Pattern Matching and Consensus Problems on Weighted Sequences and Profiles

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Strings, Partial Words, and Indeterminate Strings

Strings (solid strings):

a c a b b b

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a $\diamondsuit$ a b $\diamondsuit$ b

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Partial words (strings with don't care symbols):

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a	c	a	b	b	b
a	a	a	b	a	b
a	b	a	b	c	b

$\vdots$

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Indeterminate strings:

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Indeterminate strings:

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a	c	a	b	a	b
a	b	a	b	b	b
a	b	a	b	a	b

Weighted Sequences (PPMs) and Profiles (PWMs)

Weighted Sequences (Position Probability Matrices):

$$\mathtt{a}\quad \begin{matrix}\mathtt{b}\ 0.2\\ \mathtt{c}\ 0.8\end{matrix}\quad \mathtt{a}\quad \mathtt{b}\quad \begin{matrix}\mathtt{a}\ 0.6\\ \mathtt{b}\ 0.4\end{matrix}\quad \mathtt{b}$$

Weighted Sequences (PPMs) and Profiles (PWMs)

Weighted Sequences (Position Probability Matrices):

a	b 0.2 c 0.8	a	b	a 0.6 b 0.4	b	probability
a	c	a	b	b	b	0.32
a	c	a	b	a	b	0.48
a	b	a	b	b	b	0.08
a	b	a	b	a	b	0.12

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a 1	a 0	a 1	a 0	a 0.6	a 0	
b 0	b 0.2	b 0	b 1	b 0.4	b 1	
c 0	c 0.8	c 0	c 0	c 0	c 0	probability
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Profiles (Position Weight Matrices):

a 7 b 0 c 1	a 3 b 2 c 8	a 0 b 1 c 0	a 0 b 5 c 0	a 6 b 4 c 3	a 1 b 9 c 0

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Profiles (Position Weight Matrices):

a 7	a 3	a 0	a 0	a 6	a 1	
b 0	b 2	b 1	b 5	b 4	b 9	score
c 1	c 8	c 0	c 0	c 3	c 0	
a	c	a	b	b	b	33
a	c	a	b	a	b	35
a	b	a	b	b	b	27
a	b	a	b	a	b	29

⋮

Weighted Sequences (PPMs) and Profiles (PWMs)

Weighted Sequences (Position Probability Matrices): ← **this talk**

a 1 b 0 c 0	a 0 b 0.2 c 0.8	a 1 b 0 c 0	a 0 b 1 c 0	a 0.6 b 0.4 c 0	a 0 b 1 c 0	probability
a	c	a	b	b	b	0.32
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Profiles (Position Weight Matrices): ← **ask me or see our paper**

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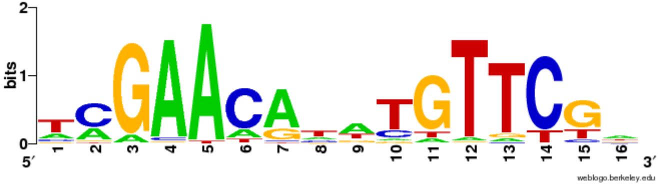
Applications of Uncertain Strings

Bioinformatics

- introduced in:

  Stormo, Schneider, Gold, and Ehrenfeucht (1982). "Use of the 'Perceptron' algorithm to distinguish translational initiation sites in E. coli". Nucleic Acids Research **10** (9): 2997–3011.

- one of the standard representations of motifs



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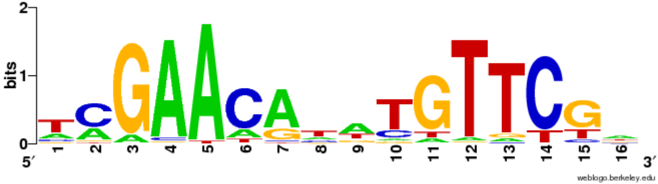

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Other applications

- annotation of noisy sensor data,
- probabilistic databases.

Threshold

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Definition

A solid string S matches a weighed sequence X if $\mathcal{P}(S, X) \geq \frac{1}{z}$ for a given threshold $\frac{1}{z}$.

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For every weighted sequence X, there are at most z strings X satisfying $\mathcal{P}(S, X) \geq \frac{1}{z}$.

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Fact

For every weighted sequence X, there are at most z strings X satisfying $\mathcal{P}(S,X) \geq \frac{1}{z}$.

- z is used as a parameter for designing the algorithms.

Algorithms for Weighted Sequences

Indexing and pattern matching

Christodoulakis et al., 2004; Iliopoulos et al., 2006;
Amir et al., 2008; Barton et al., 2016; Biswas et al., 2016

Approximate and gapped pattern matching

Zhang et al., 2004; Amir et al., 2006; Zhang et al., 2010

Repetitions and regularities discovery

Iliopoulos et al., 2005; Christodoulakis et al., 2006;
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Longest common subsequence problem

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Alignment of weighted sequences

Na et al., 2009

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Consensus and generalized pattern matching $\leftarrow$ **this work**

Weighted Pattern Matching

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$\frac{1}{z} = 0.2$ T :

a 0.25	a 1	a 0.75	a 0.5	a 1
b 0.75	b 0	b 0.25	b 0.5	b 0

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P :	a	a	b		

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$\frac{1}{z} = 0.2$	T :	a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0
$0.0625 < \frac{1}{z}$	P :	a	a	b		

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$\frac{1}{z} = 0.2$	T :	a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0
$0.375 \geq \frac{1}{z}$	P :		a	a	b	

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P :			a	a	b

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Given a weighted sequence T, a threshold $1/z$, and a solid pattern P, determine all the fragments of T matching P.

$\frac{1}{z} = 0.2$	T :	a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0
$0 < \frac{1}{z}$	P :			a	a	b

Weighted Pattern Matching: Results

- n: text length,
- R: text size (represented as a sparse matrix),
- m: pattern length,
- σ: alphabet size,
- $\frac{1}{z}$: threshold.

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this work $\mathcal{O}(R + n \log z)$ (LCP + lookahead scoring).

T :	a 0.25	a 1	a 0.75	a 0.5	a 1	$\frac{1}{z} = 0.2$
	b 0.75	b 0	b 0.25	b 0.5	b 0	

Weighted Pattern Matching: Algorithm

T :	a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0	$\frac{1}{z} = 0.2$
$\mathcal{H}(T)$:	b	a	a	b	a	

Heavy string $\mathcal{H}(T)$: most probable character at each position.

Weighted Pattern Matching: Algorithm

T :	a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0	$\frac{1}{z} = 0.2$
$\mathcal{H}(T)$:	b	a	a	b	a	0.5625
P :	a	a	b			≤ 0.5625

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- Compare P with fragments of $\mathcal{H}(T)$:
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T :	a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0	$\frac{1}{z} = 0.2$
$\mathcal{H}(T)$:	b	a	a	b	a	0.5625
P :	a	a	b			≤ 0.1875

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- Compare P with fragments of $\mathcal{H}(T)$:
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- For each mismatch:
  - update the probability bound,
  - break if it drops below threshold.

T :	a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0	$\frac{1}{z} = 0.2$
$\mathcal{H}(T)$:	b	a	a	b	a	0.375
P :		a	a	b		≤ 0.375

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$\mathcal{H}(T)$:	b	a	a	b	a	0.375
P :			a	a	b	≤ 0

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Running time analysis:

- at most $\log z$ mismatches per position (before we break);

Weighted Pattern Matching: Algorithm

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$\mathcal{H}(T)$:	b	a	a	b	a	
P :			a	a	b	

Heavy string $\mathcal{H}(T)$: most probable character at each position.

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Running time analysis:

- at most $\log z$ mismatches per position (before we break);
- $\mathcal{O}(R)$ preprocessing (for $\mathcal{H}(T)$ and the LCE data structure);

Consensus and Generalized Pattern Matching Problems

Weighted Consensus

Given two weighted sequences and a common threshold, decide if there exists a solid string matching both.

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Given two weighted sequences and a common threshold, decide if there exists a solid string matching both.

a 0.5 b 0.5	a 0.5 b 0.5	a 1 b 0	a 0 b 1	a 0.5 b 0.5

$\frac{1}{z} = 0.1$

a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0

Weighted Consensus

Given two weighted sequences and a common threshold, decide if there exists a solid string matching both.

a 0.5 b 0.5	a 0.5 b 0.5	a 1 b 0	a 0 b 1	a 0.5 b 0.5	0.125
					$\frac{1}{z} = 0.1$
a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0	0.28125

Weighted Consensus

Given two weighted sequences and a common threshold, decide if there exists a solid string matching both.

a 0.5 b 0.5	a 0.5 b 0.5	a 1 b 0	a 0 b 1	a 0.5 b 0.5	0.125
					$\frac{1}{z} = 0.1$
a 0.25 b 0.75	a 1 b 0	a 0.75 b 0.25	a 0.5 b 0.5	a 1 b 0	0.28125

General Weighted Pattern Matching

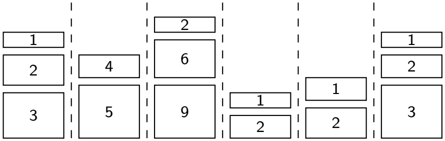
Given an weighted text T, a weighted pattern P, and a threshold $\frac{1}{z}$, decide which fragments of T have a consensus string with P.

Multichoice Knapsack Problem

Multichoice Knapsack

Multichoice Knapsack

- Input:
  - R items in n classes, each item with weight w and value v,



MULTICHOICE KNAPSACK

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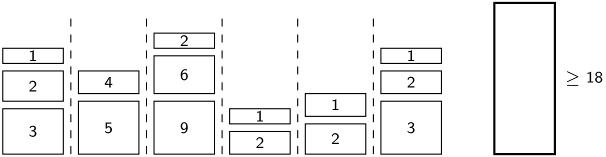

MULTICHOICE KNAPSACK

- Input:
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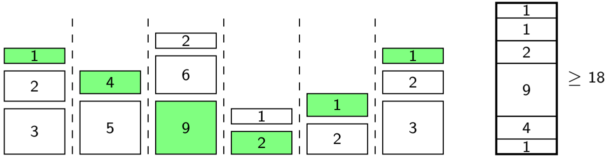
- Output: a choice of 1 item per class, with total weight $\leq W$ and total value $\geq V$.



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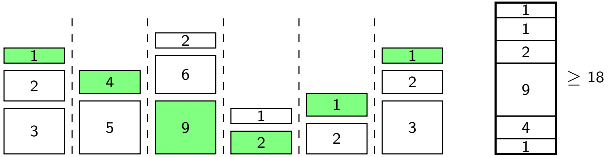

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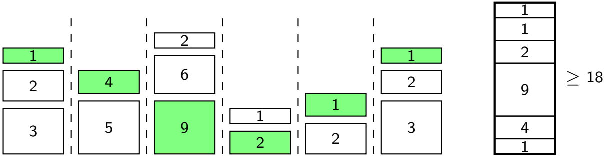

Generalizes Subset Sum and k-Sum.

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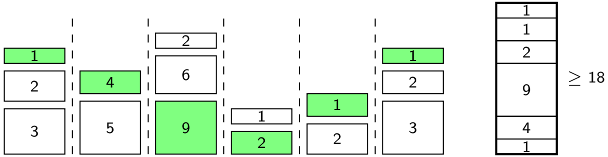

Generalizes Subset Sum and k-Sum.

Generalizes Weighted Consensus:

- classes $\leftarrow$ positions,
- items $\leftarrow$ possible characters (at a given position),
- values $\leftarrow$ log-probabilities in the first sequence,
- weights $\leftarrow$ minus log-probabilities in the second sequence.

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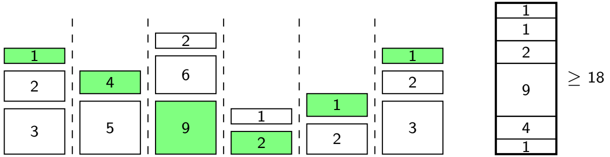
- Output: a choice of 1 item per class, with total weight $\leq W$ and total value $\geq V$.



Generalizes Subset Sum and k-Sum.

Generalizes Weighted Consensus: (also a converse reduction!)

- classes ← positions,
- items ← possible characters (at a given position),
- values ← log-probabilities in the first sequence,
- weights ← minus log-probabilities in the second sequence.

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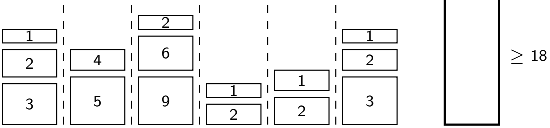

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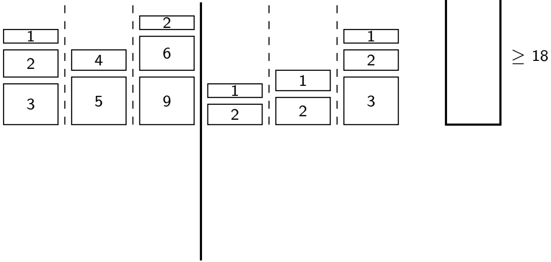

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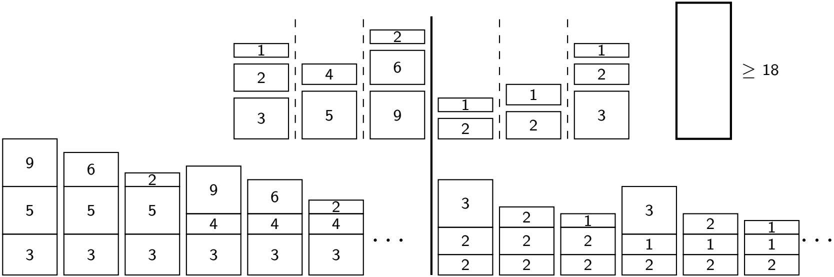

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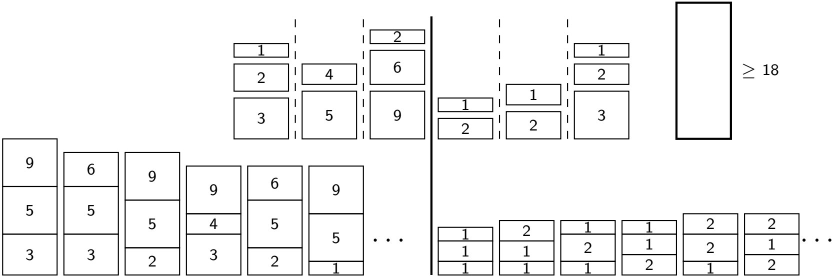

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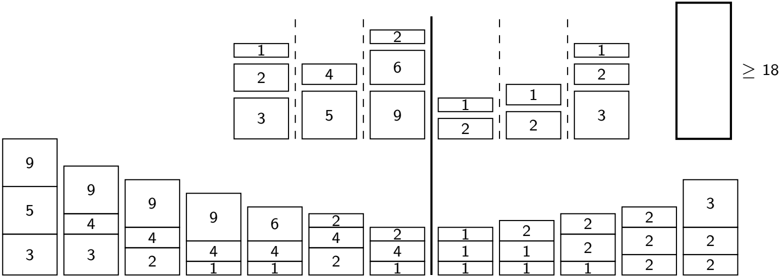

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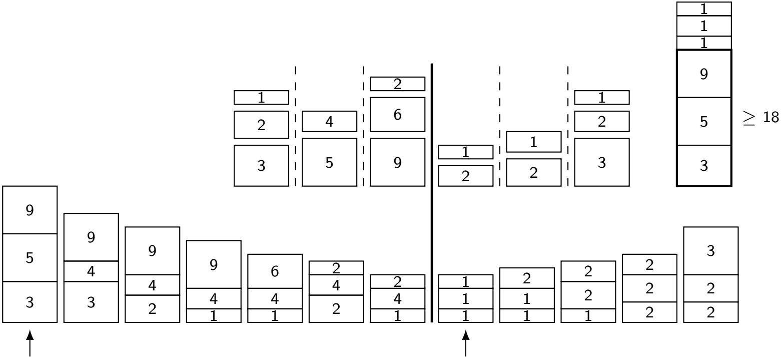

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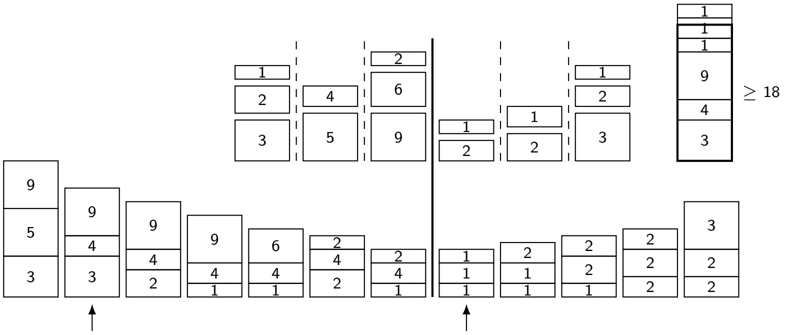

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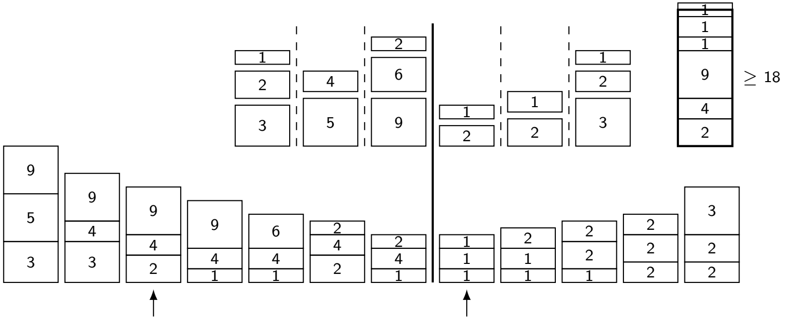

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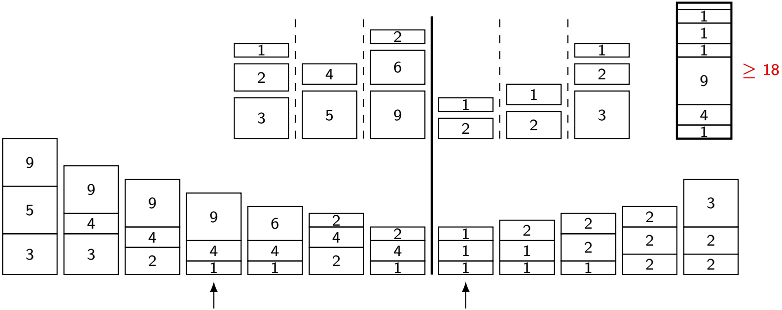

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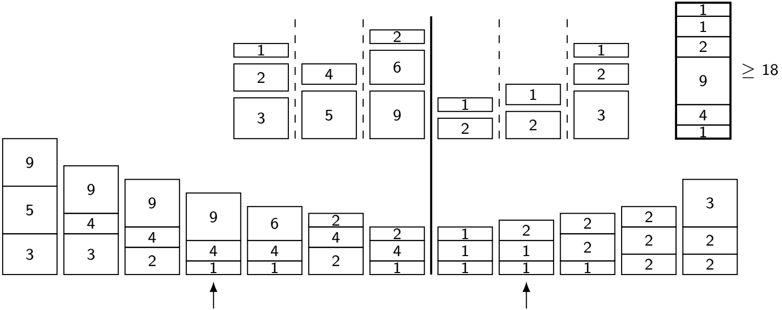

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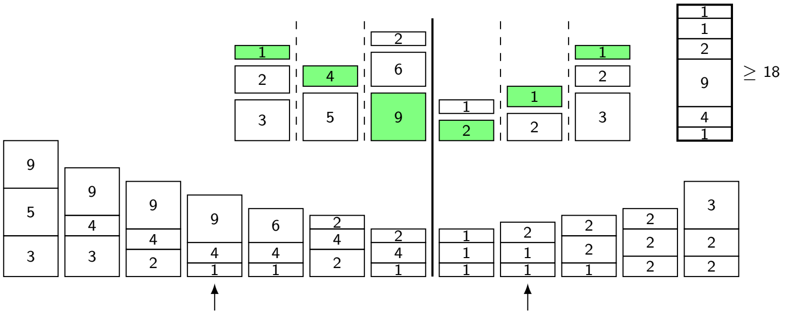

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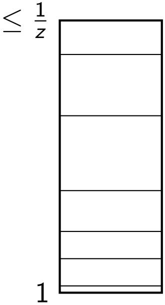

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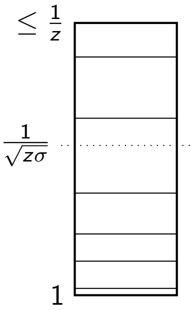

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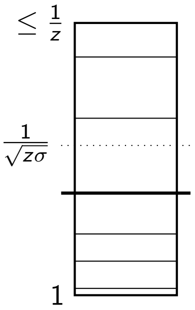

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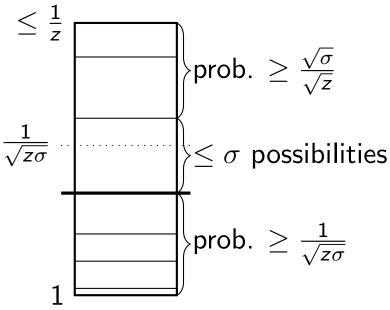

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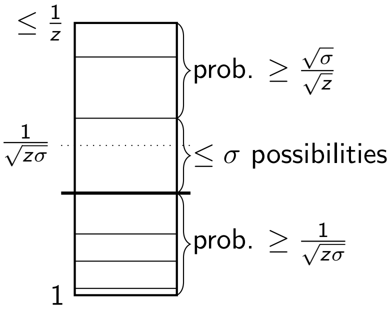

Running time $\mathcal{O}(n\sqrt{z\sigma}\log z)$:

- $\mathcal{O}(n)$ choices for the domain partitioning,
- $\mathcal{O}(\sqrt{z\sigma})$ prefixes and $\mathcal{O}(\frac{\sqrt{z}}{\sqrt{\sigma}} \cdot \sigma)$ suffixes,
- $\mathcal{O}(\log z)$ for sorting.

Summary of Our Further Results

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Improved Weighted Consensus algorithms for $z = \sigma^{\Theta(1)}$

- Conditionally optimal solutions for $z = \sigma^c$ for any constant c.
- Lower bounds wrt. k-SUM.

Questions?

Thank you for your attention!