# A Linear Time Algorithm for Seeds Computation

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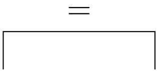
# Periodicity and quasiperiodicity

Periodicity:




a b a a a b a a a b a a a b a a

One of the key concepts in text algorithms.

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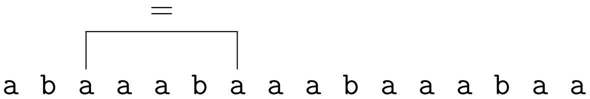


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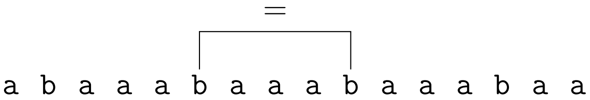


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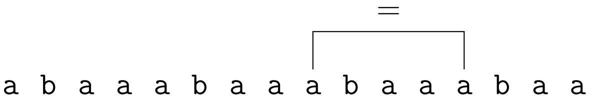


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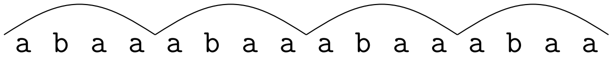


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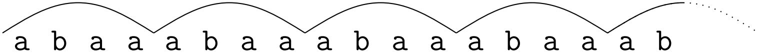


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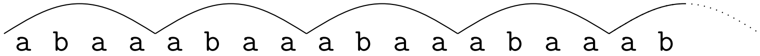


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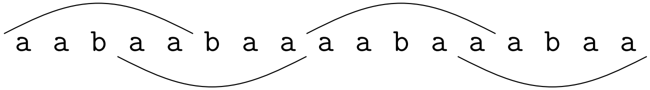

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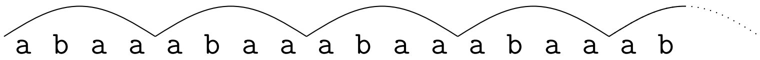


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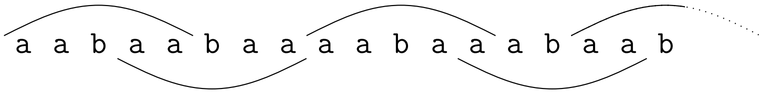

# Covers and seeds

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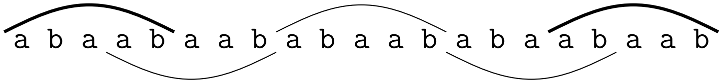
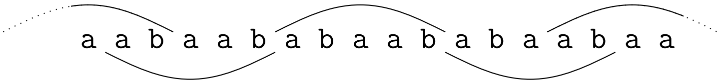



Each letter of the word is covered by an occurrence of the cover.

# Covers and seeds

Seed:



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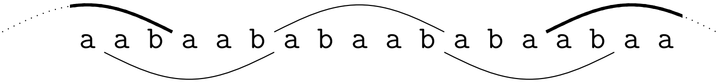


Each letter of the word is covered by an occurrence of the seed. The occurrences can be external.

# The main problem

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**Problem (All-Seeds)**

*Given a word $w$ of length $n$ over an alphabet $\Sigma$, compute an $O(n)$-sized representation of all the seeds of $w$.*

**Theorem (Our result)**

*The All-Seeds Problem for $\Sigma = \{0, 1, \ldots, n^{O(1)}\}$ can be solved in $O(n)$ time.*

# Background

- Seeds were introduced in 1993 by Iliopoulos, Moore & Park.
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- No $o(n \log n)$ algorithm even for the Shortest-Seed Problem for binary alphabet up to now.
- W.F. Smyth stated finding a linear algorithm for the All-Seeds Problem as a hard open problem in his survey (2000).

# Background

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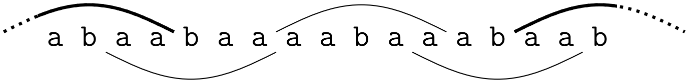
- For covers linear algorithms for similar problems are known:
    - shortest covers of each prefix (Breslauer, 1992)
    - all covers (Moore & Smyth, SODA 1994)
- Variants of seeds have been studied:
    - approximate seeds (Christodoulakis et al., 2003)
    - $\lambda$-seeds (Guo, Zhang & Iliopoulos, 2006)

# Constraints for seeds

Two different types of constraints

- Border constraints, easier



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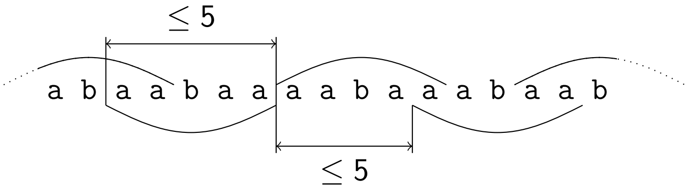
- Maxgap constrains, harder




Maxgap is a maximal distance between the starting positions of two consecutive occurrences of a given subword.

# Quasiseeds

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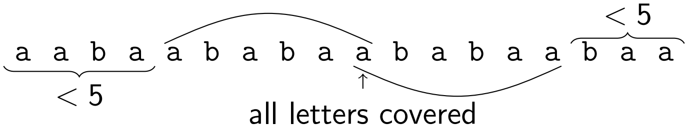
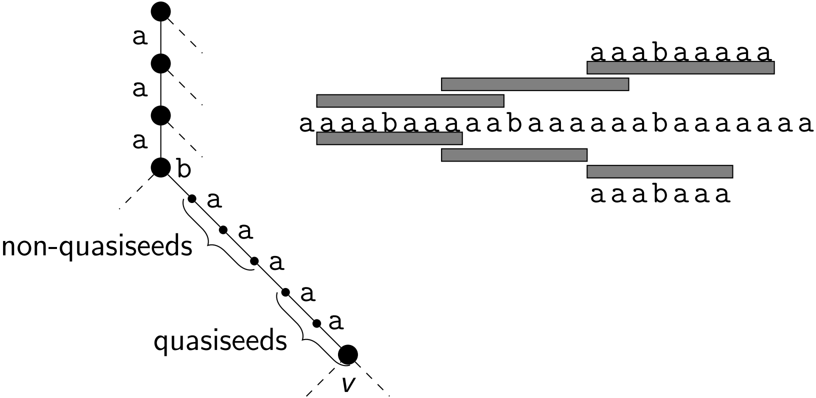
**Definition (Quasiseed)**

A subword $v$ is a *quasiseed* of $w$ if there are less than $|v|$ letters both before its first occurrence and after the last one and each letter between those two occurrences is covered by an occurrence of $v$.



# Useful properties of quasiseeds

An $O(n)$ representation on the suffix tree.



# Useful properties of quasiseeds

**Lemma (Restricted-Quasiseeds)**

*Given an integer $d$ and a word $w$ of length $n$, the representation of all quasiseeds of length in $\{d, d+1, \ldots, 2d\}$ can be found in $O(n)$ time.*

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- The All-Seeds Problem can be linearly reduced to computing (the representation of) all quasiseeds.

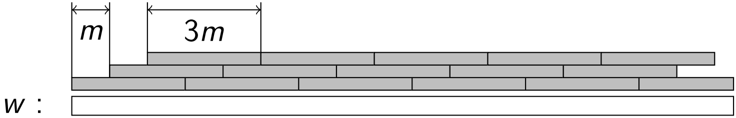
# Main problem

**Problem (All-Quasiseeds)**

*Given a word of length $n$, compute the representation of all its quasiseeds.*

# Recursive structure of the algorithm

Interval $m$-staircase



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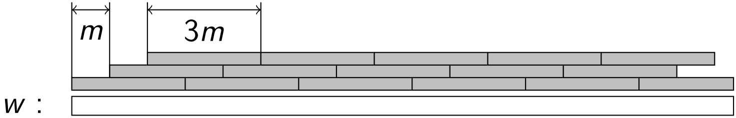


**Lemma (Short Quasiseeds)**

*A subword $v$ of length $< m$ is a quasiseed of $w$ if and only if it is a quasiseed of each subword corresponding to an $m$-staircase interval.*

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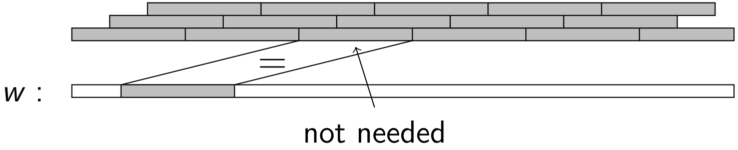

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Outline:

1. Find an appropriate reduced staircase
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**Main issue**: How to find an appropriate $m$, so that simultaneously:

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- Merging is not as easy as it may seem (RMQ and static find-union).

# $f$-factorization

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**Definition ($f$-factorization)**

An $f$-factorization $f_1 f_2 \ldots f_k$ of $w$ is constructed greedily: $f_i$ is either just the first occurrence of a letter or the longest prefix of the remaining suffix that is a subword of $f_1 \ldots f_{i-1}$.

$$\texttt{a} \,|\, \texttt{b} \,|\, \texttt{a} \,|\, \texttt{a b a} \,|\, \texttt{b a a b a} \,|\, \texttt{b a b} \,|\, \texttt{c} \,|\, \texttt{a}$$

# $f$-factorization

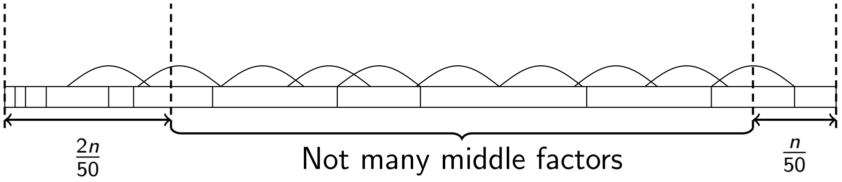
**Theorem (Crochemore, 1983; Crochemore et al. 2009)**

*The $f$-factorization over (constant) integer alphabet can be computed in $O(n)$ time.*

# Quasiseeds, staircase and factorization

**Lemma**

*Let $F$ be the $f$-factorization of $w$ ($|w| = n$) and $v$ be a quasiseed of $w$, $|v| < \frac{n}{50}$. Then at most $\left\lfloor \frac{2n}{|v|} \right\rfloor - 1$ factors from $F$ lie within $[\frac{2n}{50}, \frac{49n}{50}]$.*



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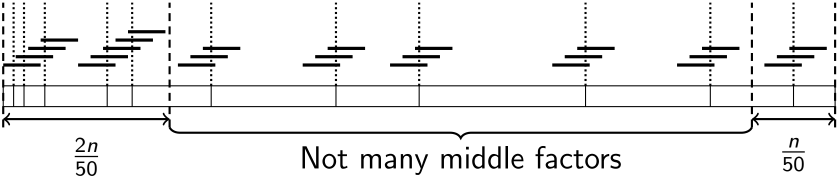


Stairs lying within a single factor are not necessary.

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**Lemma**

*If $m \leq \frac{n}{50(g+1)}$ then the size of the reduced staircase is $< \frac{n}{2}$.*

# Final structure of the algorithm

1. Find an $f$-factorization and the number of middle factors ($g$)
2. $m := \left\lfloor \frac{n}{50(g+1)} \right\rfloor$
3. Compute the reduced staircase
4. Compute the long quasiseeds (belonging to two ranges of fixed ratio)
5. If $m > 0$ compute the short quasiseeds by recursive calls and merge the results

# Conclusions

- We have presented a linear algorithm for the All-Quasiseeds Problem (over integer alphabet).
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