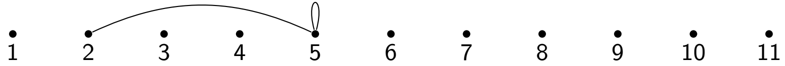

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Theorem (Our main technical result)

The LCE *problem can be solved in* $\mathcal{O}(n\alpha(n))$ *time in the general alphabet model if args* $\{i, j\}$ *of the* LCE *queries are non-crossing.*

Properties of Non-Crossing Families

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A family of non-crossing pairs from $\{1, \ldots, n\}$ *contains less than* $3n$ *distinct pairs.*

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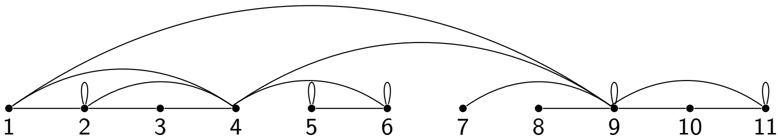

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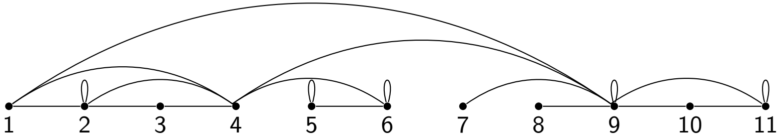

Proof. Pairs $\{i, j\}$ form the edge set of an outerplanar graph:

- at most n loops
- simple outerplanar graph has less than $2n$ edges.

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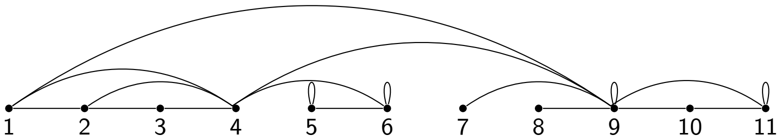
Let us partition $\{1, \ldots, n\}$ into b contiguous blocks.
A family of non-crossing pairs involves less than $3b$ pairs of blocks (block-pairs).



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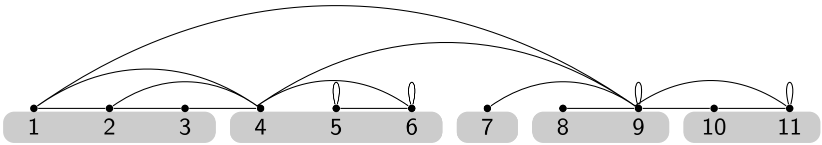

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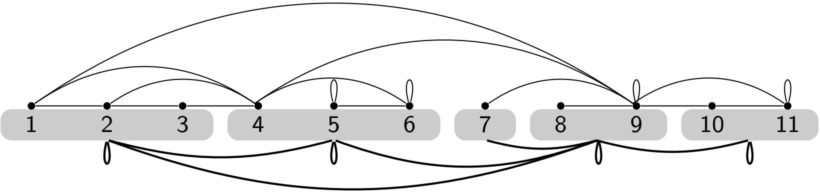

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Lemma (Gawrychowski et al., CPM 2016)

A sequence of $\mathrm{LCE}_{\leq \ell_q}(i_q, j_q)$ queries can be answered on-line in $\mathcal{O}((n + \sum \log \ell_q) \cdot \alpha(n))$ time in the general alphabet model.

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Non-crossing LCE queries in $\mathcal{O}(n\alpha(n))$ time:

- apply the idea above to $\mathcal{O}(\log n)$ levels,
- blocks of length 2^k in level k ($k = 0, \ldots, \log n$).

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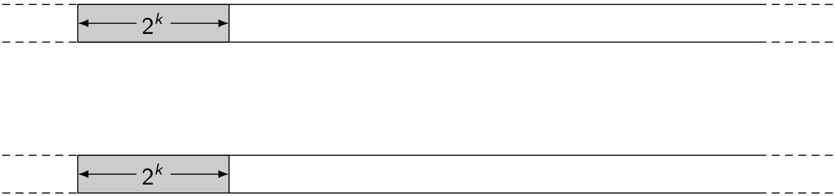

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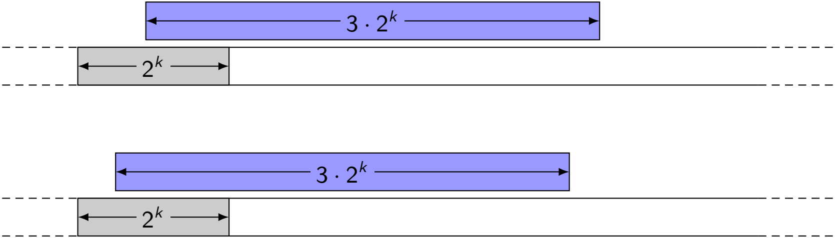

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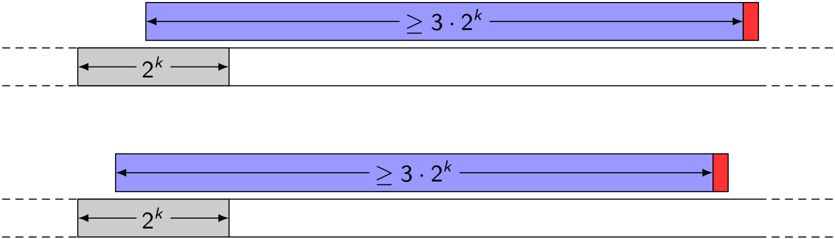

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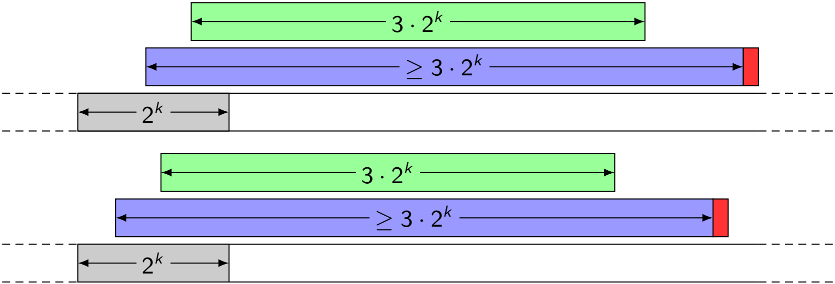
- The result can be exploited for queries with the same shift.



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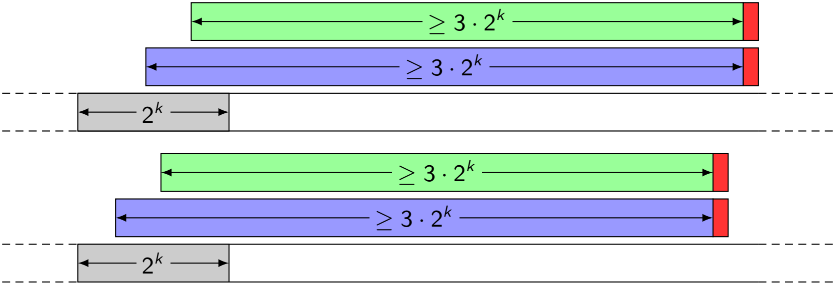

Answering Long Queries Involving a Block-Pair

Intuition:

- **learn** the structure using $\mathcal{O}(1)$ (unlimited) LCE queries,
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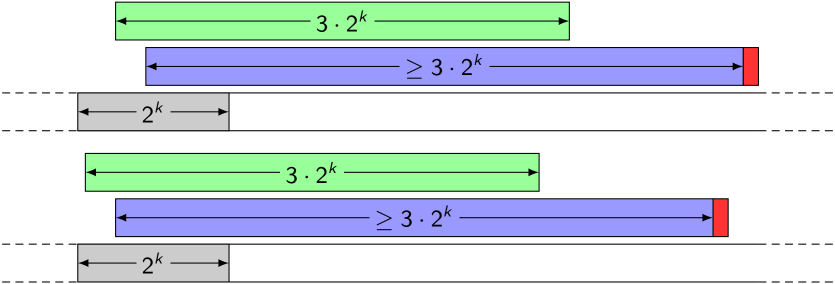

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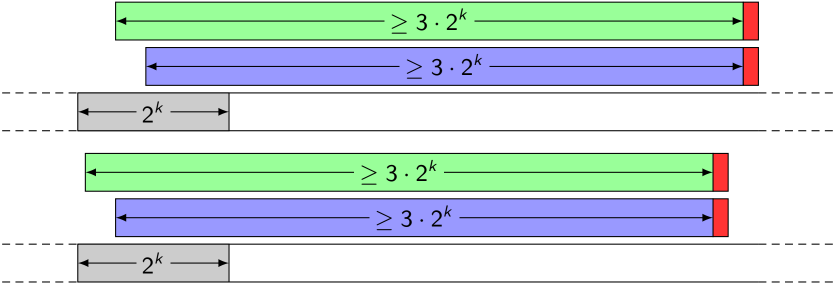

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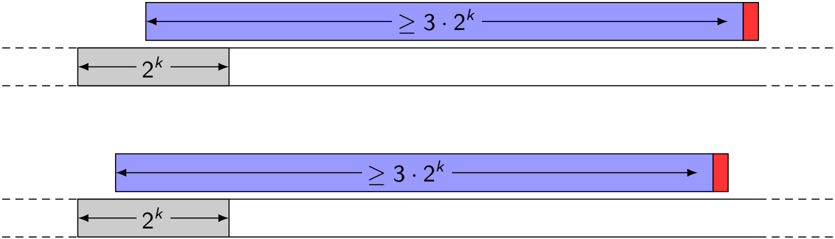

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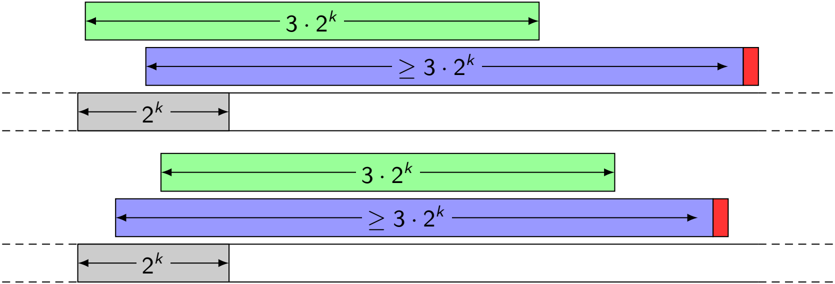

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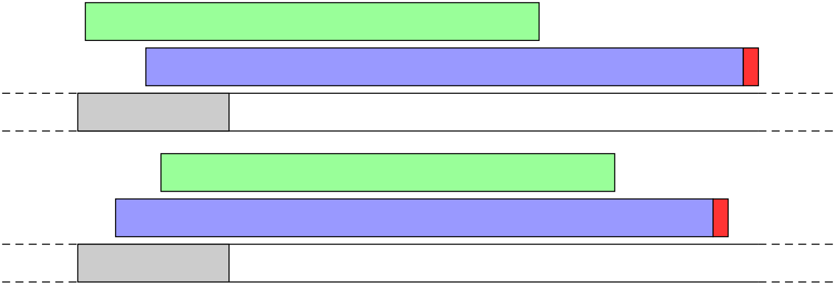

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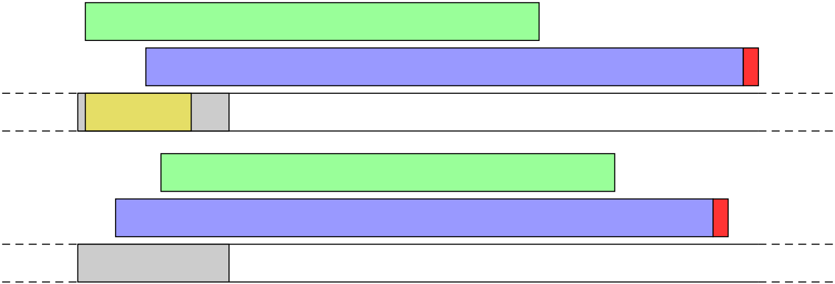

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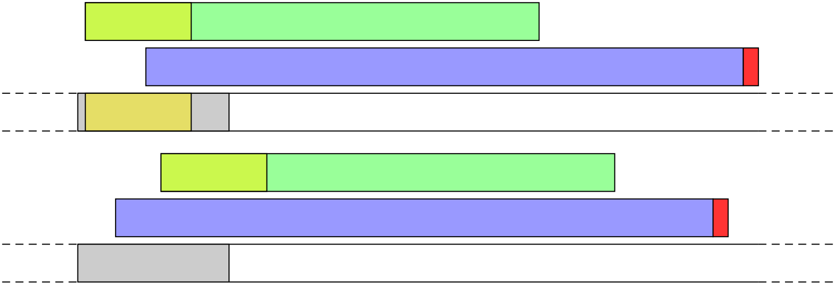

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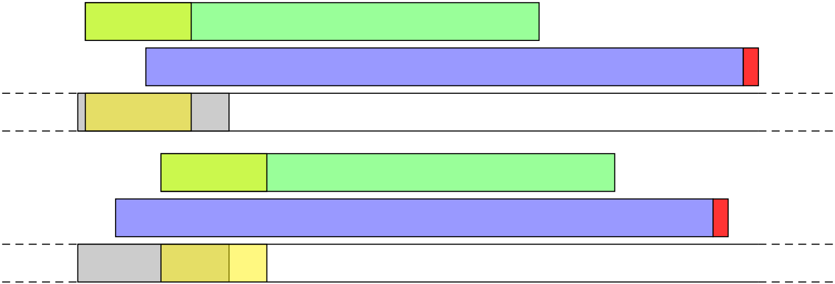

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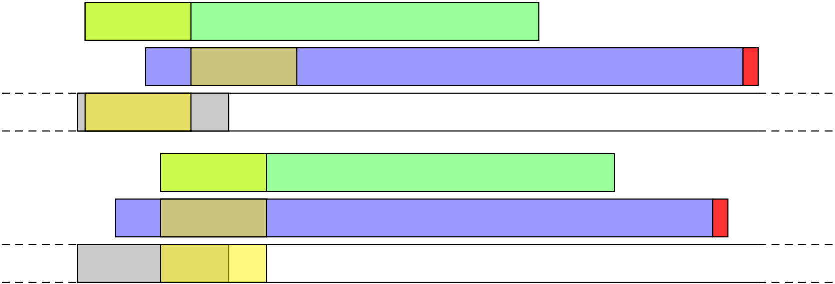

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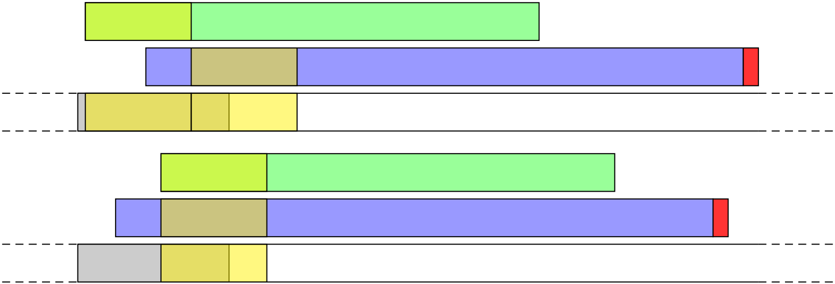

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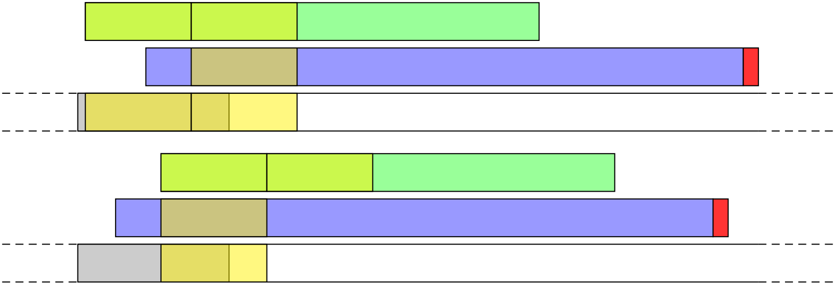

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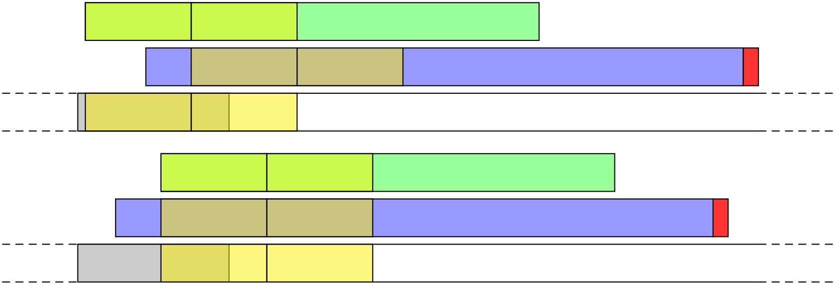

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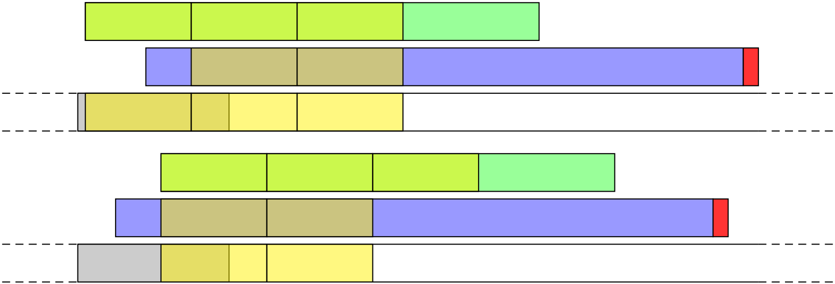

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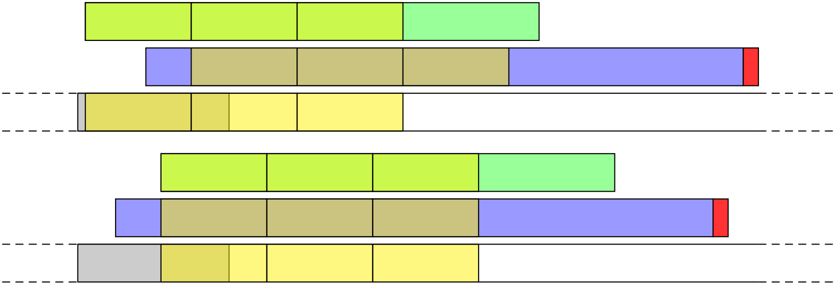

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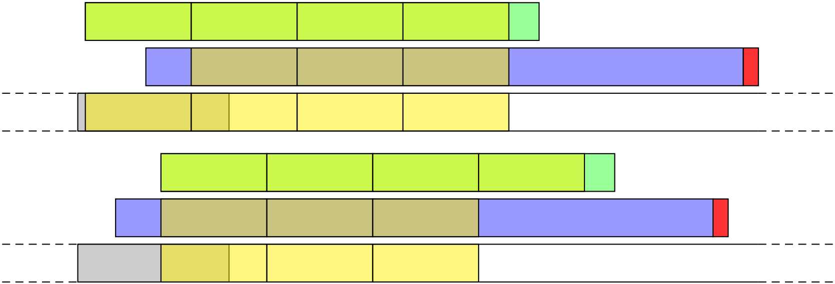

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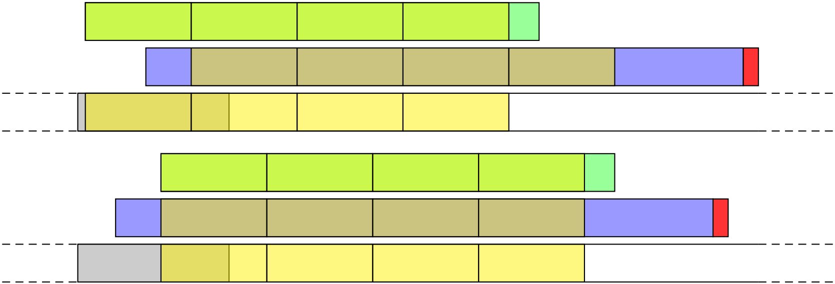

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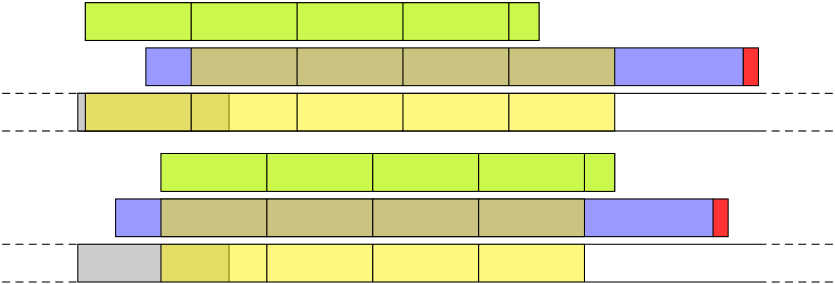

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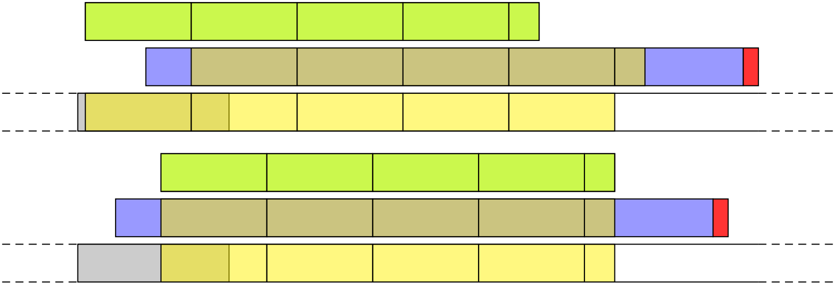

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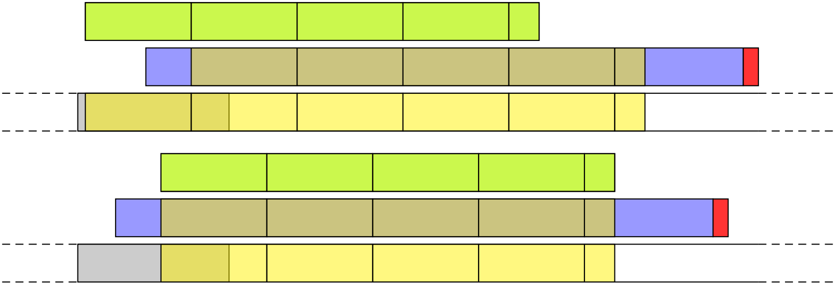

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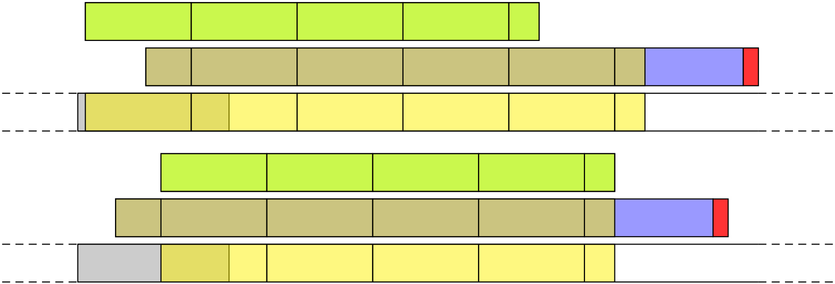

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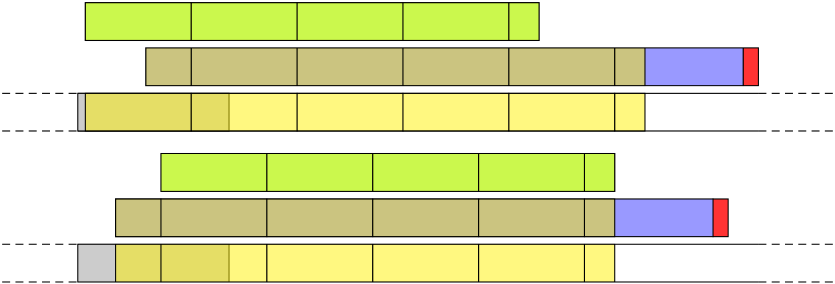

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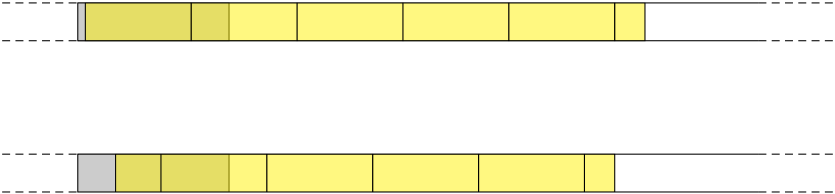

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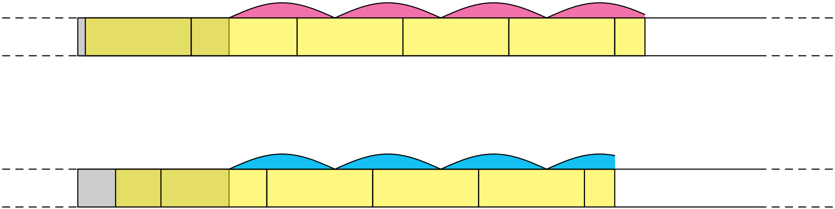

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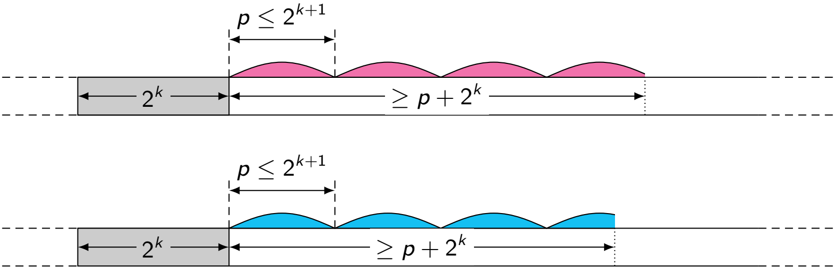

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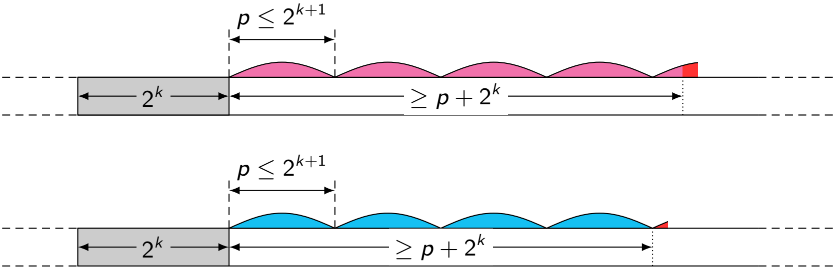

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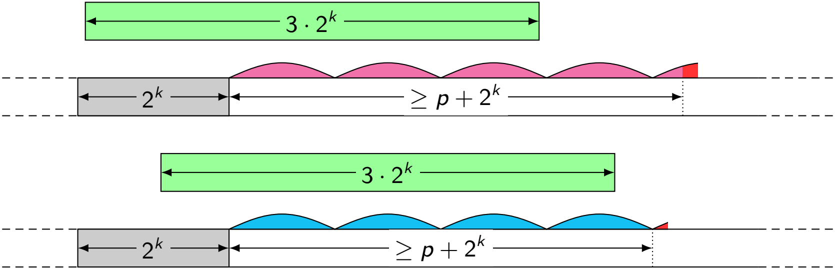

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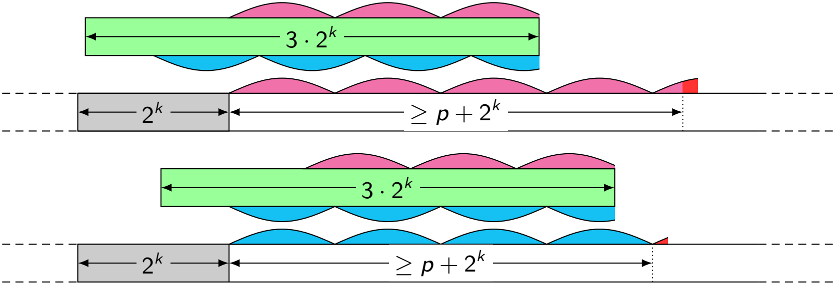

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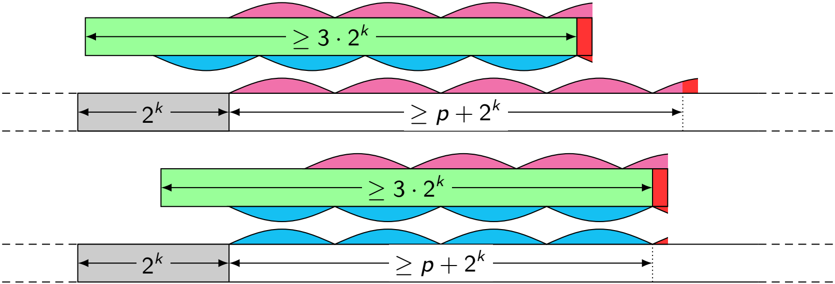

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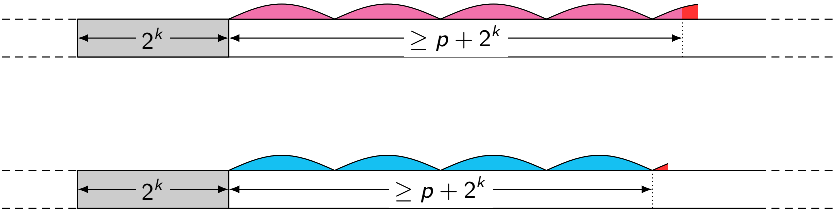

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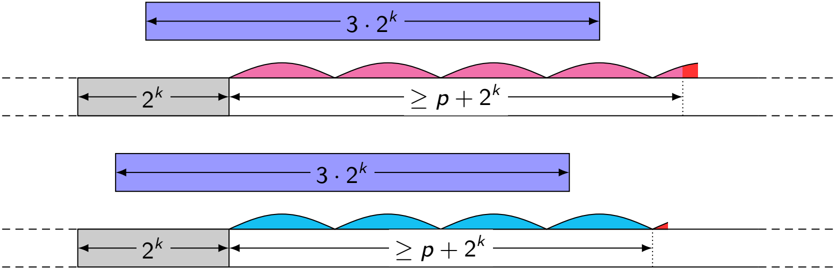

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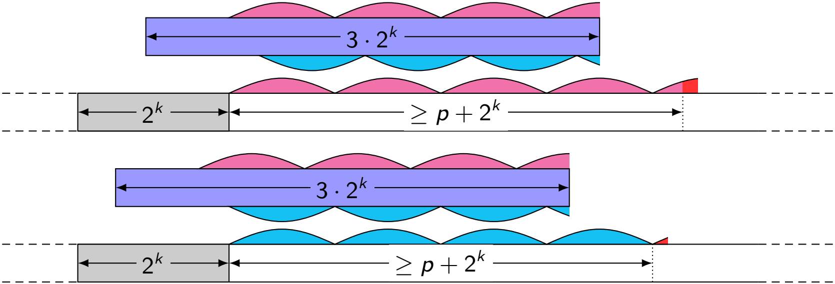

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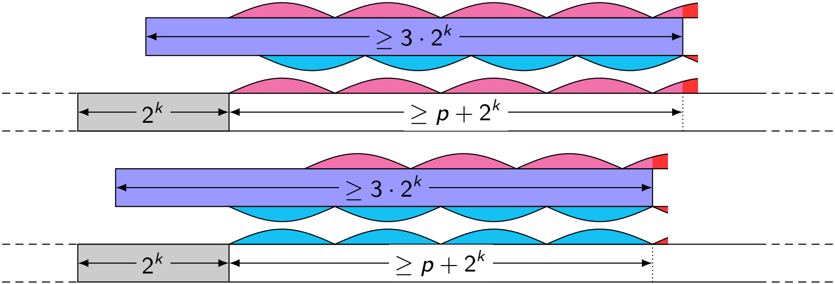

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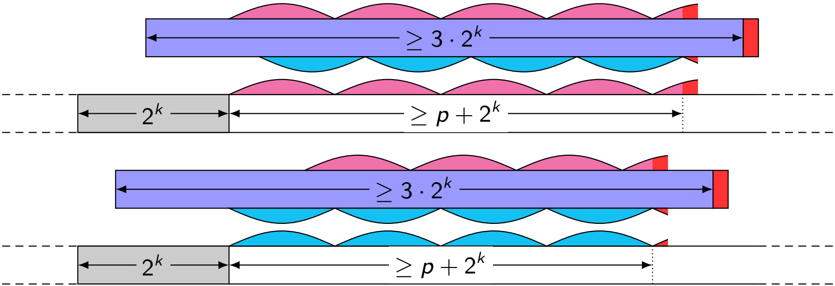

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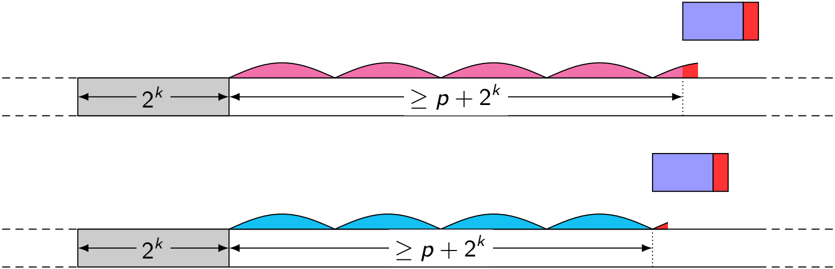

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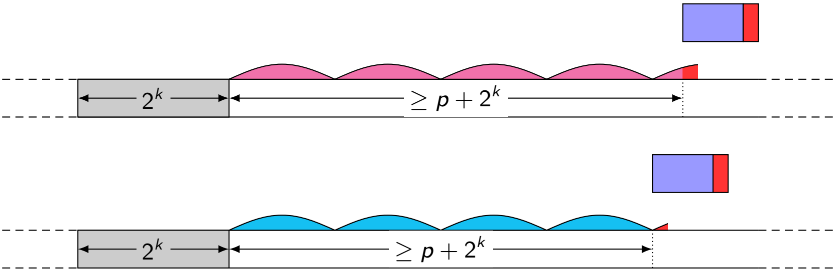
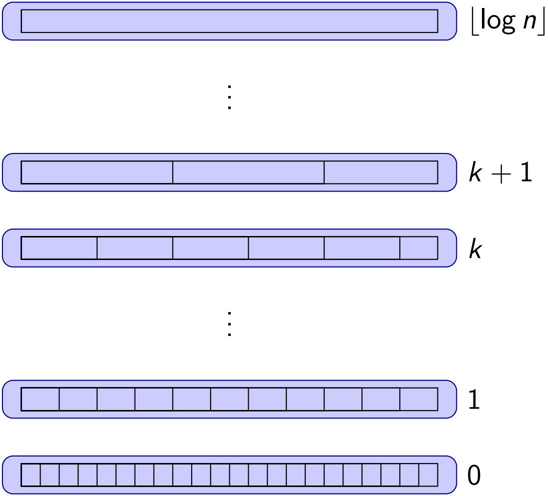


Answering Non-Crossing LCE Queries: Overview



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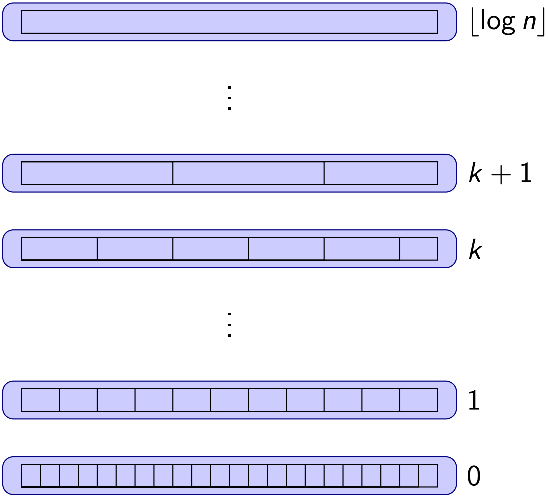

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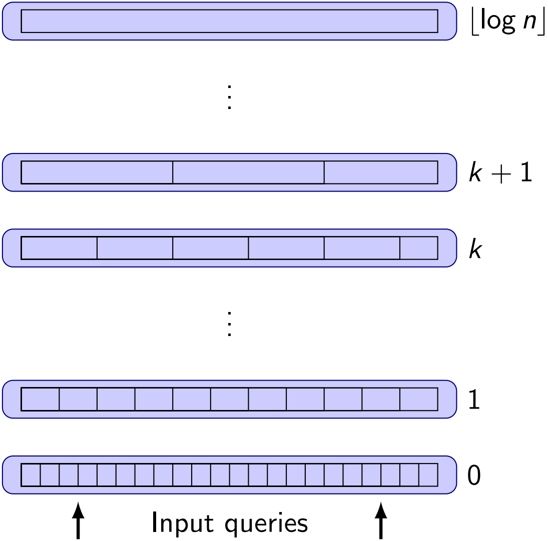

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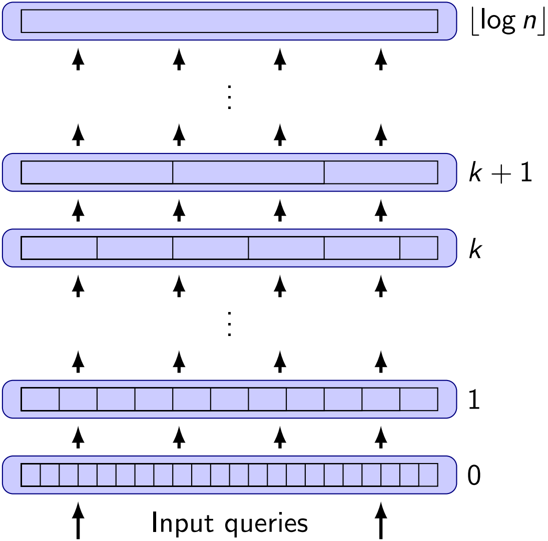

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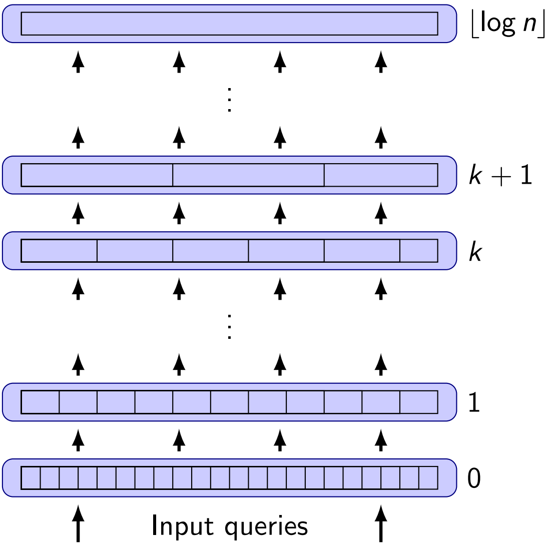

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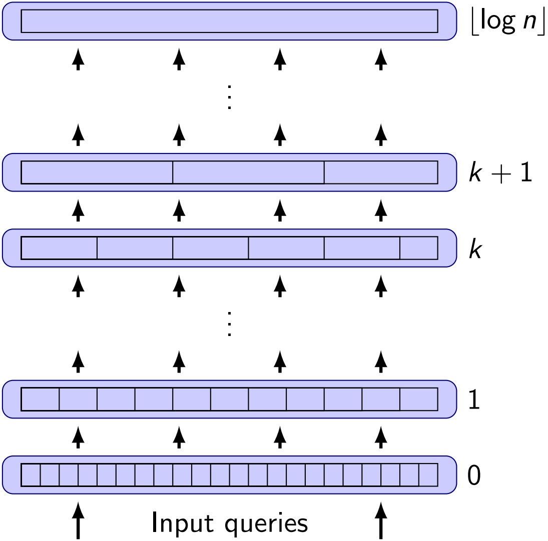

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Theorem (Our main technical result)

The LCE problem can be solved in $\mathcal{O}(n\alpha(n))$ time in the general alphabet model if the $\mathrm{LCE}(i,j)$ queries are non-crossing.

Runs Computation

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4. Check which nodes correspond to Lyndon roots of runs.
   - $\mathcal{O}(n)$ non-crossing LCE queries in w (extension to the right),
   - $\mathcal{O}(n)$ non-crossing LCE queries in w^R (extension to the left).

Theorem

Runs in a word of length n over a general ordered alphabet can be computed in $\mathcal{O}(n\alpha(n))$ time.

Questions?

Thank you for your attention!