LECTURE 9
NEW HINT
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TIME SPLITTING[1]

Instead of introduction: a very simple example.
Consider the initial value problem for Ordinary Differential Equations (**ODE**)

$$\frac{du}{dt} = f(u) + g(u), \quad t \in [t_n, t_n + \tau], \quad u(t_n) = a \tag{1}$$

and the following two initial value problems:

$$\frac{dv}{dt} = f(v), \quad t \in [t_n, t_n + \tau], \quad v(t_n) = a \tag{2}$$

$$\frac{dw}{dt} = g(w), \quad t \in [t_n, t_n + \tau], \quad w(t_n) = v(t_n + \tau). \tag{3}$$

What can we expect concerning the difference:

$$u(t) - w(t), \quad t \in [t_n, t_n + \tau]?$$

May be it is **small**? Answer is:

YES! and in many cases,
providing that the time step τ is reasonably small.

Sometimes, for certain reasons it may be better, to solve the system (2)(3), than to solve the equation (1). We can try, in such a case, to approximate the solution u of the equation (1), by the solution w of the equation (3)!

THIS IS THE TIME SPLITTING METHOD

[1] In, fact idea of the method here described is relatively old: probably it was applied for the first time by D.W.Paceman and H.H Rachford jr. in 1955.

Look at the **SPLITTING DIAGRAM**:

$$\begin{array}{llllll} \textbf{time} & \textbf{level} & \mathbf{n+1} & v(t_n+\tau) & & w(t_n+\tau) \approx u(t_n+\tau) \\ & & & \uparrow & \searrow & \uparrow \\ \textbf{time} & \textbf{level} & \mathbf{n} & v(t_n) & & v(t_n+\tau) \end{array}$$

The left uparrow corresponds to solving (2), while right one corresponds to solving (3).

EXAMPLE.

The **TIME SPLITTING** may be applied for partial differential equations. Let us consider the transport equation in two space dimensions:

$$(4) \qquad u_t + \underline{\alpha}\nabla u = 0, \quad \underline{\alpha} = [\alpha_1, \alpha_2], \quad u = u(t, x_1, x_2).$$

Here $x_1 \in [0, L_1]$, $x_2 \in [0, L_2]$, and $\nabla u = [u_{x_1}, u_{x_2}]^T$. Let us rewrite the equation (4) in the following equivalent form

$$(5) \qquad u_t + \alpha_1 u_{x_1} + \alpha_2 u_{x_2} = 0.$$

For equation (4)(5) we have the initial condition:

$$u(0, x_1, x_2) = \phi(x_1, x_2).$$

The boundary conditions (depending on the sign of α_1 and α_2) are also needed.

We can split equation (4)(5) in the followig way:

- $$(6) \qquad u_{1,t}(t, x_1, x_2) + \alpha_1 u_{1,x_1}(t, x_1, x_2) = 0,$$

  We have initial condition for (6): $u_1(0, x_1, x_2) = \phi(x_1, x_2)$ This equation needs the boundary condition at one of ends of the interval $[0, L_1]$. Observe that x_1 is an 'active' variable of differential equation, while x_2 is a parameter. It runs over the interval $[0, L_2]$.

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$$u_{2,t}(t,x_1,x_2)+\alpha_2 u_{2,x_2}(t,x_1,x_2)=0. \tag{7}$$

For (7) we have the same initial conddition as for (6): $u_2(0,x_1,x_2)=\phi(x_1,x_2)$. Here also the boundary condition at one of ends but of the interval $[0,L_2]$ is necessary; now x_1 is a parameter runing over $[0,L_1]$, while x_2 is the active variable of the differential equation.

A TEOREM ON APPROXIMATION

The theorem below concerns the case of spliting: **(1)=>(2)(3)**.

THEOREM.
Let:

- $(X,|\cdot|)$ *be a linear normed space*
- $f,g:X\to X$
- $u,v,w:[t_n,t_{n+1}]\to X,\quad t_{n+1}=t_n+\tau,\ \tau>0$
- $(X_\tau,\|\cdot\|_\infty)$ *be a linear, B-space of functions* $x:[t_n,t_{n+1}]\to X$, *such that* $\|x\|_\infty=\sup_{t\in[t_n,t_{n+1}]}|x(t)|<\infty$

We assume about (1), (2), (3) *that:*

- $u,\ v,\ w\in X_\tau$
- *there exist positive constants* $L_f,\ L_g,\ K_f,\ K_g$ *such that if* $x=u,\ v,\ w$ *then* $\|f(x)\|_\infty\le K_f$, $\|g(x)\|_\infty\le K_g$, *and that for all* $x,\ y\in X$ *the following Lipschitz conditions*

  $$|f(x)-f(y)|\le L_f|x-y|,\quad |g(x)-g(y)|\le L_g|x-y|$$

  are satisfied.

Under above conditions, for solutions o $u,\ v,\ w$ *of equations* (1)(2)(3) *following estimates hold:*

$$\|u-v\|_\infty\le\frac{\tau K_g}{1-\tau L_f}, \tag{8}$$

$$(9) \qquad \|u-w\|_\infty \le \tau \frac{K_f}{1-\tau L_g} + \tau^2 \frac{K_g L_f}{(1-\tau L_f)(1-\tau L_g)} = O(\tau).$$

However it is to note that:

$$(10) \qquad |u(t_{n+1}) - w(t_{n+1})| \le$$

$$\le \tau^2 \frac{K_g L_f}{1-\tau L_f} + \tau^2 \frac{L_g K_f}{1-\tau Lg} + \tau^3 \frac{K_g L_f L_g}{(1-\tau L_f)(1-\tau L_g)} = O(\tau^2).$$

The inequqlity **(10)** *says that at the point* t_{n+1} *the* **'SUPER-CONVERGENCE'** *occurs.* **This means that the order of approximation at the point t_{n+1} is greater then the order of approximation in norm.**

CONCLUSION. From above investigations it follows, that for the time spplitting

$$(1) \approx (3)(2)$$

the choice $w(t_{n+1})$ is not only the most intuitive, but really the best!

Proof. We have:

$$(1') \qquad u(t) = a + \int_{t_n}^{t} f(u(s))ds + \int_{t_n}^{t} g(u(s)ds$$

$$(2') \qquad v(t) = a + \int_{t_n}^{t} f(v(s))ds$$

$$(3') \qquad w(t) = a + \int_{t_n}^{t_{n+1}} f(v(s))ds + \int_{t_n}^{t} g(w(s))ds$$

To prove **(8)** observe that

$$u(t) - v(t) = \int_{t_n}^{t} [f(u(s) - f(v(s))]ds + \int_{t_n}^{t} g(u(s))ds$$

hence $\forall t \in [t_n, t_{n+1}]$

$$|u(t) - v(t)| \le \tau L_f \|u-v\|_\infty + \tau K_g.$$

hence

$$\|u - v\|_\infty \leq \frac{\tau K_g}{1 - \tau L_f}. \tag{8}$$

Now

$$w(t) - u(t) = \int_{t_n}^{t_{n+1}} f(v(s))ds + \int_{t_n}^{t} g(w(s))ds$$
$$- \int_{t_n}^{t} f(u(s))ds - \int_{t_n}^{t} g(u(s))ds$$
$$- \int_{t_n}^{t} f(v(s))ds + \int_{t_n}^{t} f(v(s))ds$$

this gives

$$w(t) - u(t) = \int_{t}^{t_{n+1}} f(v(s))ds + \int_{t_n}^{t} [f(v(s)) - f(u(s))]ds$$
$$+ \int_{t_n}^{t} [g(w(s)) - g(u(s))]ds$$

Now applying **(8)** we get

$$(*) \qquad |w(t) - u(t)| \leq |t_{n+1} - t| K_f + \tau^2 \frac{K_g L_f}{1 - \tau L_f} + \tau L_g \|u - w\|_\infty$$

Tacking 'sup' over both sides of this last inequality we get:

$$\|w - u\|_\infty \leq \tau K_f + \tau^2 \frac{K_g L_f}{1 - \tau L_f} + \tau L_g \|u - w\|_\infty$$

Finally:

$$\|u - w\|_\infty \leq \tau \frac{K_f}{1 - \tau L_g} + \tau^2 \frac{K_g L_f}{(1 - \tau L_f)(1 - \tau L_g)} = O(\tau) \tag{9}$$

Let us go back to the formula **(∗)**. If we put $t = t_{n+1}$, then the first term on the RHS desapears. Now applying **(9)**, we get **(10).** □