Minimal Suffix and Rotation of a Substring in Optimal Time

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University of Warsaw, Poland

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Minimal & Maximal Suffix and Rotation Queries

Preprocess a text T of length n.

T :	a	b	a	a	b	a	b	a	a	b	a	a	b	a	b
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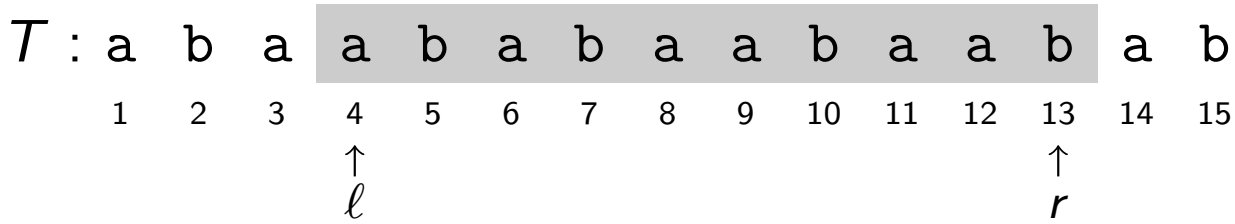

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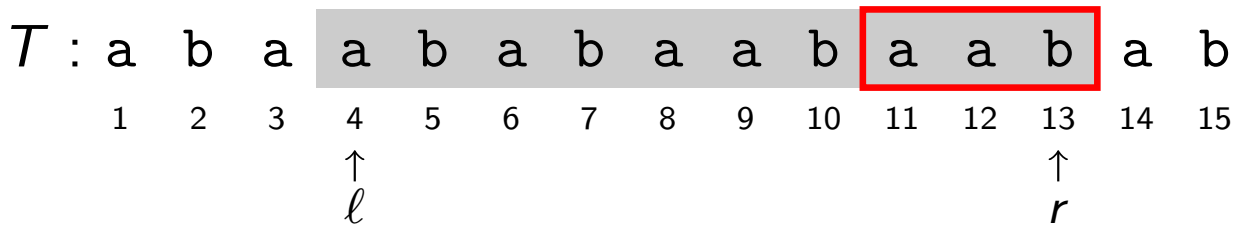

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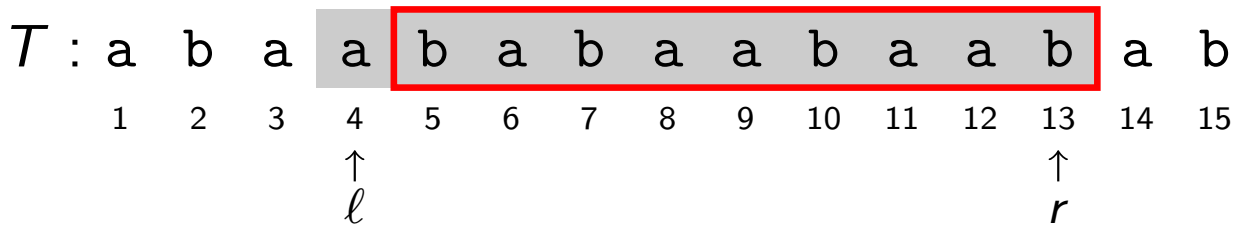

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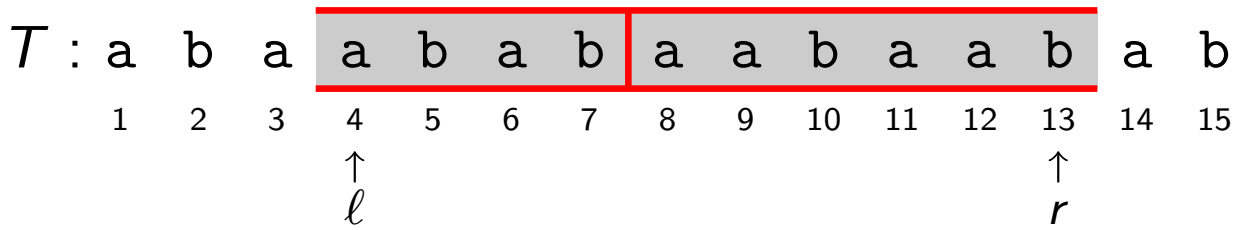

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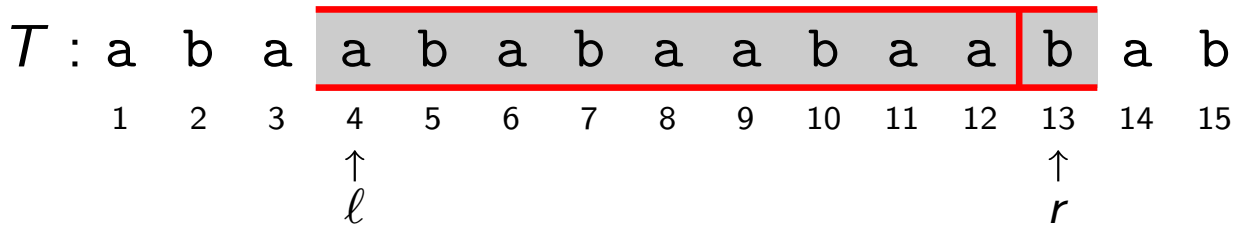

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Previous & our results

Problem	Preprocessing	Query	Reference

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Common features:

- $\mathcal{O}(n)$ space,
- word RAM model with words of $\Omega(\log n)$ bits,
- integer alphabet $\{1, \ldots, n^{\mathcal{O}(1)}\}$.

Generalized minimal suffix queries

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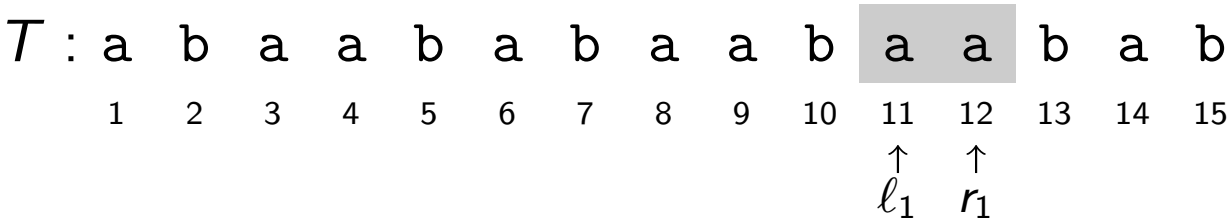

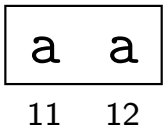

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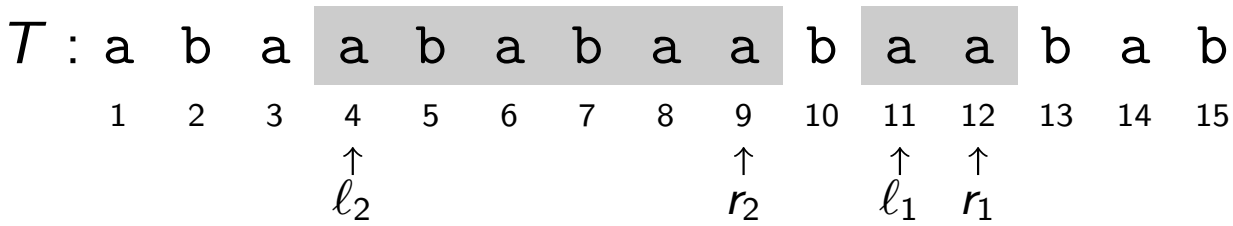

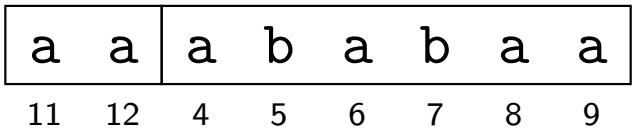

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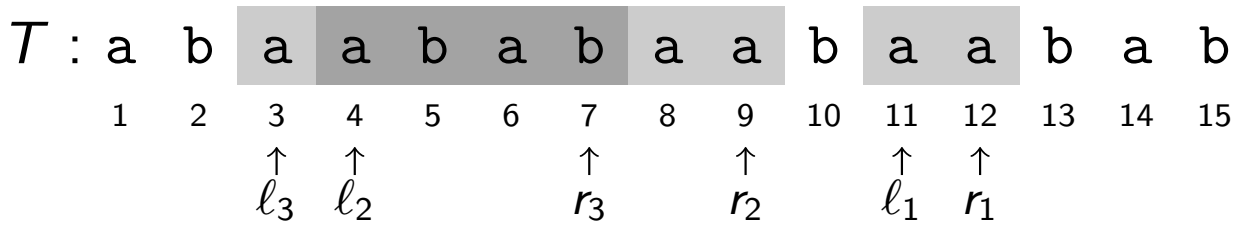

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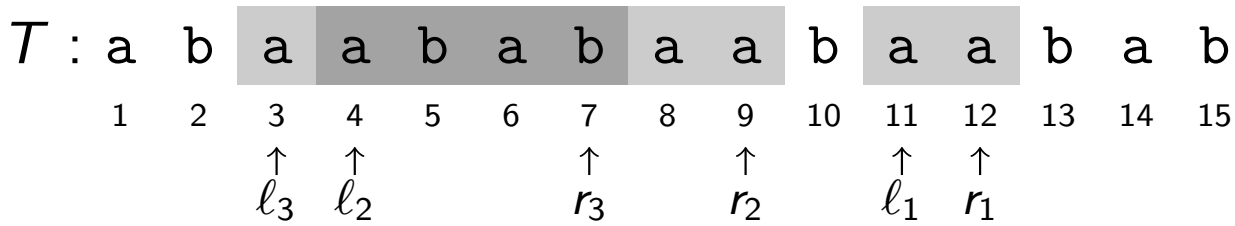

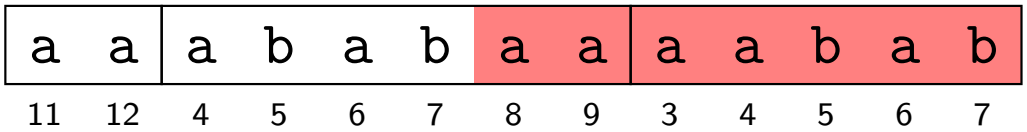

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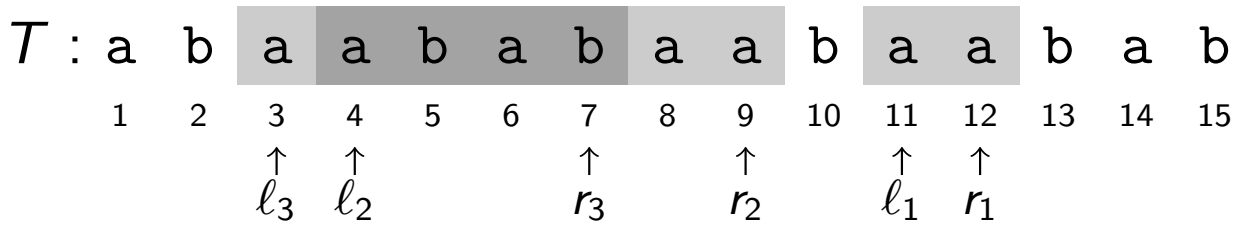

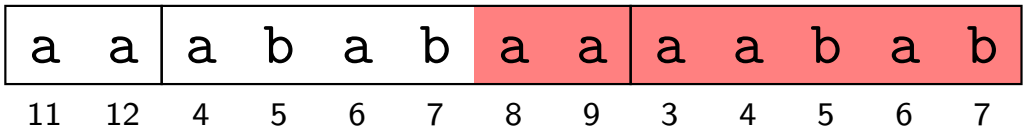

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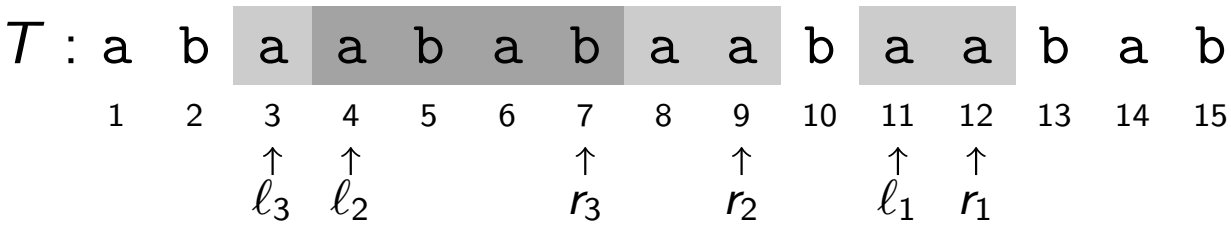

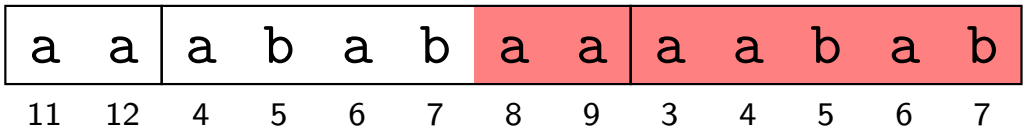

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- $\mathrm{MaxSuf}^R(v)\$ = \mathrm{MinSuf}(v\$)$ where $\$ > c$ for every $c \in \Sigma$.
- $\mathrm{MinRot}(v) = \mathrm{MinSuf}(vv\$)[1..|v|]$.

Applications

Lyndon word: $w = \mathrm{MinSuf}(w)$

Lyndon factorization: $w = w_1^{\alpha_1} \cdots w_m^{\alpha_m}$ such that:

- $w_1 > \cdots > w_m$,
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Our result:
After $\mathcal{O}(n)$-time preprocessing, Lyndon factorization of the concatenation of k fragments can be computed in $\mathcal{O}(k^2 m)$ time.

Combinatorial tools

Significant suffixes

$\mathrm{MinSuf}(v, w) = \min\{sw : s \text{ is a suffix of } v\}$

Observation: $\mathrm{MinSuf}(vw) \in \{\mathrm{MinSuf}(v, w), \mathrm{MinSuf}(w)\}$.

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Definition (I et al., CPM'13, SPIRE'13)

A suffix s of v is **significant** if $sw = \mathrm{MinSuf}(v, w)$ for some $w \neq \varepsilon$.

Notation: $\Lambda(v) = \{s_0, \ldots, s_m\}$ (the set of significant suffixes of v).

- $s_0 = \varepsilon$, $|s_i| > 2|s_{i-1}|$
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- $x_0^\infty < \cdots < x_{m-1}^\infty$.

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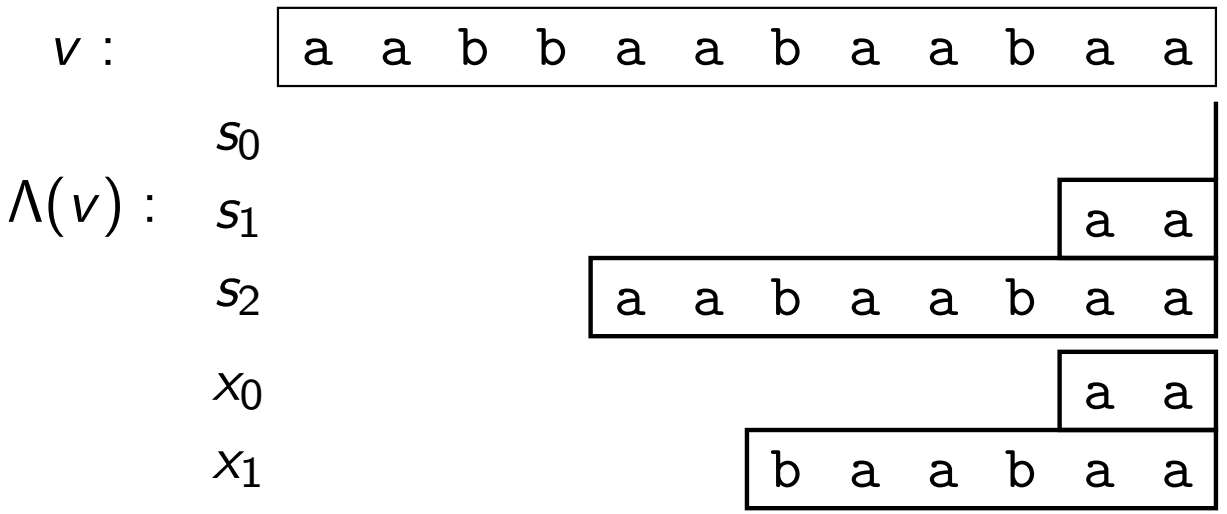

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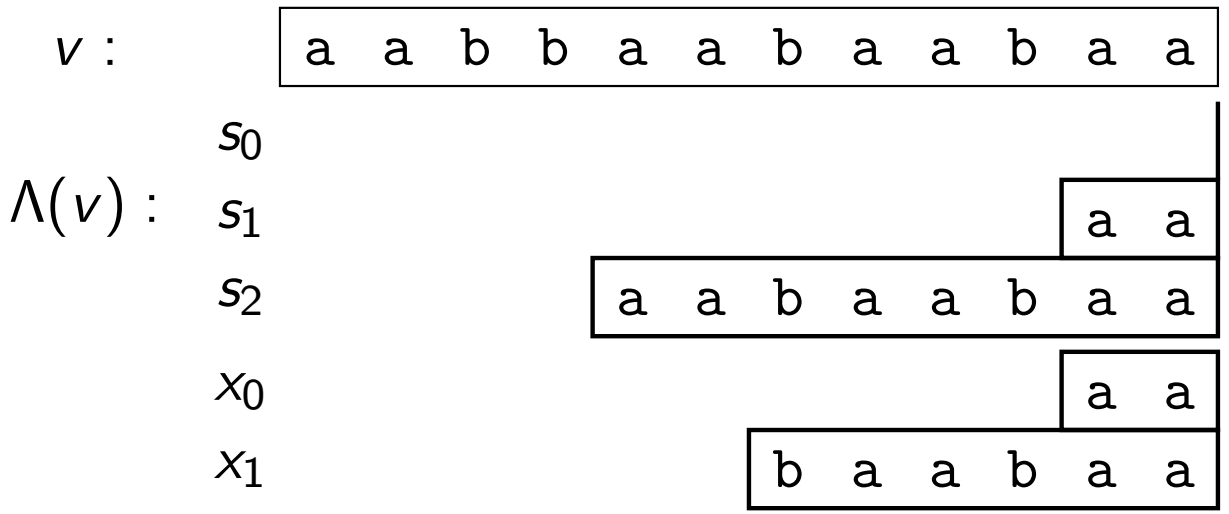

$$\mathrm{MinSuf}(v, w) = \begin{cases} w & \text{if } w < (aa)^\infty \\ aaw & \text{if } (aa)^\infty < w < (baabaa)^\infty \\ aabaabaaw & \text{if } (baabaa)^\infty < w \end{cases}$$

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$\mathrm{MaxSuf}^R(u, v')$ *can be computed in $\mathcal{O}(1)$ time using* ***augmented suffix array*** *(SA, ISA+RMQ, LCP+RMQ built for T and T^R).*

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Corollary

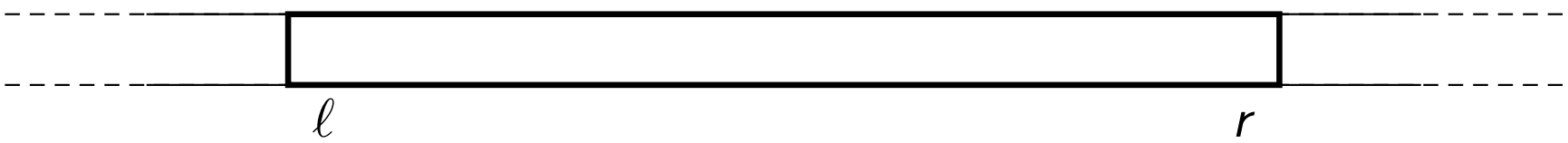
$\Lambda(v)$ *can be computed in $\mathcal{O}(\log |v|)$ time using augmented SA.*

Recurse on v' and filter $\{\mathrm{MaxSuf}^R(u, v'), \Lambda(v')\}$.

Data structure

Inefficient solutions

$\mathcal{O}(n)$-time preprocessing, $\mathcal{O}(\log n)$-time queries:



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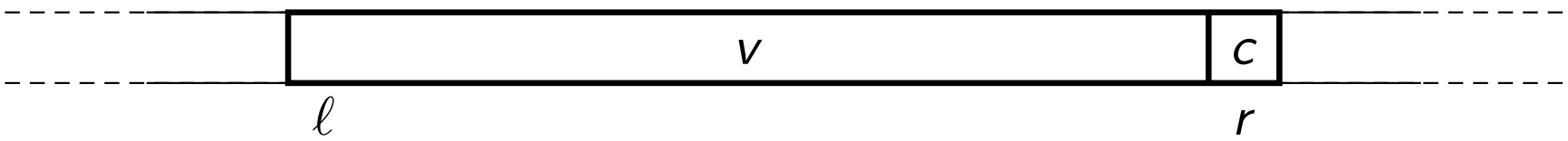
1. Let $T[\ell..r] = vc$ where $|c| = 1$.



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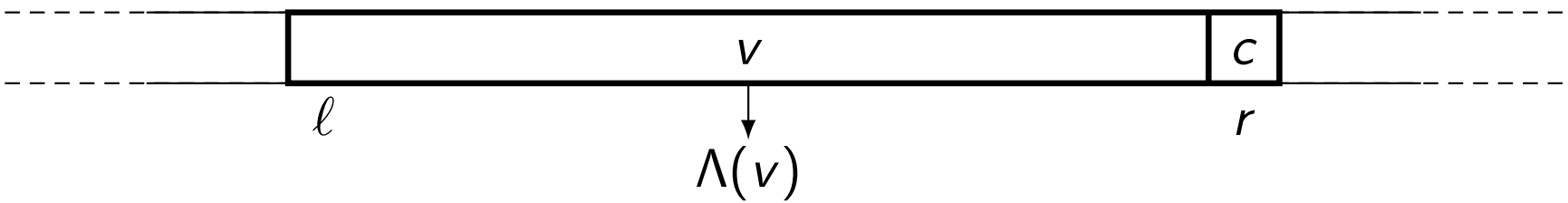

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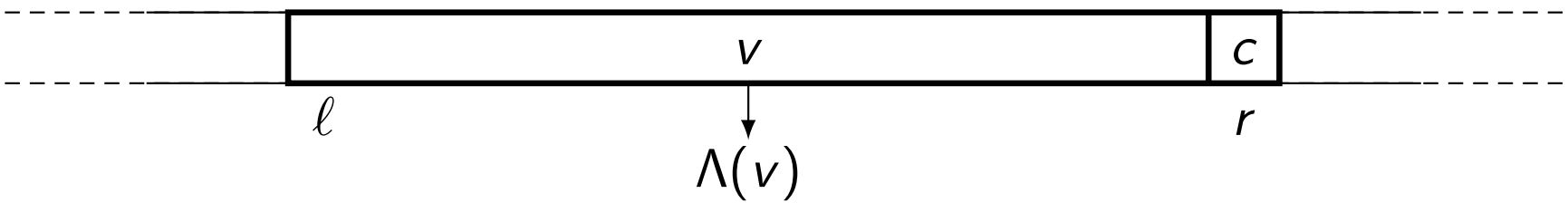

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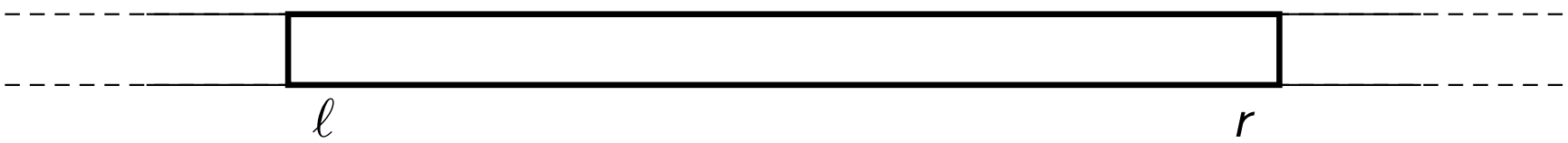

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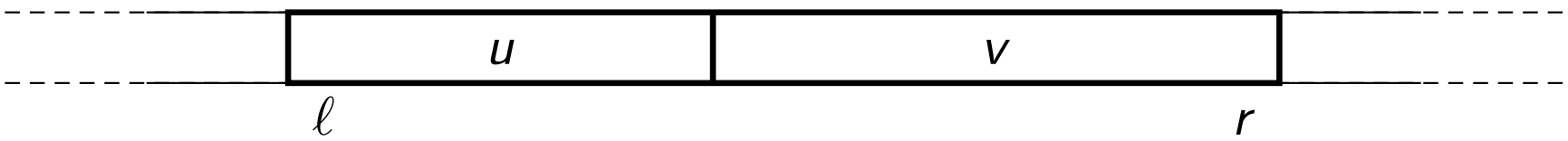

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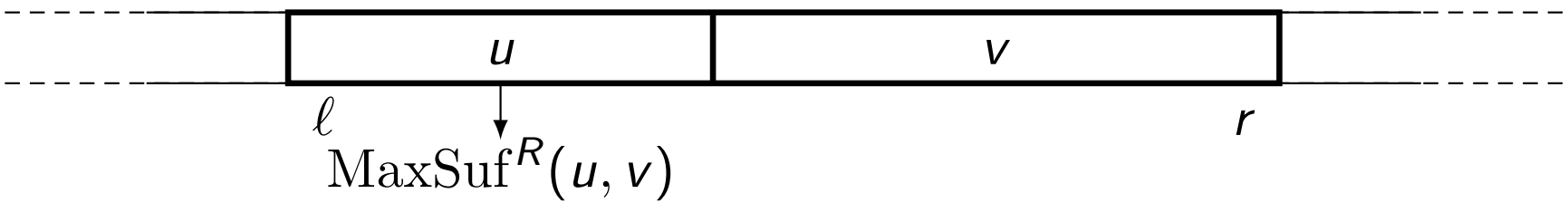
1. Let $T[\ell..r] = uv$ where $|u| + 1 \leq |v| \leq |u| + 2$.
2. Compute $S_1 = \mathrm{MaxSuf}^R(u, v)$ using the augmented SA.



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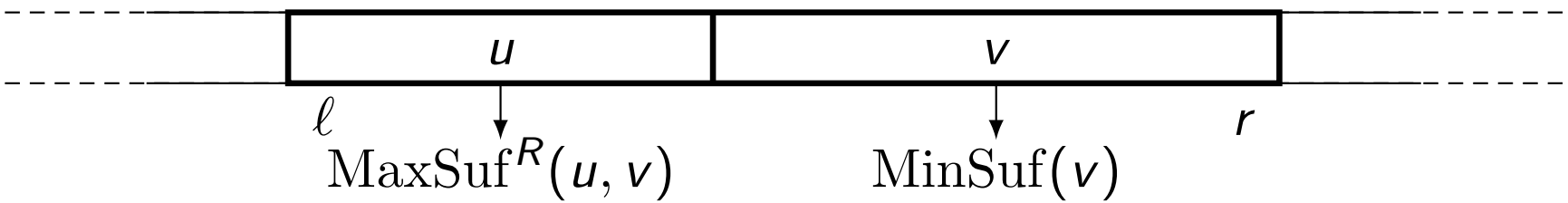

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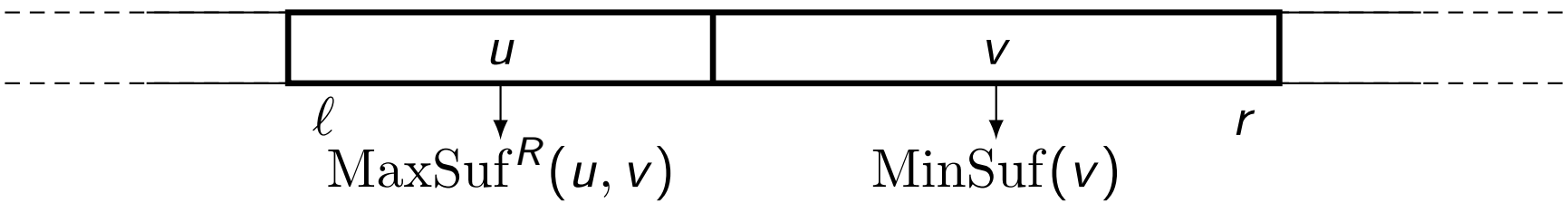

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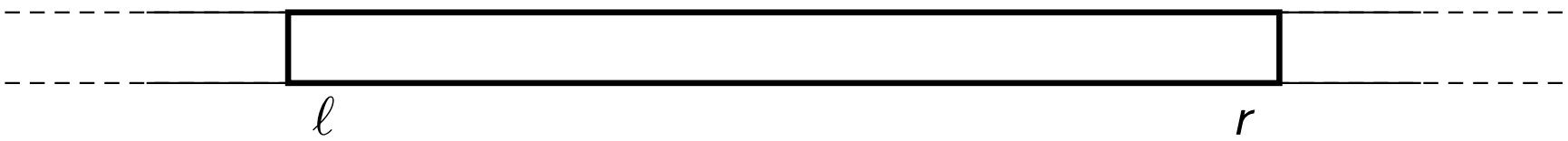

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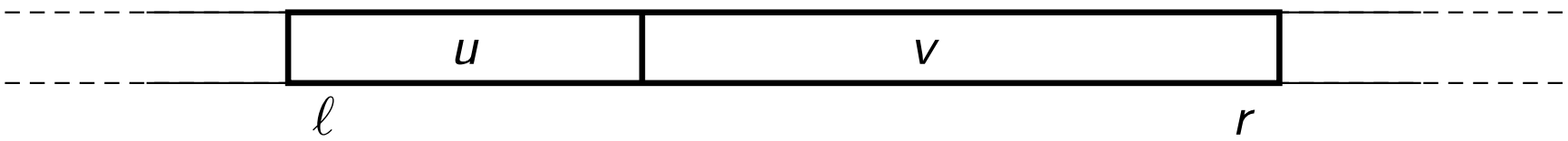

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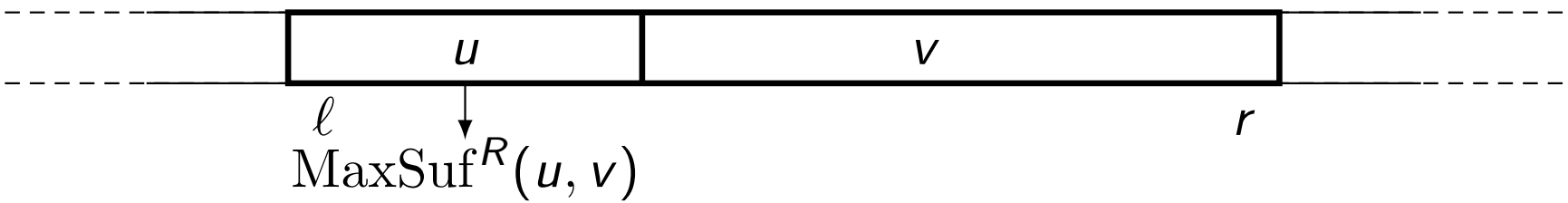

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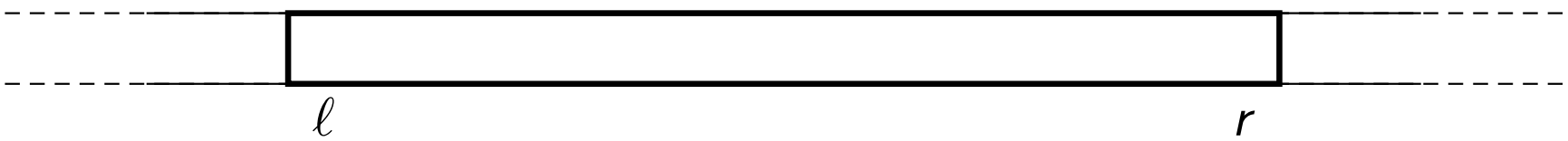

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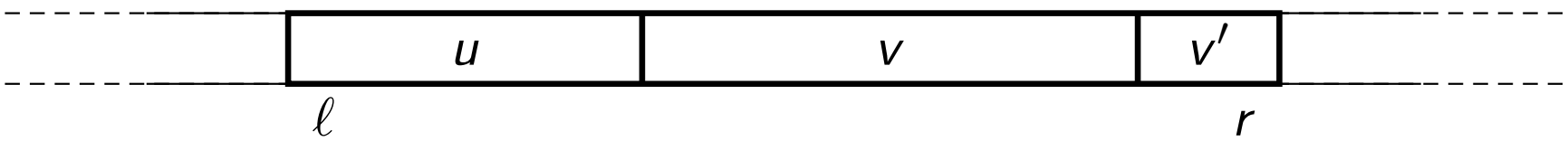

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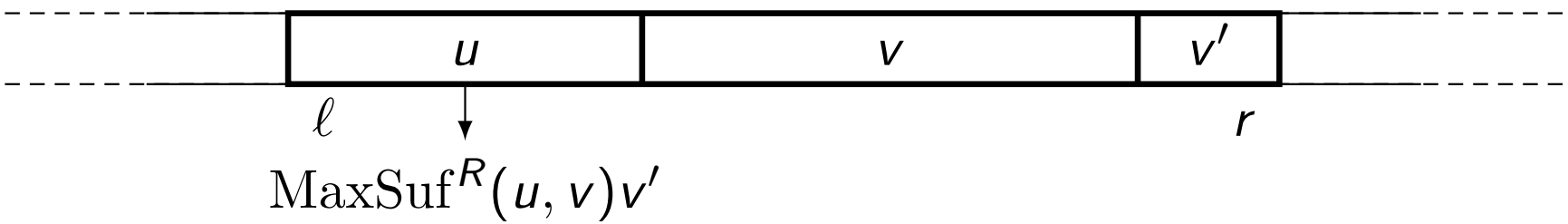

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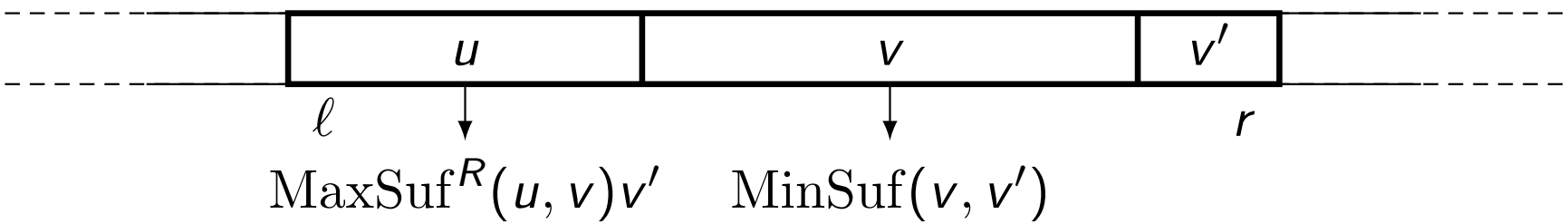

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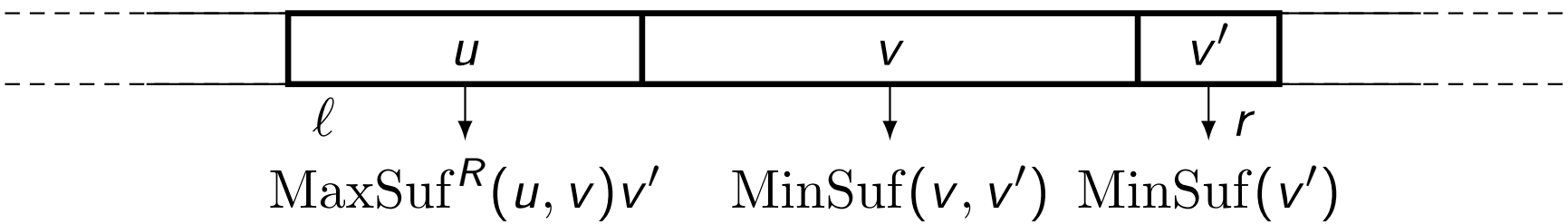

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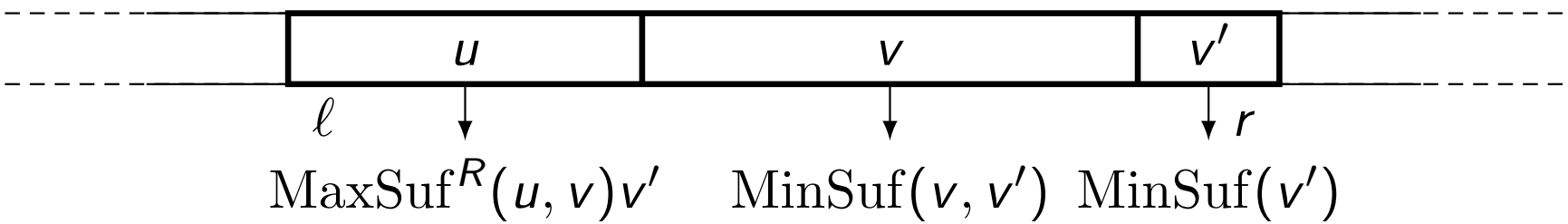

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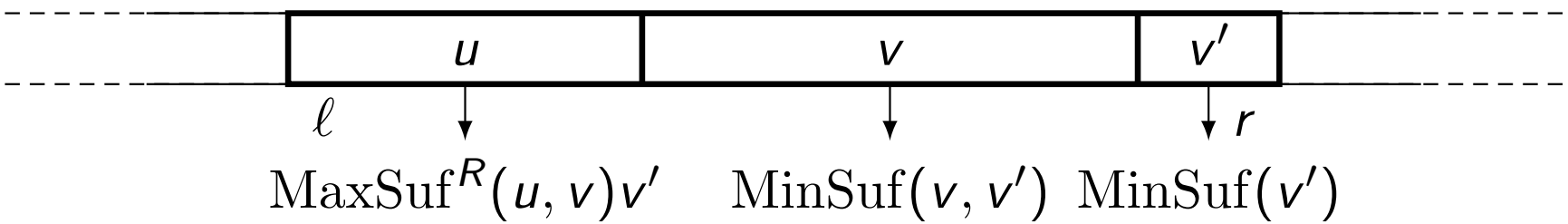

Depth of recursion: $\mathcal{O}(\log^* n)$ since $f(f(x)) = \log^{o(1)} x = o(\log x)$.

Main component

Problem:

Preprocess $u = T[\ell..r]$ to determine $\mathrm{MinSuf}(u, T[r+1..r'])$.

Combinatorial tool: Significant suffixes $\Lambda(u) = \{s_0, \ldots, s_m\}$

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Algorithmic tool: fusion trees [Pătraşcu & Thorup, FOCS 2015]
$\mathcal{O}(1)$-time queries and $\mathcal{O}(R(u))$-time construction
(since $|R(u)| = W^{\mathcal{O}(1)}$ where $W = \Omega(\log n)$ is machine word size).

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Solution:

1. Consider $\mathcal{O}(n/f(f(n)))$ fragments of length $2f(f(n))$.
2. Group them by order isomorphism (e.g. using fusion trees).
3. Memorize all the answers for a representative of each class.

Summary & open problems

Our results:

Minimal Suffix Queries

$\mathcal{O}(1)$-time queries, $\mathcal{O}(n)$-time preprocessing,

Generalized Minimal Suffix Queries

minimal suffix of the concatenation of k substrings

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Open problems:

- Encoding with $o(n \log n)$ bits for min/max suffix queries?
- Efficient $o(n \log n)$-bit data structure for constant alphabets?
- Lyndon factorization of RLZ-compressed text in $o(k^2 m)$ time?

Questions?

Thank you for your attention!