A LITLE BIT ABOUT NONLINEAR PROBLEMS

SOME IMPORTANT EXAMPLES

- **BURGERS EQUATION**

  (1) $$u_t + uu_x = 0$$

  or

  $$u_t + (\frac{u^2}{2})_x = 0$$

  may by considered as a kind of nonlinear transport equation, when u is interpreted as velocity.

- **STOKES OPERATOR AND STOKES EQUATION**

  $$\frac{\partial}{\partial t} + \underline{v}\nabla$$

  (2) $$\frac{\partial \underline{v}}{\partial t} + \underline{v}\nabla\underline{v} = 0$$

  For $\underline{v} = [v_x, v_y, v_z]$ equation (2) may be writen as follows

  $$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = 0$$

  $$\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z} = 0$$

  $$\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z} = 0$$

  Equation (2) may be treated as 3-dimensional version of the Burgers equation.

- The Burgers equation (1) is a typical equation in the so called **'form of conservation law'**. General form of such an equation is as follows:

  (3) $$u_t + F(u)_x = 0,$$

where $\boldsymbol{F}$ is a given function. To the equation **(3)** in one-dimensional case, the **Box-scheme** may be applied. In order to get the Box-scheme for **(1)** let us form **the grid** $\boldsymbol{x_k = kh}$, $\boldsymbol{t_n = n\tau}$ with the space-step $\boldsymbol{h > 0}$ and the time-step $\boldsymbol{\tau > 0}$ on the $\boldsymbol{t - x}$ plane. Integrating **(3)** over the **grid-box**

$$\begin{array}{ccccc} n+1 & & - & & \\ & | & & | & \\ n & k & - & k+1 & \end{array}$$

$$\int_{t_n}^{t_{n+1}} \int_{x_k}^{x_{k+1}} [u_t + F(u)_x] dx dt = 0,$$

and approximating one integral of each term using the trapezoidal rule we get for example (if we want to compute $\boldsymbol{u_{k+1}^{n+1}}$) the nonlinear equation of the form

$$u_{k+1}^{n+1} + \lambda F(u_{k+1}^{n+1}) = A, \quad \lambda = \frac{\tau}{h} \tag{4}$$

where $\boldsymbol{A}$ stands for the sum of all known terms (see the linear version of the Box-scheme). Of course this version of the Box-scheme is applicable only under certain conditions concerning function $\boldsymbol{F}$ (compare the $\boldsymbol{L} => \boldsymbol{R}$ and $\boldsymbol{R} => \boldsymbol{L}$ versions in the linear case)

How to solve approximately this nonlinear equation?

1. In the scalar case the **Bisection Method** may be applied
2. If the function F is sufficiently regular (for example if the Lipschitz condition is satisfied) the **Simple Iteration Method (Banach Fixed Point Theorem)** can be applied. Put

   $$x_{p+1} = A - \lambda F(x_p) \tag{5}$$

   If $\boldsymbol{x = A - \lambda F(x)}$ (the solution $\boldsymbol{x}$ of **(5)** does exist because of the Lipschitz condition) hence we have for the error at the $\boldsymbol{p+1}$ -st step of iteration $\boldsymbol{e_{p+1} = x - x_{p+1}}$:

   $$|e_{p+1}| = \lambda |F(x) - F(x_p)| \leq \lambda L |e_p|$$

   where $\boldsymbol{L}$ is the Lipschitz constant. Choosing the steps $\boldsymbol{h}$ and $\boldsymbol{\tau}$ so that $\boldsymbol{\lambda = \frac{\tau}{h}}$ is small enough we get convergence $\boldsymbol{e_p \to 0}$ when

$p \to \infty$. In dependece of the properties of the function F the version $L => R$ or $R => L$ of the BOX algorithm has to be applied. What can we say about the Burgers equation in this context?

- The very important in applications, generalization of Burgers and Stokes equations, is the **Navier-Stokes equation**:

$$\rho \frac{\partial v}{\partial t} + \rho v \nabla v = \nabla \mathcal{P} + \rho f$$

  here ρ is the mass density function, v is the velocity vector, at the right hand side is the stress tensor and the body forces.

- The very universal in its applications **Ludwik Boltzmann Equation** have to be mentioned here. Also because its linear part defines exactely the well known for us, linear transport equation.

$$\rho_t + \alpha \nabla \rho = \int_{\mathcal{A}} M(\cdot, \alpha, \beta) d\beta$$

  Here $\rho(t, x, \alpha)$ is some kind of mass density function where t is the time, $x \in \Omega \subset R^3$, is the space point, α, $\beta \in \mathcal{A} \subset R^3$ is the velocity vector which here is one of independent variables, (β is an 'additional copy of α') used for integration in the right hand side term. This term is called 'collision term'. Similar in the form but with different function M in the right hand side term is the turbulent flow model equation of Prof. Marek Burnat from our department of the Warsaw University. In this last model the right hand side term defines the so called **mixer**. The Marek Burnat equation is as follows

$$\rho_t + \alpha \nabla \rho - \nu \Delta \rho = \int_{\mathcal{A}} M(\cdot, \alpha, \beta) d\beta$$

  where $\nu > 0$ is the small positive coefficient of the laplasian Δ. Laplasian is the added viscosity term, helpful in numerical realization of this model.

- Perhaps it is a good moment to analyse the results of application of the Laplace Operator to a given function f. Consider a function f and its **Fourier Transform**

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

as well as its retransform

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\frac{s}{2\pi}) e^{isx} ds$$

where $s = 2\pi\xi$. Observe that $c(s) = \hat{f}(\frac{s}{2\pi})$ is a 'Fourrier coefficient' of the function f, depending on the frequency $s \in R$. Apply now the Laplace operator to f:

$$f_{xx}(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)(-s^2) d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\frac{s}{2\pi}) e^{isx} (-s^2) ds$$

Finally

$$f_{xx}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(s)(-s^2) e^{isx} ds. \tag{6}$$

We can read the the formula **(6)** in the following way: for low frequencies $|s| < 1$ the corresponding Fourier coefficients $-c(s)s^2$ of the Laplace Operator f_{xx} of f diminish with respect to the Fourier coefficients of the original function f. However for high frequencies $|s| > 1$, the corresponding Fourier coefficients grows. In other words, taking the Laplace Operator of f, we filter out the low frequency Fourier components while we amplify the high frequecy ones.

- **How to menage with the nonlinear, non differential term in our evolution equation?**

  We met such a term in Boltzmann equation and in Burnat Model equation. If the **Time-Splitting Method** is applied, we have to solve the initial value problem for nonlinear ordinary differential equation (**ODE**). For example

$$u_t(t, \cdot) = F(t, \cdot, u), \quad u(t_n, \cdot) = \phi(t_n, \cdot) \tag{7}$$

  In many applications, this simple problem may have nasty properties: for example our **ODE** can be **STIFF**.

  In order to explain what the **STIFFNESS** is, consider the following simple example. Let λ_1 and λ_2 be two complex numbers, **both with negative real parts**. Assume that $|\Re(\lambda_1)| \gg |\Re(\lambda_2)|$.

Suppose that we are interested in the behaviour of solution for $t \in [0, T]$ for some $T > 0$ Take now the system of two **ODE's**:

$$\frac{du_1}{dt} = \lambda_1 u_1$$

$$\frac{du_2}{dt} = \lambda_2 u_2 \tag{8}$$

Solution of this system is as follows

$$u_1(t) = C_1 e^{\lambda_1 t}, \quad u_2(t) = C_2 e^{\lambda_2 t}.$$

But if $|\Re(\lambda_1)|$ **is much greater than** $|\Re(\lambda_2)|$, the function u_1 very quickly desapears, and then the solution of our system can be represented by the function u_2 alone. On the other hand **the right hand side** of the system contains λ_1, and very large $|\lambda_1|$ may badly influence stability of the whole system. Similar situation may occur also in many cases of the equation not of the form (8), and when it occurs, we say that such ODE is **STIFF**.

- **What can we do to not have stability problems if the stiffness occurs?** The question is in the right choice of the numerical method. Let us look shortly at this problem.

  There is lot of various numerical methods for solving initial value problems of ODE. A large class are so called **Multistep Methods**. Suppose we are solving following initial value problem:

$$\frac{dx}{dt} = f(t, x), \quad x(0) = x_0 \tag{9}$$

The general q-step method for problem (8) is as follows:

$$\sum_{j=0}^{q} \alpha_j x_{k+j} = h \sum_{j=0}^{q} \beta_j f_{k+j} \tag{10}$$

where $x_i \approx x(ih)$, and $f_i = f(t_i, x_i), \quad t_i = ih$. Since we are interested in **STIFFNESS** we observe what happens if we put $f(t, x) = ax$

for any complex a, such that $\Re(a) < 0$ (see (8)). Now we get from (10)

$$\sum_{j=0}^{q}(\alpha_j - z\beta_j)x_{k+j} = 0 \tag{11}$$

where $z = ah$ is an arbitrary complex number with $\Re(z) < 0$. It is known that the general solution of the **finite difference** equation (11) can be expressed with help of so called **characteristic equation of (11)** which is of the following form

$$\sum_{j=0}^{q}(\alpha_j - z\beta_j)\xi^j = 0 \tag{12}$$

Assume that the roots $\xi_j, \quad j = 1, 2, \cdots q$ of the polinomial equation **(12) are all single**. In this case

$$x_i = \sum_{j=1}^{q} c_j \xi_j^i$$

with arbitrary constants $c_1, c_2, \cdots, c_q$.

DEFINITION 1. *We say that the q-step method* (10) *is* **absolutly stable** *for given complex z if*

$$|\xi_j| \leq 1, \quad j = 1, 2, \cdots, q$$

DEFINITION 2 *Let $\Omega \subset \mathcal{C}$ be some subset of the complex plane $\mathcal{C}$. We say that* Ω **is the domain of absolute stability of (10) iff** $\forall z \in \Omega$ *the q-step method* (10) *is* **absolutly stable**.

DEFINITION 3 *The q-step method* (10) *is* **A-stable** *if its domain of absolute stability Ω contains the set $Re(z) \leq 0$.*

Comment. A-stable q-step method is resistant against the stiffness.

Consider the 1-step implicit method **"the Trapezoidal Rule":**

$$\textbf{(12)} \qquad u_{k+1} - u_k = \frac{h}{2}(f_k + f_{k+1}).$$

It is easy to find its **domain of absolute stability.**

We have in this case:

$$\alpha_0 = -1, \quad \alpha_1 = 1$$

$$\beta_0 = \beta_1 = \frac{1}{2}$$

and if $z = r + is$ with $r \leq 0$ then

$$\xi = \frac{2+z}{2-z}, \quad |\xi|^2 = \frac{(r+2)^2 + s^2}{(r-2)^2 + s^2} \leq 1$$

this means that **the Trapezoidal Rule is A-stable.** It can be proven that it is the unique A-stable method in his class.

Moreover observe that if f satisfies the Lipschitz Condition, then it is allways possible to chose the **step h** so that the simple iteration method for iterative solution of the nonlinear equation **(12)** converges.