Fast Algorithm for Partial Covers in Words

Tomasz Kociumaka[1], Solon P. Pissis[2,3],
Jakub Radoszewski[1], Wojciech Rytter[1],
Tomasz Waleń[4,1]

[1]University of Warsaw
[2]Heidelberg Institute for Theoretical Studies
[3]University of Florida
[4]International Institute of Molecular and Cell Biology in Warsaw

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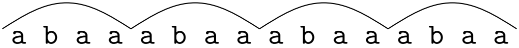
Periodicity and Quasiperiodicity

Periodicity: occurrences are aligned.



a b a a a b a a a b a a a b a a

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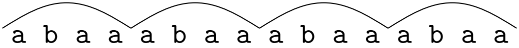
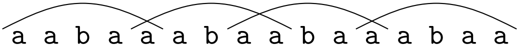


a b a a a b a a a b a a a b a a

Quasiperiodicity: occurrences may overlap.

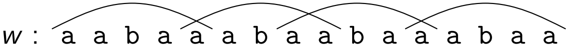


a a b a a a b a a b a a a b a a

Covers

Definition (Apostolico, Farach, Iliopoulos, 1991)

Let u be a factor of w. We say that u is a *cover* of w, if each position (letter) in w lies within some occurrence of u in w.


w : a a b a a a b a a b a a a b a a

The covers of w are aabaa

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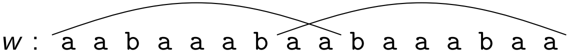

The covers of w are `aabaa`, `aabaaabaa`

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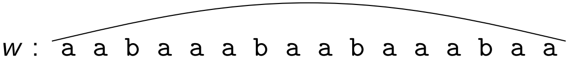

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The covers of w are aabaa, aabaaabaa and aabaaabaabaaabaa.

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w : `a a b a a a b a a b a a a b a a`

The covers of w are `aabaa`, `aabaaabaa`
and `aabaaabaabaaabaa`.
The whole word is always a cover of itself,
most words do not have any other cover.

Cover Index, Partial Covers

Definition

The *cover index* of u in w is the number of positions in w which lie within some occurrence of u in w.

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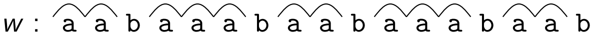

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The cover index of a is 12

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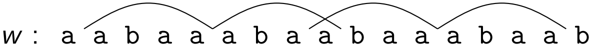

w : a a b a a a b a a b a a a b a a b

The cover index of a is 12, of abaa is 15

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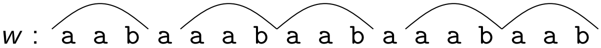

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The cover index of a is 12, of abaa is 15,
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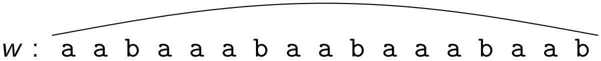

w : `a a b a a a b a a b a a a b a a b`

The cover index of `a` is 12, of `abaa` is 15,
of `aab` is 15, only of `aabaaabaabaaabaab` is 17.

Cover Index, Partial Covers

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The cover index of a is 12, of abaa is 15,
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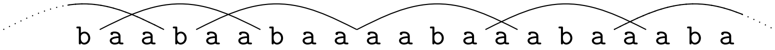
Definition

For a positive integer α an α-partial cover of w is a factor of w with cover index at least α.

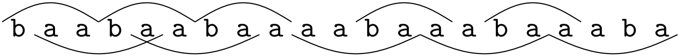
Other variants of covers



b a a b a a b a a a a b a a a b a a a b a

- *seeds* (Iliopoulos, Moore, Park; 1996) – covers of a superstring

Other variants of covers



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- *k-covers* (Iliopoulos, Smyth; 1998) – each position lies within an occurrence of at least one of k factors, together being a k-cover

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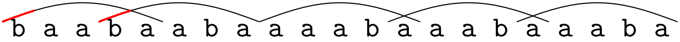

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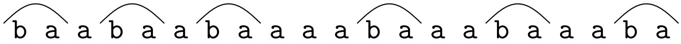

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- *enhanced covers* (Flouri, Iliopoulos, K., Pissis, Puglisi, Smyth, Tyczyński; 2012) – as partial covers with an additional requirement of being simultaneously a border

Our Results

Problem (PartialCovers)

Given a word w and a positive integer α, identify all shortest α-partial covers of w.

Theorem

The PartialCovers *problem can be solved in $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ space, where $n = |w|$.*

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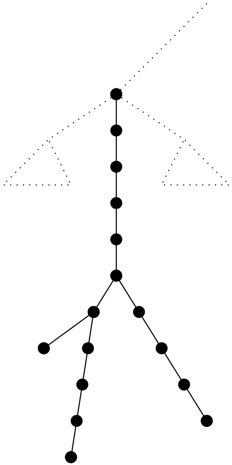
The PartialCovers *problem can be solved in $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ space, where $n = |w|$.*

Theorem

For any word w of length n exists a data structure of size $\mathcal{O}(n)$, which given u can find the cover index of u in $\mathcal{O}(|u|)$ time. It can be built in $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ space. If $u = w[i..j]$ is given as a pair of integers i, j, then $\mathcal{O}(\log \log |u|)$ query time can be achieved.

Suffix Trees: Notation

- The *suffix trie* of w for each factor u of w has a node corresponding to u, called the *locus* of u.



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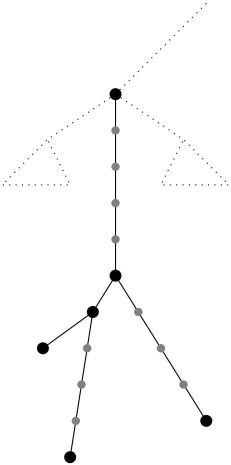

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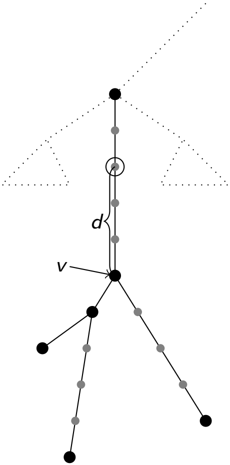

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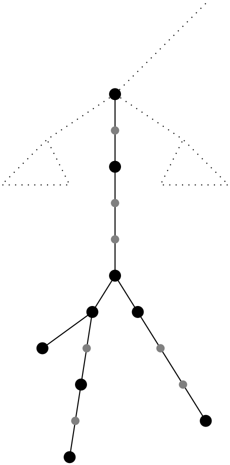

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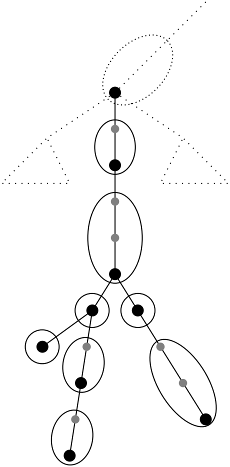
- An *edge* of a tree contains all implicit nodes and the lower explicit end.



Auxiliary definitions

Definition

A factor u is a *primitive square* if $u = v^2$ for some v, but $u \neq v^{2k}$ for any v and $k \geq 2$.

Examples: `aa`, `abaaba`. Non-examples: `ababa`, `abababab`.

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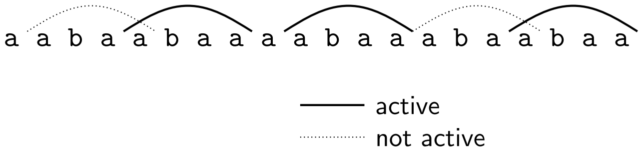
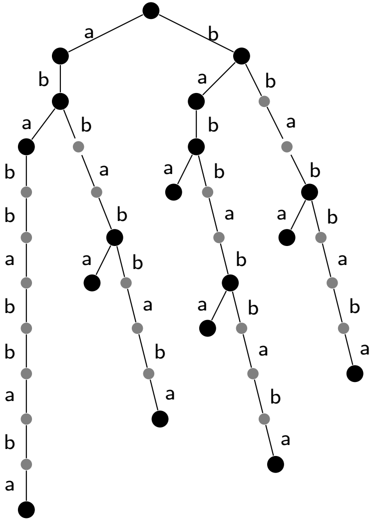
Definition

An occurrence of u in w is *active* if no other occurrence of u in w starts within it.



Cover Suffix Tree

The cover suffix tree of w $CST(w)$ is the suffix tree of w:



$CST($ababbabbaba$)$

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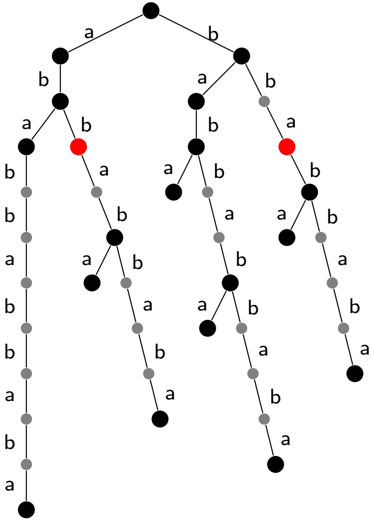

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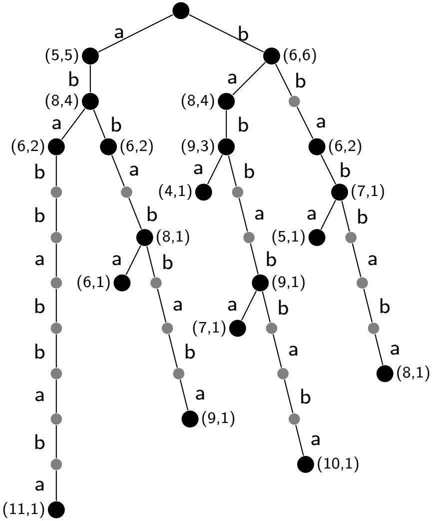

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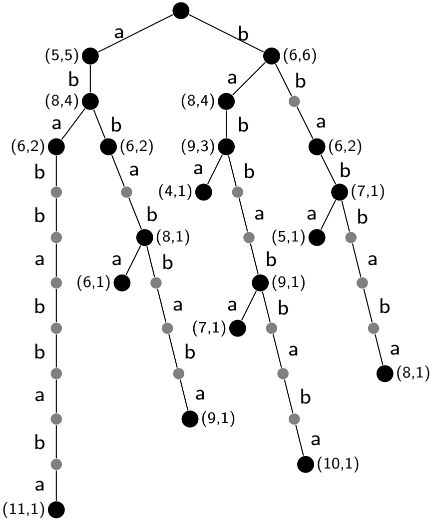
The number of square factors is linear (Fraenkel, Simpson, 1998), so the size of $CST(w)$ is $O(|w|)$.



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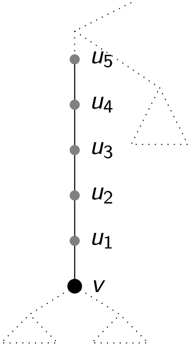
Crucial Lemma

Lemma

Let $v = u_0, u_1, \ldots, u_k$ *be the nodes of an edge of* $CST(w)$ *with* v *being the lowest node. Then* $cv(u_i) = cv(v) - i\Delta(v)$*.*



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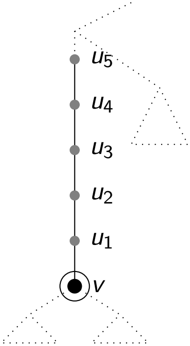

Proof.

Recall that if u, u' are on the same edge of the suffix tree, then occurrences of u and u' start at the same positions. In $CST(w)$ also the active occurrences agree. Thus, u_{i+1} covers $\Delta(u_i) = \Delta(v)$ positions less than u_i. ☐

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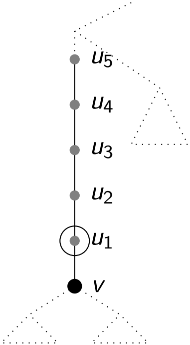

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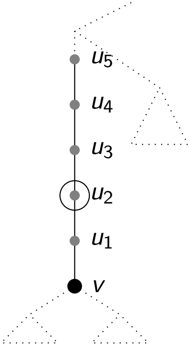

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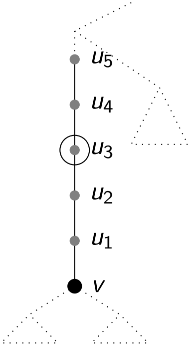

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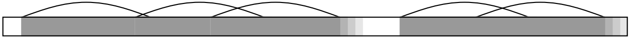

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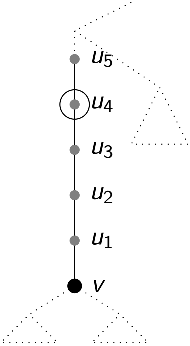

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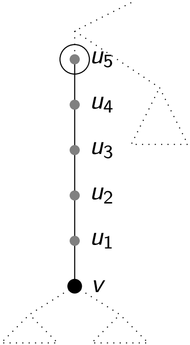

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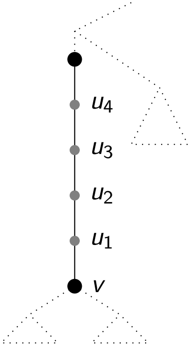

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Answering Queries

- Recall that we have defined the locus of u as a pair (v, d), where v is the highest explicit descendant of u.
- The Lemma proves that $cv(u) = cv(v) - d\Delta(v)$, so computing the cover index of u given its locus in $CST(w)$ is trivial.

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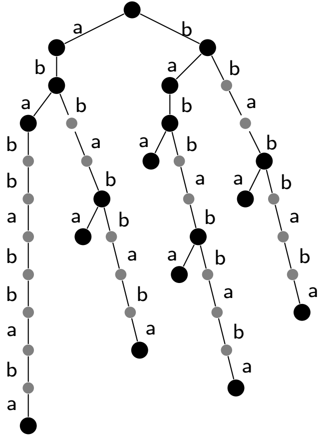
- If $u = w[i..j]$ is given as a pair of indices, use the weighted ancestors data structure ($\mathcal{O}(\log\log|u|)$ time).
- Finding the shortest α-partial covers reduces to solving one linear inequality per edge ($cv(v) - d\Delta(v) \geq \alpha$), this takes linear time once $CST(w)$ is given.

Construction Algorithm

The structure resembles an $\mathcal{O}(n \log n)$-time construction of a similar data structure, MAST (Brodal et al.; 2002).

- We start with the suffix tree.



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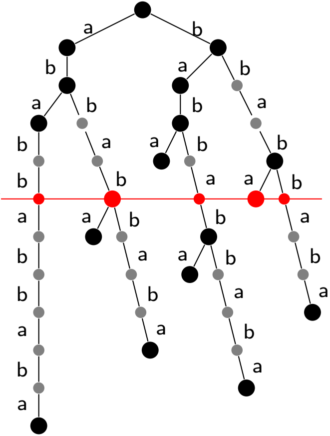
- We start with the suffix tree.
- We process the nodes in the decreasing order of the corresponding factors' lengths.
- While at level d, for each factor of length d we implicitly keep a *sorted* linked list of its occurrences.



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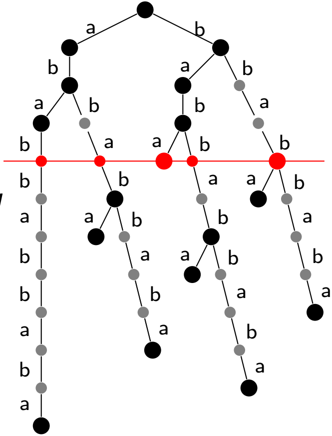

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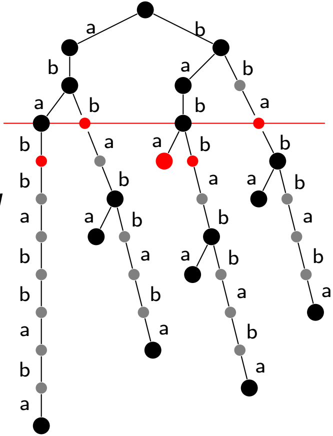

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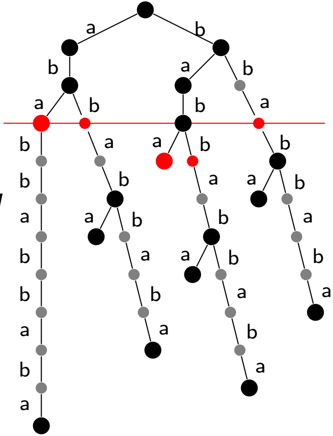

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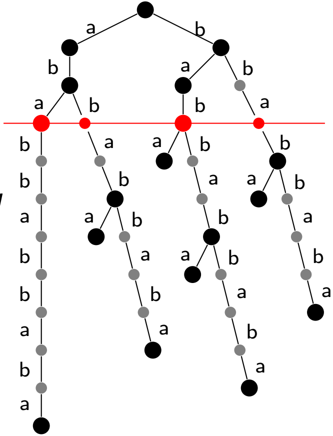

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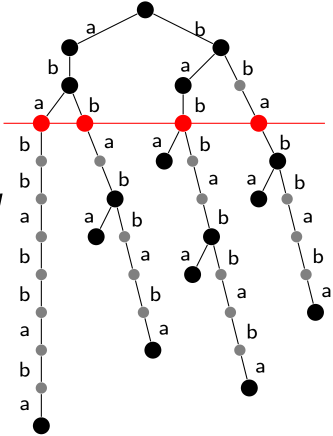
- We also need to add extra nodes.



Change Sets

Definition

Let $\mathcal{P}$ be a partition of $[n] = \{1, \ldots, n\}$. For any $a \in [n]$ we define the *successor* of a in $\mathcal{P}$ as $\min\{b \in P : b > a\}$ where $P \in \mathcal{P}$ is the partition class containing a.
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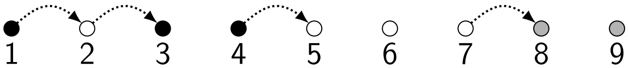
Definition

Consider two partitions $\mathcal{P}$, $\mathcal{P}'$ of $[n]$. The *change set* of $\mathcal{P}$ and $\mathcal{P}'$ is the family of pairs (i, j) such that j is the successor of i in $\mathcal{P}'$, but not in $\mathcal{P}$.

Let $\mathcal{P} = \{\{1, 3, 4\}, \{2, 5, 6, 7\}, \{8, 9\}\}$ and $\mathcal{P}' = \{\{1, \ldots, 9\}\}$.
The change set is $\{(1, 2), (2, 3), (4, 5), (7, 8)\}$.



Ordered Disjoint Sets

Problem (Ordered Disjoint Sets)

Maintain a partition of $[n]$*, support the following operations:*

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There is a data structure of size $\mathcal{O}(n)$*, which handles a sequence of* q *operations in* $\mathcal{O}(q + n \log n)$ *total time. Moreover, the total size of change sets is* $\mathcal{O}(n \log n)$*.*

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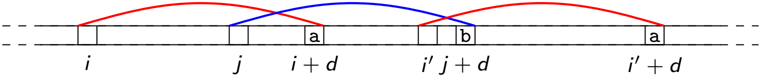
We use this data structure to store a partition into equivalence classes of the $w[i..i+d-1] = w[j..j+d-1]$ relation.

Construction Algorithm

We handle two types of events:

- A branching node, the lists need to be merged.



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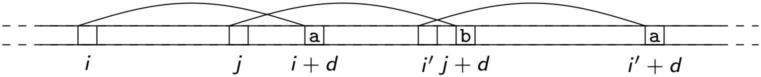

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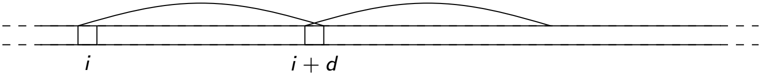

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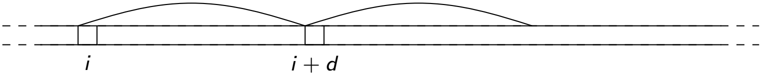

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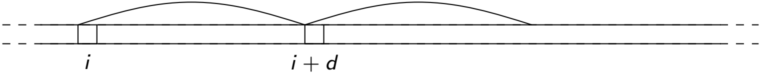

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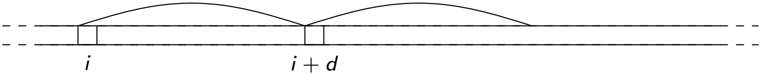

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The complexity of the construction algorithm is amortized by the total size of change sets, which gives

Theorem

Given a word w of length n, its Cover Suffix Tree $CST(w)$ can be constructed in $\mathcal{O}(n \log n)$ time and $\mathcal{O}(n)$ space.

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Can *CST* be constructed faster?
A linear time algorithm might be difficult, all seeds can be easily found in linear time using *CST*.

Thank you for your attention!