The Least Fibred Lifting and the Expressivity of Coalgebraic Modal Logic

Bartek Klin

University of Sussex,
Warsaw University

Abstract Every endofunctor B on the category **Set** can be lifted to a fibred functor on the category (fibred over **Set**) of equivalence relations and relation-preserving functions. In this paper, the least (fibre-wise) of such liftings, $L(B)$, is characterized for essentially any B. The lifting has all the useful properties of the relation lifting due to Jacobs, without the usual assumption of weak pullback preservation; if B preserves weak pullbacks, the two liftings coincide.
Equivalence relations can be viewed as Boolean algebras of subsets (predicates, tests). This correspondence relates $L(B)$ to the least test suite lifting $T(B)$, which is defined in the spirit of predicate lifting as used in coalgebraic modal logic. Properties of $T(B)$ translate to a general expressivity result for a modal logic for B-coalgebras. In the resulting logic, modal operators of any arity can appear.

1 Introduction

Coalgebras are used as models for various kinds of transition systems, offering a general view on the notions of coinduction, bisimulation, and on logics used to reason about systems. For example, a finitely branching labelled transition system with carrier X is a coalgebra $h : X \to \mathcal{P}_{\mathrm{f}}(A \times X)$, where A is a set of labels. Replacing the behaviour functor $\mathcal{P}_{\mathrm{f}}(A \times -)$ with various functors on the category **Set** of sets and functions, one models other kinds of systems [16].

Final coalgebras are abstract models of the behaviour of systems. For a coalgebra $h : X \to BX$, two processes in X are considered behaviourally equivalent if they are identified by the unique morphism from h to a final B-coalgebra. If $B = \mathcal{P}_{\mathrm{f}}(A \times -)$, behavioural equivalence coincides with bisimilarity.

Usually, bisimilarity is defined as the greatest bisimulation on a transition system. A natural coalgebraic generalization of the classical notion of bisimulation [13,11] is based on spans of coalgebras [16]. There, a bisimulation is a relation that lifts to a span of coalgebra morphisms. If the behaviour functor B preserves weak pullbacks, the greatest bisimulation exists and coincides with behavioural equivalence. Another, closely related approach was presented by Jacobs et al. [4,7,9]. There, the functor B is provided with a relation lifting $J(B)$, which is a functor on the category of binary relations that behaves as B on the underlying sets. A bisimulation is then a $J(B)$-coalgebra. Provided B preserves weak pullbacks, bisimulations satisfy many useful properties, e.g., they are closed under unions and preserved by coalgebra morphisms; also, as in the span approach, the greatest bisimulation coincides with behavioural equivalence.

In the case of labelled transition systems, bisimilarity is characterized by Hennessy-Milner logic [3]. The coalgebraic framework provides a more general perspective, where a modal logic can be derived for coalgebras for any functor. It is expected that coalgebraic modal logic is invariant under behavioural equivalence. Moreover, it should be expressive: logically indistinguishable states should be behaviourally equivalent. Several approaches to deriving coalgebraic modal logics have been proposed [6,10,12,14,15,17], based on different abstract notions of modal operators (modalities).

In [12], a single modal operator is associated with every functor B. The resulting logic is expressive for essentially all functors. However, the modal operator involved is rather complex and difficult to relate to modalities usually considered in particular cases (e.g., the box and diamond modalities of Hennessy-Milner logic). In [14], modalities are defined to be predicate liftings, i.e., natural transformations $\lambda : 2^- \to 2^B$, which transform predicates on any set X to predicates on BX. These correspond more closely to the modal operators usually considered in modal logics. However, the logic thus obtained fails to be expressive for many functors. In [10], modalities are defined to be test constructors, i.e. functions $w : B2 \to 2$.

Very recently, Schröder [17] provided a characterization of functors which admit an expressive logic based on predicate liftings. He also observed that predicate liftings and test constructors are in a one-to-one correspondence. To enhance expressivity, he then introduced polyadic predicate liftings, corresponding to functions $w : B(2^\kappa) \to 2$, and proved that a modal logic based on those is expressive for all accessible functors.

In this paper, we treat both relation lifting and coalgebraic modal logics in a fibrational setting. The relation lifting $J(B)$ is viewed as a lifting of B to a fibred functor on the category **Rel** of binary relations and relation-preserving functions. Similarly, any family of predicate liftings (test constructors) induces a fibred lifting of B to the category **TS** of test suites, i.e., families of predicates.

Our first observation is that, when one wants to lift a functor B to a fibred functor on **Rel**, the choice of $J(B)$ among many other liftings is not entirely clear. Despite its many useful properties, it does not really seem canonical. We therefore study the *least* fibred liftings of functors. The least fibred liftings to **Rel** turn out to be trivial, but when restricted to the category **ERel** of equivalence relations, the least fibred lifting $L(B)$ is nontrivial, and it enjoys all the useful properties of $J(B)$, without the usual assumption of weak pullback preservation. Indeed, if B preserves weak pullbacks, both liftings coincide. It seems, therefore, that $L(B)$ is the canonical choice of relation lifting, at least to equivalence relations, and that $J(B)$ inherits its properties from it. Our lifting $L(B)$ is defined only for equivalence relations, but this does not seem a serious limitation in this context, since behavioural equivalence, which one aims to model by bisimulations, is always an equivalence relation. (Note, however, that the lifting $J(B)$ has been used in other contexts [8], where arbitrary relations are essential.)

The category **ERel** is isomorphic to the category **BTS** of test suites closed under union, intersection and complementation. This isomorphism relates $L(B)$ corresponds to the least fibred lifting $T(B)$ of B to **TS**. Lifting functors to test suites is in the spirit of coalgebraic modal logic, where families of predicate liftings induce such constructions. From a characterization of $T(B)$ we derive a modal logic for any functor on **Set**, and prove an expressivity result for a wide class of functors (we give proofs for ω-continuous functors here, but all results generalize to all accessible functors, as will be shown in the full version of this paper). The logic we obtain is exactly the same as the logic of polyadic modalities of [17]. This provides further insight into the canonicity and importance of Schröder's results.

The paper is organized as follows. Section 2 treats preliminaries. In Section 3, basic definitions concerning fibrations are given together with two examples used throughout the paper. In Section 4, the least relation lifting $L(B)$ is characterized and compared with Jacobs's lifting $J(B)$. In Section 5, the least test suite lifting $T(B)$ is characterized and studied, and in Section 6 we define the modal logic related to it. We also give some examples suggesting that polyadic modalities can be useful to describe practically important cases.

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2 Preliminaries

Let **Set** denote the category of sets and functions, and **Pos** the category of partial orders and monotonic functions. For an endofunctor B on any category, a B-coalgebra is a morphism $h : X \to BX$. A coalgebra morphism from $g : X \to BX$ to $h : Y \to BY$ is a morphism $f : X \to Y$ such that $Bf \circ g = h \circ f$. For any B, all B-coalgebras and their morphisms form a category. For any $h : X \to BX$, the unique morphism to the final B-coalgebra (if it exists) is called the coinductive extension of h.

Many functors on **Set** admit final coalgebras; in particular, all ω-continuous ones do. A functor B is called ω-continuous if it preserves limits of diagrams of the shape

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow X_3 \cdots\cdots$$

The carrier Z of a final B-coalgebra arises from the limiting cone

$$\begin{tikzcd} & & Z \arrow[dll, "p_0"'] \arrow[dl, "p_1"'] \arrow[d, "p_2"] \arrow[dr, "p_3"] \arrow[drr, dotted, no head] & & \\ 1 & B1 \arrow[l, "!"] & B^2 1 \arrow[l, "B!"] & B^3 1 \arrow[l, "B^2 !"] & \cdots \end{tikzcd} \tag{1}$$

where 1 is the final object in the underlying category, and $! : B1 \to 1$ is the unique map from $B1$.

Many kinds of state-based systems can be modelled as coalgebras. For example, finitely branching labelled transition systems are coalgebras $h : X \to \mathcal{P}_{\mathrm{f}}(A \times X)$, where $\mathcal{P}_{\mathrm{f}}$ is the covariant finite powerset functor, and A is a fixed set of labels. Coalgebras for this functor are described by Hennessy-Milner logic [3], which is defined by the grammar:

$$\phi ::= \mathtt{tt} \mid \phi \wedge \phi \mid \neg\phi \mid \langle a \rangle\, \phi$$

Given any coalgebra $h : X \to \mathcal{P}_{\mathrm{f}}(A \times X)$, every formula of this logic is interpreted as a predicate on X or, equivalently, as a function $[\![\phi]\!] : X \to 2$ (here and in the following, 2 denotes the two-element set $\{\mathtt{tt}, \mathtt{ff}\}$), defined inductively in the natural way, with the only interesting clause:

$$[\![\langle a \rangle\, \phi]\!](x) = \mathtt{tt} \iff \exists\, \langle a, y \rangle \in hx.\ [\![\phi]\!](y) = \mathtt{tt}$$

This logic is invariant under behavioural equivalence. Moreover, it is expressive: if two states are not distinguished by Hennessy-Milner formulae, they are identified by the coinductive extension of h.

3 Fibrations and Lifting

Any functor $(-)^* : \mathbf{Set}^{op} \to \mathbf{Pos}$ gives rise to a *total category* $\mathbf{Set}^*$, via the so-called Grothendieck construction: objects of $\mathbf{Set}^*$ are pairs $\langle X, R \rangle$ such that $R \in X^*$, and morphisms $f : \langle X, R \rangle \to \langle Y, S \rangle$ are functions $f : X \to Y$ such that $R \leq f^* S$, where $\leq$ is the ordering relation in X^*.

The obvious forgetful functor $p : \mathbf{Set}^* \to \mathbf{Set}$, mapping a pair $\langle X, R \rangle$ to its underlying set X, is then a *fibration*. This is only a special case of a more general definition used in fibration theory (see [5] for a detailed presentation), but only examples of this kind are considered in this paper. Slightly abusing the terminology, we will simply call functors $(-)^* : \mathbf{Set}^{op} \to \mathbf{Pos}$ fibrations. The partial order X^* is called the *fibre* over X. The map $f^* : Y^* \to X^*$ is called the *reindexing* along f.

Example 1. Define $(-)^\dagger : \mathbf{Set}^{op} \to \mathbf{Pos}$ by

$$\begin{aligned} X^\dagger &= \langle \mathcal{P}(X \times X), \subseteq \rangle \\ f^\dagger(S) &= \{\, \langle x, y \rangle \in X \times X \;:\; (fx)S(fy) \,\} \end{aligned}$$

for any set X, function $f : X \to Y$ and relation $S \subseteq Y \times Y$. The proof of functoriality is very easy and omitted here. In the total category arising from this fibration, objects are pairs $\langle X, R \rangle$ with R a binary relation on X, and a function $f : X \to Y$ is a morphism between $f : \langle X, R \rangle \to \langle Y, S \rangle$ if and only if xRy implies $(fx)S(fy)$. The total category is therefore the category $\mathbf{Rel}$ of binary relations and relation-preserving functions. The fibre over a set X is the set of all binary relations on X, partially ordered by inclusion.

Note that for any $f : X \to Y$, the reindexing $f^\dagger : Y^\dagger \to X^\dagger$ maps equivalence relations to equivalence relations. It is therefore possible to restrict the functor

$(-)^\dagger$ to yield equivalence relations only; the total category arising from this fibration is the category **ERel** of equivalence relations and relation-preserving functions.

Example 2. Denote $2 = \{\mathtt{tt}, \mathtt{ff}\}$. A *test* on a set X is a function $V : X \to 2$. (A test can be also seen as a subset of its domain, but the functional representation will make some definitions and results in Section 5 look more natural.) A *test suite* on X is simply a set of tests on X. To say that θ is a test suite on a set X, we write $\theta : X \rightrightarrows 2$. Define $(-)^\ddagger : \mathbf{Set}^{op} \to \mathbf{Pos}$ by

$$\begin{aligned} X^\ddagger &= \langle \{\theta : X \rightrightarrows 2\}, \supseteq \rangle \\ f^\ddagger(\vartheta) &= \{\, V \circ f \;:\; V \in \vartheta \,\} \end{aligned}$$

for any set X, function $f : X \to Y$ and test suite $\vartheta : Y \rightrightarrows 2$. Note that fibres $X^\ddagger$ are ordered by reverse inclusion. The proof of functoriality is again very easy and omitted here. The total category arising from this fibration is denoted by **TS**. As object in **TS** is a pair $\langle X, \theta \rangle$ where $\theta : X \rightrightarrows 2$, and a morphism $f : \langle X, \theta \rangle \to \langle Y, \vartheta \rangle$ is a function $f : X \to Y$ such that for any test $V \in \vartheta$, one has $V \circ f \in \theta$.

To get some intuition, note that any topology can be seen as a test suite subject to additional closure conditions. Morphisms in **TS** between such test suites are then exactly continuous functions between topological spaces.

Intersections, unions and complements of tests are defined in the obvious way. It is easy to see that if a test suite $\vartheta : Y \rightrightarrows 2$ is a complete field of sets (i.e., it is closed under arbitrary intersection, union and complementation), then for any $f : X \to Y$, the test suite $f^\ddagger \vartheta$ is also a complete field of sets. It is therefore possible to restrict $(-)^\ddagger$ to complete fields of sets only; the total category arising from this fibration will be denoted **BTS**.

Proposition 3. The categories **ERel** and **BTS** are isomorphic; the two maps defined on objects as

$$\begin{aligned} \langle X, \theta \rangle &\mapsto \langle X, \{\, \langle x, y \rangle \in X \times X \;:\; \forall V \in \theta.\ Vx = Vy \,\} \rangle \\ \langle X, R \rangle &\mapsto \langle X, \{\, V : X \to 2 \;:\; \forall x, y \in X.\ xRy \Rightarrow Vx = Vy \,\} \rangle \end{aligned} \tag{2}$$

and as identities on morphisms are mutually inverse functors.

□

For any fibration $(-)^* : \mathbf{Set}^{op} \to \mathbf{Pos}$, say that a functor $B^* : \mathbf{Set}^* \to \mathbf{Set}^*$ *lifts* a functor $B : \mathbf{Set} \to \mathbf{Set}$ if

$$p \circ B^* = B \circ p$$

where $p : \mathbf{Set}^* \to \mathbf{Set}$ is the forgetful functor associated to $(-)^*$. Given a functor $B : \mathbf{Set} \to \mathbf{Set}$, a lifting of B to B^* is uniquely determined by its action on the elements of fibres, i.e., by a family (indexed by sets X) of functions

$$B_X : X^* \to (BX)^*$$

For any such family, the map defined by

$$\begin{aligned} B^* \langle X, R \rangle &= \langle BX, B_X R \rangle \\ B^* f &= Bf \end{aligned} \tag{3}$$

is a functor on $\mathbf{Set}^*$ if and only if all functions B_X are monotonic and, for any function $f : X \to Y$ and any $S \in Y^*$,

$$B_X(f^* S) \le (Bf)^*(B_Y S)$$

If, moreover, the above inequality holds as equality:

$$B_X(f^* S) = (Bf)^*(B_Y S) \tag{4}$$

we say that B^* is *fibred*.

In the following, an iterated version of B_X will be useful. For any $R \in X^*$, define $B^n R \in (B^n X)^*$ by induction:

$$\begin{aligned} B^0_X R &= R \\ B^{n+1}_X R &= B_{B^n X} B^n_X R \end{aligned}$$

The following result, used in Section 5, is a characterization of final coalgebras for fibred liftings of ω-continuous functors.

Proposition 4. Assume a fibration $(-)^* : \mathbf{Set}^{op} \to \mathbf{Pos}$ such that

- every fibre has arbitrary intersections $\bigwedge$, and
- reindexing functions preserve intersections.

Every fibred functor $B^* : \mathbf{Set}^* \to \mathbf{Set}^*$ lifting an ω-continuous functor $B : \mathbf{Set} \to \mathbf{Set}$ admits a final coalgebra

$$\phi : \langle Z, \zeta \rangle \to \langle BZ, B_Z \zeta \rangle$$

where $\phi : Z \to BZ$ is the final B-coalgebra obtained as in (1), and

$$\zeta = \bigwedge_{n \in \mathbb{N}} p^*_n (B^n_1 T)$$

where T is the largest element (i.e., the empty intersection) in 1^*.

4 The Least Relation Lifting

In [4,7,9], Jacobs et al. showed how to lift any functor $B : \mathbf{Set} \to \mathbf{Set}$ to a functor $J(B) : \mathbf{Rel} \to \mathbf{Rel}$ (in fact they work in a slightly more general setting, with relations of the type $X \times Y$ rather than $X \times X$; here their definition is simplified to fit into the setting of this paper), defined by the following action:

$$J(B)_X(R) = \{ \langle \alpha, \beta \rangle \in BX \times BX \ : \exists w \in BR.(B\pi_1)w = \alpha, B(\pi_2)w = \beta \}$$

where $\pi_1, \pi_2 : R \to X$ are projections. They then define a bisimulation (on the underlying B-coalgebra) to be a $J(B)$-coalgebra. If B preserves weak pullbacks, $J(B)$ has many useful properties. For example, it is fibred, bisimulations are closed under unions, and (the relation component of the carrier of) the greatest bisimulation on the final B-coalgebra is the equality relation.

The definition of $J(B)$ is elegant in that it does not depend on the structure of B. It is also closely related to the coalgebra span approach to bisimulation [1,16]. However, speaking in terms of fibrations, it does not really seem canonical: it is not immediately clear why one should choose this particular lifting from many available ones, and only the numerous useful properties of $J(B)$ convince one that the lifting is "right".

In search for a canonical lifting of a functor B to **Rel** it is natural to look at the least and the greatest (fibre-wise) fibred liftings. These turn out to be trivial: as is easy to check, the least (greatest) fibred lifting of any B to **Rel** is defined by the action mapping any relation on X to the empty (resp. full) relation on BX. However, when one restricts to the category **ERel** of equivalence relations, the least fibred lifting becomes more interesting. In the remainder of this section we shall see the construction of this lifting, denoted $L(B)$, and some of its properties. The construction works for any functor that preserves monos. This is a much weaker assumption than that of weak pullback preservation. Practically all functors on **Set** preserve monos, and every functor on **Set** preserves all monos with nonempty domains.

Observe that $L(B)$ is fully determined by its action on equality relations. Indeed, take any set X and an equivalence R on X, and consider the abstraction function

$$[-]_R : X \to X/_R$$

Note that $[-]_R^\dagger \varDelta_{X/_R} = R$ (here and in the following, $\varDelta_X$ denotes the equality relation on X.) Pictorially:

$$\begin{tikzcd} R \arrow[from=r, mapsto, "{[-]_R^\dagger}"'] & \varDelta_{X/_R} \\ X \arrow[r, "{[-]_R}"'] & X/_R \end{tikzcd}$$

Then

$$L(B)_X R = L(B)_X([-]_R^\dagger(\varDelta_{X/_R})) = (B[-]_R)^\dagger(L(B)_{X/_R}(\varDelta_{X/_R}))$$

The second equality holds by the fibredness condition (4) on $L(B)_X$. Note that $(B[-]_R)^\dagger$ in the rightmost expression does not depend on the lifting $L(B)_X$, so the left hand side is fully determined by $L(B)_{X/_R}(\varDelta_{X/_R})$.

The least possible value of the latter is $\varDelta_{B(X/_R)}$, as this is the smallest equivalence relation on $B(X/_R)$. Then one has

$$\begin{aligned} L(B)_X R &= (B[-]_R)^\dagger(\varDelta_{B(X/_R)}) = \\ &= \{ \langle \alpha, \beta \rangle \subseteq BX \times BX \; : \; (B[-]_R)\alpha = (B[-]_R)\beta \} \end{aligned} \tag{5}$$

From the above reasoning it is clear that every fibred lifting of B is fibre-wise larger than $L(B)$ defined as in (5). It is also obvious that for any equivalence relation R on X, $L(B)_X R$ is also an equivalence relation. It must yet be checked that $L(B)_X$ defines a fibred functor. This follows from a collection of properties analogous to those of $J(B)$ as listed in [9]:

Proposition 5. If B preserves monos, then the maps $L(B)_X$:

(i) preserve equality relations: $L(B)(\Delta_X) = \Delta_{BX}$;
(ii) half-preserve the transitive closure of relational composition: for R, S equivalence relations on X, the relational composition $S \circ R = \{ \langle x, z \rangle \,:\, \exists y.\ xRy, ySz \}$ satisfies: $L(B)_X(\overline{S \circ R}) \supseteq \overline{L(B)_X R \circ L(B)_X S}$;
(iii) are monotonic: if $R \subseteq S$ then $L(B)_X R \subseteq L(B)_X S$;
(iv) preserve reversals: $L(B)_X(R^{op}) = (L(B)_X R)^{op}$;
(v) preserve reindexing: for any $f : X \to Y$ and any relation S on Y, the condition (4) from Section 3 holds.

Note that the property (iv) above is trivially satisfied, and is included here for a comparison with an analogous result in [9].

Corollary 6. If B preserves monos, then $L(B)$ defined as in (3) from maps $L(B)_X$ defined in (5) is a fibred functor on **ERel**.

Proof. : Immediate from properties (iii) and (v) in Proposition 5. $\square$

By a *bisimulation* on a B-coalgebra $h : X \to BX$ we will mean simply an $L(B)$-coalgebra

$$h : \langle X, R \rangle \to \langle BX, L(B)_X R \rangle$$

In other words, it is an equivalence relation R on X such that

$$xRy \Longrightarrow (B[-]_R)(hx) = (B[-]_R)(hy)$$

The next theorem states some properties of bisimulations, analogous to those mentioned in [9] (see also [16]).

Proposition 7. If B preserves monos, then for any coalgebras $g : X \to BX$, $h : Y \to BY$:

(i) Bisimulations are closed under transitive closures of arbitrary unions, hence there exists a greatest bisimulation, called *bisimilarity.*
(ii) Equality relations are bisimulations.
(iii) If $f : X \to Y$ is a B-coalgebra homomorphism from g to h, then any x is bisimilar to fx in the coalgebra

$$[B\iota_1 \circ g, B\iota_2 \circ h] : X + Y \to B(X + Y)$$

(iv) If $f : X \to Y$ is a coalgebra homomorphism from g to h then x is bisimilar to x' in X if and only if fx is bisimilar to fx' in Y.
(v) If B is ω-continuous, then (a final B-coalgebra exists and) bisimilarity on the final B-coalgebra is equality.

Propositions 5 and 7 show that $L(B)$ satisfies all the useful properties of $J(B)$, without the assumption that B preserves weak pullbacks. Moreover:

Proposition 8. If B preserves weak pullbacks, then $L(B) = J(B)$.

Therefore $J(B)$, when restricted to equivalence relations, is essentially a special case of the canonical lifting $L(B)$.

5 The Least Test Suite Lifting

In Section 3, it was noted that the category **ERel** of equivalence relations and relation-preserving functions is isomorphic to the category **BTS** of test suites that are complete fields of sets. This means that for any $B : \mathbf{Set} \to \mathbf{Set}$, the least fibred lifting $L(B)$ to **ERel** corresponds to the least fibred lifting to **BTS**; its concrete description can be obtained from (5) and the isomorphisms (2). However, the resulting definition is rather unwieldy. Since least fibred liftings of functors to test suites will be of use in Section 6, instead we use techniques as in Section 4 to derive a characterization of the least fibred lifting to **TS**, denoted $T(B)$ in the following. It turns out that the new characterization does not require even the very mild condition of B preserving monos.

For any set X, denote the test suite of all tests on X by $\mathbb{T}_X$. Begin by observing that any fibred lifting of $B : \mathbf{Set} \to \mathbf{Set}$ to **TS** is fully determined by its values on the $\mathbb{T}_X$. Indeed, consider any set X and any test suite $\theta : X \rightrightarrows 2$. Define a function $e_\theta : X \to 2^\theta$ by

$$e_\theta(x)(V) = Vx \qquad \text{for } V \in \theta$$

Lemma 9. For any $\theta : X \rightrightarrows 2$, $\theta = e_\theta^\ddagger \mathbb{T}_{2^\theta}$; pictorially,

$$\begin{tikzcd}
\theta \arrow[d, squiggly, no head] & \mathbb{T}_{2^\theta} \arrow[l, "e_\theta^\ddagger"', mapsto] \arrow[d, squiggly, no head] \\
X \arrow[r, "e_\theta"'] & 2^\theta
\end{tikzcd}$$

□

Then, by the fibredness condition (4), for any $\theta : X \rightrightarrows 2$ we have

$$T(B)_X\theta = T(B)_X(e_\theta^\ddagger \mathbb{T}_{2^\theta}) = (Be_\theta)^\ddagger T(B)_{2^\theta}\mathbb{T}_{2^\theta}$$

The least (wrt. the fibre ordering) possible value for $T(B)_{2^\theta}\mathbb{T}_{2^\theta}$ is $\mathbb{T}_{B(2^\theta)}$, hence the least candidate for a fibred lifting $T(B)_X$ of B is defined by

$$T(B)_X\theta = (Be_\theta)^\ddagger \mathbb{T}_{B(2^\theta)} = \left\{ W \circ Be_\theta \ : \ W : B(2^\theta) \to 2 \right\} \tag{6}$$

Theorem 10. For any functor B on **Set**, the above action $T(B)_X$ defines a fibred functor on **TS** as in (3).

Proof. For any X, the action $T(B)_X$ is clearly monotonic, therefore the only condition to check is that for any $f : X \to Y$ and any $\theta : Y \rightrightarrows 2$ the equality

$$(Bf)^\ddagger T(B)_Y \theta = T(B)_X(f^\ddagger \theta) \tag{7}$$

holds. To prove this, we begin with two easy lemmas, assuming arbitrary $f : X \to Y$ and $\theta : Y \rightrightarrows 2$.

Lemma 11. For any $Z : 2^{f^\ddagger \theta} \to 2$ there exists $W : 2^\theta \to 2$ such that

$$Z = W \circ 2^{(-\circ f)}$$

Proof. The function $2^{(-\circ f)} : 2^{f^\ddagger \theta} \to 2^\theta$ is always a mono, so (since its domain is nonempty) it is a section. Take $W = Z \circ u$, where $u : 2^\theta \to 2^{f^\ddagger \theta}$ is the corresponding retraction. □

Lemma 12. The following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "e_{f^\ddagger\theta}"] \arrow[d, "f"'] & 2^{f^\ddagger\theta} \arrow[d, "2^{(-\circ f)}"] \\ Y \arrow[r, "e_\theta"'] & 2^\theta \end{tikzcd}$$

Proof. Calculate, for any $x \in X$ and $V \in \theta$,

$$\begin{aligned} (2^{(-\circ f)}(e_{f^\ddagger\theta}(x)))(V) &= ((e_{f^\ddagger\theta}(x)) \circ (- \circ f))(V) \\ &= e_{f^\ddagger\theta}(x)(V \circ f) \\ &= (V \circ f)(x) = V(f(x)) = e_\theta(f(x))(V) \end{aligned}$$

□

We are now ready to prove (7). Calculate

$$\begin{aligned} (Bf)^\ddagger T(B)_Y \theta &= \left\{ W \circ Be_\theta \circ Bf \ : \ W : B(2^\theta) \to 2 \right\} \\ &= \left\{ W \circ B(2^{(-\circ f)}) \circ Be_{f^\ddagger\theta} \ : \ W : B(2^\theta) \to 2 \right\} \\ &= \left\{ Z \circ Be_{f^\ddagger\theta} \ : \ Z : B(2^{f^\ddagger\theta}) \to 2 \right\} = T(B)_X(f^\ddagger\theta) \end{aligned}$$

using two lemmas above. □

Note that the characterization of $T(B)$ does not require B to preserve monos. From properties of $L(B)$ and $T(B)$ it follows that:

Corollary 13. If B preserves monos then $T(B)$, restricted to **BTS**, coincides with $L(B)$ along the isomorphisms between **BTS** and **ERel**.

Moreover, the properties of $L(B)$ as stated in Propositions 5 and 7 translate to analogous properties of $T(B)$. Two of these properties (corresponding to properties (iv) and (v) from Proposition 7) will be useful in Section 6, so we restate them here, and we provide independent proofs.

In the following theorem, given any $h : X \to BX$, elements $x, y \in X$ are called *bisimilar* if there exists a $T(B)$-coalgebra $h : \langle X, \theta \rangle \to \langle BX, T(B)_X \theta \rangle$ such that x, y are not distinguishable by tests from θ, which is then called a bisimulation suite.

Theorem 14. If $f : X \to Y$ is a B-coalgebra homomorphism from g to h then x is bisimilar to x' in X if and only if fx is bisimilar to fx' in Y.

Proof (sketch). For any bisimulation suite θ on h such that $V(fx) = V(fx')$ for all $V \in \theta$, $f^{\ddagger}\theta$ is a bisimulation suite on g. Moreover, $Vx = Vx'$ for all $V \in f^{\ddagger}\theta$. Similarly, for any bisimulation suite θ on g such that $Vx = Vx'$ for all $V \in \theta$,

$$f_{\ddagger} = \{\, V : Y \to 2 \,:\, V \circ f \in \theta \,\}$$

is a bisimulation suite on h, and $V(fx) = V(fx')$ for all $V \in f_{\ddagger}\theta$. $\square$

Theorem 15. (Expressivity) Assume $B : \mathbf{Set} \to \mathbf{Set}$ is ω-continuous and preserves monos. Then a final $T(B)$-coalgebra

$$\phi : \langle Z, \zeta \rangle \to \langle BZ, T(B)_Z \zeta \rangle$$

exists, and the test suite $\zeta : Z \rightrightarrows 2$ is jointly monic, i.e., any two elements of Z are distinguished by a test from ζ.

Proof. Since B is ω-continuous, a final B-coalgebra

$$\phi : Z \to BZ$$

exists and is obtained as in (1). Since $T(B)$ is fibred, by Proposition 4 a final $T(B)$-coalgebra as above exists and

$$\zeta = \bigcup_{n \in \mathbb{N}} p_n^{\ddagger}(T(B)_1^n \emptyset)$$

where $\emptyset : 1 \rightrightarrows 2$ is the final test suite, i.e., the empty test suite on 1. Pictorially:

$$\begin{tikzcd}
 & & Z \arrow[dll, "p_0"'] \arrow[dl, "p_1"] \arrow[d, "p_2"] \arrow[dr, "p_3"] \arrow[drr, dotted, no head] & & \\
1 \arrow[d, shift left] \arrow[d, shift right, "\emptyset"'] & B1 \arrow[l, "!"'] \arrow[d, shift left, "T(B)_1\emptyset"] \arrow[d, shift right] & B^2 1 \arrow[l, "B!"'] \arrow[d, shift left, "T(B)_1^2\emptyset"] \arrow[d, shift right] & B^3 1 \arrow[l, "B^2!"'] \arrow[d, shift left] \arrow[d, shift right] \arrow[r, dotted, no head] & {} \\
2 & 2 & 2 & 2 \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

The family of functions $\{\, p_n : Z \to B^n 1 \,:\, n \in \mathbb{N} \,\}$, being a limiting cone, is jointly monic. Since compositions of jointly monic families is always jointly monic, it is enough to ensure that all test suites $T(B)_1^n \emptyset$ are jointly monic. The empty test suite on 1 is jointly monic, so it is enough to show $T(B)$ preserves joint monicity of test suites: whenever $\theta : X \rightrightarrows 2$ is jointly monic, then $T(B)_X \theta : BX \rightrightarrows 2$ is jointly monic.

To show that, recall that

$$T(B)_X\theta = \{ W \circ Be_\theta : W : B(2^\theta) \to 2 \}$$

First, if θ is jointly monic then e_θ is a mono. Indeed, assume any $x, y \in X$ such that $e_\theta(x) = e_\theta(y)$. This, by definition of e_θ, means that for all $V \in \theta$, $Vx = Vy$. Since θ is jointly monic, $x = y$.

Since B preserves monos, also Be_θ is a mono. Moreover, the family of all functions from $B(2^\theta)$ to 2 is jointly monic, as 2 is a cogenerator in **Set**. Precomposing a jointly monic family with a mono yields a jointly monic family, therefore $T(B)_X\theta$ is jointly monic. This completes the proof. □

Results and examples of Section 6 should give some intuition on how the final $T(B)$-coalgebra looks in concrete cases.

6 Application to Coalgebraic Modal Logic

We proceed to show how the least fibred lifting $T(B)$ of an ω-continuous functor B to **TS** gives rise to an expressive modal logic for B-coalgebras.

Given a functor $B : \mathbf{Set} \to \mathbf{Set}$, define the set $\mathcal{L}$ of *formulae* inductively: for any $k \in \mathbb{N}$, and for any function $U : B(2^k) \to 2$, if $\phi_1, \ldots, \phi_k \in \mathcal{L}$ then

$$[U](\phi_1, \ldots, \phi_k) \in \mathcal{L}$$

The interpretation of $\mathcal{L}$ in a coalgebra $h : X \to BX$ is a function

$$[\![-]\!] : \mathcal{L} \to (X \to 2)$$

defined inductively by

$$[\![[U](\phi_1, \ldots, \phi_k)]\!] = U \circ B([\![\phi_1]\!] \times \cdots \times [\![\phi_k]\!]) \circ B\Delta_k \circ h$$

where $\Delta_k : X \to X^k$ is the diagonal map, defined by $\Delta_k(i)(x) = x$.

The relation of this logic to the lifting $T(B)$ from Section 5 is made apparent in the following theorem:

Theorem 16. If B is ω-continuous and preserves monos, then for any coalgebra $h : X \to BX$ with the coinductive extension $f : X \to Z$,

$$\{ [\![\phi]\!] : \phi \in \mathcal{L} \} = f^\ddagger\zeta$$

where $\zeta : Z \rightrightarrows 2$ comes from the final $T(B)$-coalgebra as in Proposition 4.

Proof. (Sketch). To prove the left-to right inclusion, proceed by structural induction on $\mathcal{L}$. Consider any $\phi = [U](\phi_1, \ldots, \phi_k)$, for some $U : B(2^k) \to 2$. By the inductive assumption, $V_1, \ldots, V_k \in f^\ddagger\zeta$, hence there exists a function $\rho : k \to f^\ddagger\zeta$. Define $W : B(2^{f^\ddagger\zeta}) \to 2$ by

$$W = U \circ B(2^\rho)$$

Note that for any $x \in X$, $i \leq k$,

$$(2^\rho(e_{f^\ddagger\zeta}(x))(i) = e_{f^\ddagger\zeta}(x)(\rho(i)) = V_i x$$

therefore

$$2^\rho \circ e_{f^\ddagger\zeta} = (V_1 \times \cdots \times V_k) \circ \Delta_k$$

Then calculate

$$\begin{aligned}
[\![\phi]\!] &= U \circ B(V_1 \times \cdots \times V_k) \circ B\Delta_k \circ h = \\
&= U \circ B2^\rho \circ Be_{f^\ddagger\zeta} \circ h = \\
&= W \circ Be_{f^\ddagger\zeta} \circ h \in h^\ddagger T(B)_X(f^\ddagger\zeta) \subseteq f^\ddagger\zeta
\end{aligned}$$

To prove the right-to-left inclusion, define $\theta_n = f^\ddagger(p_n^\ddagger(T(B)_1^n\emptyset))$ (see the proof of Theorem 15). Then $f^\dagger\zeta = \bigcup_{n\in\mathbb{N}} \theta_n$. One shows by induction on n that every test from θ_n is an interpretation of a formula from $\mathcal{L}$. To this end, prove from fibredness of $T(B)$ that

$$\theta_n = h^\ddagger(T(B)_X(\theta_{n-1}))$$

therefore each test from θ_n is of the form

$$V = W \circ B(e_{\theta_{n-1}}) \circ h$$

for some $W : B(2^{\theta_{n-1}}) \to 2$. One proves by induction that all θ_n are finite. Take $k = |\theta_{n-1}|$, fix a bijection $\sigma : k \to \theta_{n-1}$, and define

$$U = W \circ B(2^\sigma)$$

Consider the formula

$$\phi = [U](\phi_1, \ldots, \phi_k)$$

such that $[\![\phi_i]\!] = \sigma(i)$ (such ϕ_i exist by the inductive assumption). Then calculate, similarly as above,

$$\begin{aligned}
[\![\phi]\!] &= U \circ B(\sigma(1) \times \cdots \times \sigma(k)) \circ B\Delta_k \circ h = \\
&= W \circ B2^\sigma \circ B(\sigma(1) \times \cdots \times \sigma(k)) \circ B\Delta_k \circ h = \\
&= W \circ Be_{\theta_{n-1}} \circ h = V
\end{aligned}$$

□

Corollary 17. If B is ω-continuous and preserves monos, the logic $\mathcal{L}$ is adequate and expressive.

Proof. Combine Theorems 14, 15 and 16. □

The logic $\mathcal{L}$ and its interpretation is essentially the expressive logic of polyadic predicate liftings as defined in [17], so the above expressivity result should come as no surprise. The purpose of this section was to show a strong link of the logic to the least fibred lifting of B, thus providing further insight into the canonicity and importance of results from [17].

Example 18. Consider $B = \mathcal{P}_f(A \times X)$, where A is a fixed set of labels. Coalgebras for B are finitely branching labelled transition systems. The finitary Hennessy-Milner logic, adequate and expressive for such coalgebras, can be equivalently defined inductively as follows: for any logical operator $b : 2^k \to 2$, and for any labels $a_1, \dots, a_k \in A$ and for any formulae $\phi_1, \dots, \phi_k$,

$$b[\langle a_1 \rangle \phi_1, \dots, \langle a_k \rangle \phi_k]$$

is a formula. The interpretation of formulae in a coalgebra $h : X \to BX$ is a function from formulae to $X \to 2$ defined inductively:

$$[\![b[\langle a_1 \rangle \phi_1, \dots, \langle a_k \rangle \phi_k]]\!](x) = b(v_1, \dots, v_k)$$

where $v_i = \mathtt{tt}$ iff $\langle a_i, y \rangle \in hx$ such that $[\![\phi_i]\!] y = \mathtt{tt}$. An interpretation-preserving translation between this and the usual representation of Hennessy-Milner logic is straightforward. The translation is not bijective; for example, Hennessy-Milner formulae $\mathtt{tt} \wedge \langle a \rangle (\mathtt{tt} \vee \langle a \rangle \mathtt{tt})$ and $\langle a \rangle \mathtt{tt}$ are mapped to the same formula. However, all identified formulae are logically equivalent.

This version of Hennessy-Milner logic can be translated bijectively to the logic $\mathcal{L}$ defined above. The translation, denoted γ, is defined by induction:

$$\gamma(b[\langle a_1 \rangle \phi_1, \dots, \langle a_k \rangle \phi_k]) = [U](\gamma(\phi_1), \dots, \gamma(\phi_k))$$

where $U : \mathcal{P}_f(A \times 2^k) \to 2$ is defined by

$$U(\beta) = b(u_1, \dots, u_k)$$

where $u_i = \mathtt{tt}$ iff $\langle a_i, \mathtt{tt} \rangle \in \beta$. It is straightforward to check that this translation is bijective and preserves interpretation of formulae: $[\![\phi]\!] = [\![\gamma(\phi)]\!]$.

Example 19. In [17], Schröder proves that if, for a functor B, the family of functions

$$(Bf : BX \to B2)_{f : X \to 2}$$

is jointly monic for any set X, then B admits an expressive modal logic with unary modalities. He also shows a few functors for which this condition fails. Here is another example, maybe simpler and more natural: the functor $BX = \mathcal{P}_f(A \times X \times X)$, for A a nonempty fixed set of labels. Indeed, for $X = \{x, y, z\}$, the set

$$\{\langle a, x, x \rangle, \langle a, x, y \rangle, \langle a, y, z \rangle, \langle a, x, z \rangle\}$$

is identified with

$$\{\langle a, x, x \rangle, \langle a, x, y \rangle, \langle a, y, z \rangle\}$$

under Bf for any $f : X \to 2$. This means that the expressive logic $\mathcal{L}$ for this functor makes essential use of modalities of multiple arity. Proceeding as in Example 18 above, one shows a correspondence between $\mathcal{L}$ and a logic similar to Hennessy-Milner logic and defined by:

$$\phi ::= \mathtt{tt} \mid \phi \wedge \phi \mid \neg \phi \mid \langle a \rangle (\phi, \phi)$$

(where a ranges over A), with an interpretation (for a given $h : X \to BX$) as for Hennessy-Milner logic, except:

$$[\![\langle a \rangle (\phi_1, \phi_2)]\!](\beta) = \texttt{tt} \text{ iff } \langle a, x, y \rangle \in \beta \text{ s.t. } [\![\phi_1]\!]x = \texttt{tt}, [\![\phi_2]\!]x = \texttt{tt}$$

This logic has actually been used in the literature, to describe a simple calculus with process passing [18]. This example shows that the extension of coalgebraic modal logic to modalities of arbitrary arity, as done first in [17], is of practical importance. Note, however, that allowing multiple sorts of logical formulae one can define an expressive logic for this functor with only unary modalities, along the lines of [2].

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