LECTURE 6

SOME FINITE DIFFERENCE SCHEMES FOR TRANSPORT AND HEAT EQUATIONS IN 1D

TRANSPORT EQUATION $u_t + \alpha u_x = 0$.

Grid functions: $u^n_k \approx u(t_n, x_k)$, $t_n = \tau n$, $x_k = kh$, $\lambda = \frac{\tau}{h}$.

1. BOX LEFT=>$RIGHT (implicit-explicit)

(6.1) $au^{n+1}_{k+1} + bu^{n+1}_k = bu^n_{k+1} + au^n_k$

where $a = (1+\alpha \lambda)$, $b = (1-\alpha \lambda)$. For $\alpha > 0$ equation (6.1) is solved in $u^n_{k+1}$. Scheme of order $O(\tau) = O(h)$, stability test $u^n_k = \gamma^ne^{-isk}$ unconditionally successful.

2. UPWIND SCHEMES (explicit)

(6.2) $u^{n+1}_k = u^n_k(1 + \alpha \lambda) - \alpha \lambda u^n_{k+1}$, $\alpha < 0$

(6.3) $u^{n+1}_k = u^n_k(1 - \alpha \lambda) + \alpha \lambda u^n_{k-1}$, $\alpha \geq 0$

Schemes of order $O(\tau) = O(h)$. Stability test $u^n_k = \gamma^ne^{-isk}$ successful if $\lambda \leq \frac{1}{|\alpha|}$.

3. LAX-FRIEDRICHES (explicit)

(6.4) $\frac{1}{\tau} (u^{n+1}_k - \frac{u^n_{k-1} + u^n_{k+1}}{2}) + \alpha \frac{u^n_{k+1} - u^n_{k-1}}{2h} = 0$

or

(6.5) $u^{n+1}_k = \frac{(1 + \alpha \lambda)}{2} u^n_{k-1} + \frac{(1 - \alpha \lambda)}{2} u^n_{k+1} = 0$

Scheme is of order $O(\tau) = O(h)$, stability test $u^n_k \gamma^ne^{-isk}$ is successful if $\lambda \leq \frac{1}{|\alpha|}$, independently of the sign of $\alpha$. 


4. Be careful! In the following scheme

\[
(6.6) \quad \frac{u_{k+1}^n - u_k^n}{\tau} + \frac{u_{k+1}^n - u_{k-1}^n}{2h} = 0
\]

the derivative \( u_x \) is approximated by the so called central finite difference of the order \( O(h^2) \), hence it seems to be 'beter' then, for example, the schemes of the type "upwind" (6.2) (6.3). Unfortunately, the stability test \( u_k^n = \gamma^n e^{-isk} \) for the scheme (6.6) never passes with success.

HEAT EQUATION \( u_t = \nu u_{xx}, \nu > 0 \) (will follow)