Behavioural differential equations: an introduction

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Part the NWO project 'CoRE' (Coinductive Calculi of Regular Expressions)

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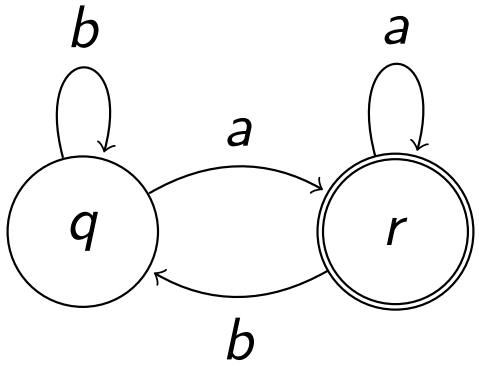
3. ...and if time allows even some of my own research.
4. (Almost) no category theory.

A fresh look at automata

An *automaton* is a triple (Q, o, δ)



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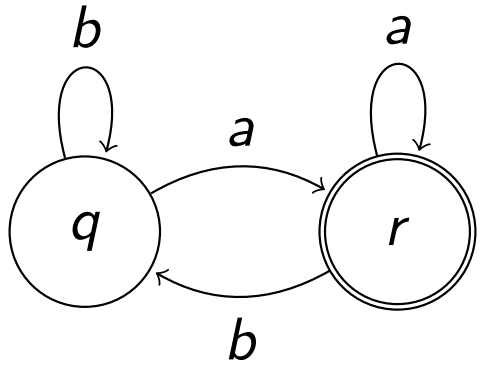


$$Q = \{q, r\}$$

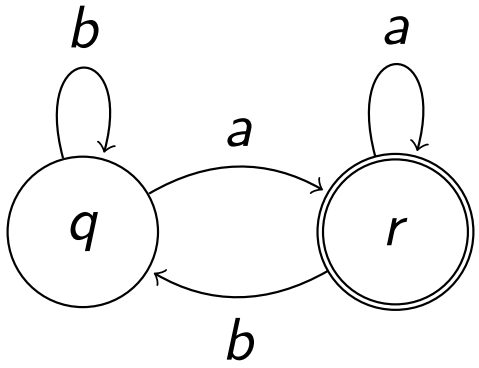
We represent o and δ with a system of *behavioural differential equations*:

$$\begin{array}{lllllllll} o(q) & = & 0 & \quad q_a & = & r & \quad q_b & = & q \\ o(r) & = & 1 & \quad r_a & = & r & \quad r_b & = & q \end{array}$$

Derivatives...

- q_a: shorthand for $\delta(q, a)$.
- Called the *a-derivative* of q...
- ...has some properties similar to the familiar derivative from calculus!

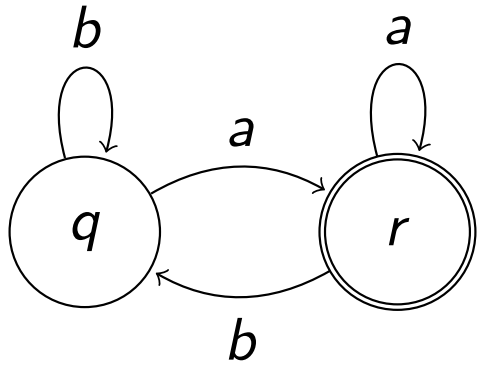
Word derivatives



We can extend the notion of derivatives from alphabet symbols to words inductively (1 denotes the empty word):

$$\begin{aligned} q_1 &= q \\ q_{a \cdot w} &= (q_a)_w \end{aligned}$$

Word derivatives



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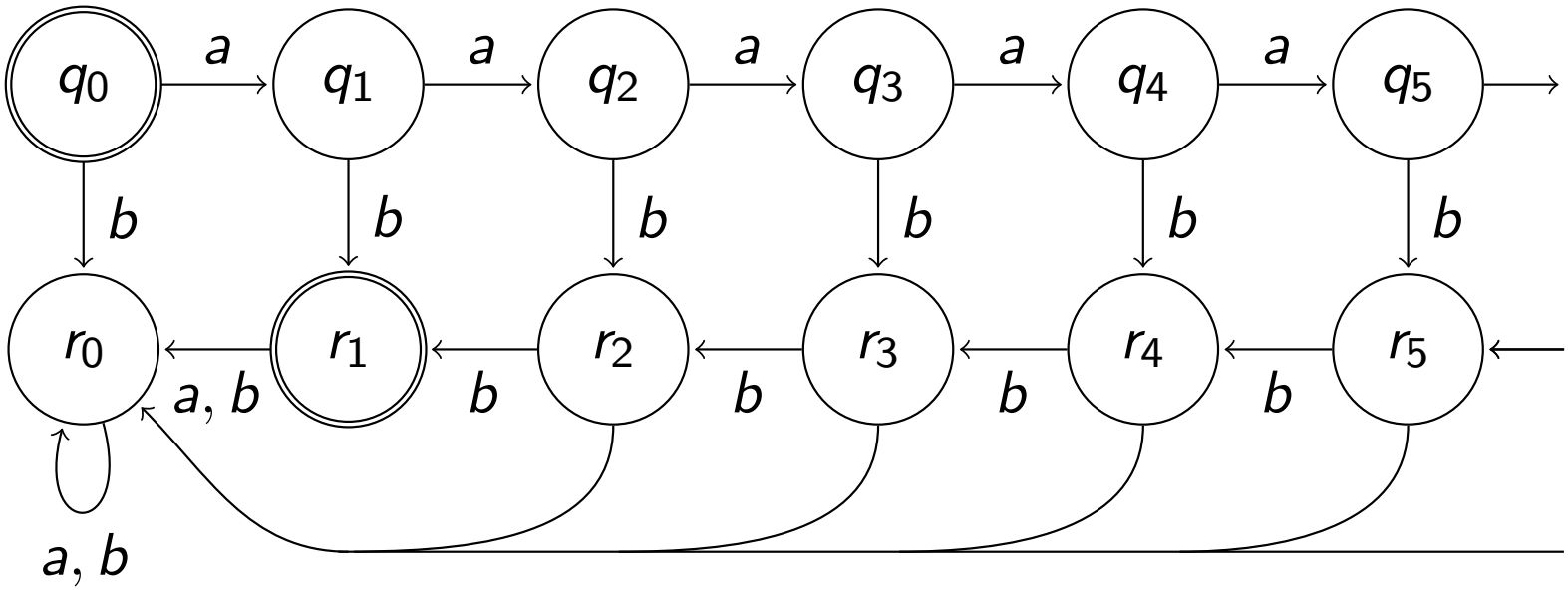
In the above automaton, we have e.g.

$$q_{abb} = q$$

An infinite automaton...



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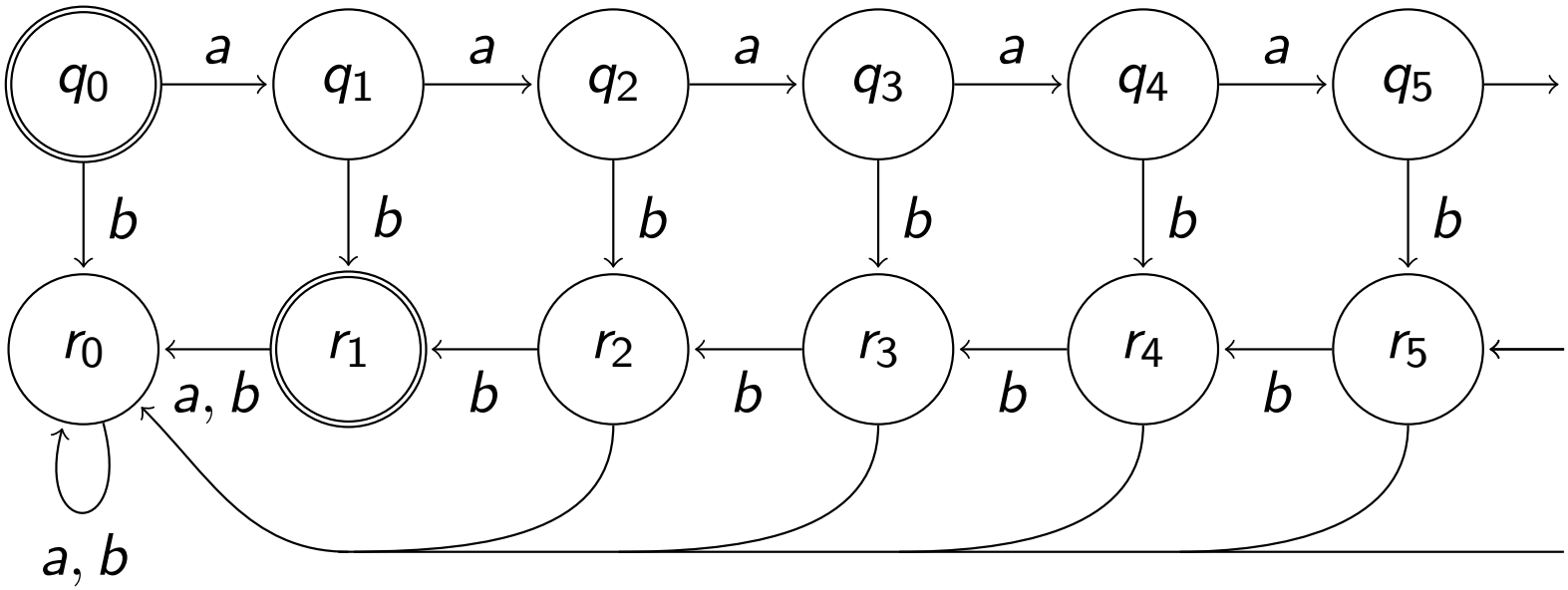

Corresponding system of b.d.e.'s:

$$
\begin{array}{rclcrclcrcl}
o(q_i) & = & 0 & & (q_i)_a & = & q_{i+1} & & (q_i)_b & = & r_i \\
o(r_0) & = & 0 & & (r_0)_a & = & r_0 & & (r_0)_b & = & r_0 \\
o(r_1) & = & 1 & & (r_1)_a & = & r_0 & & (r_1)_b & = & r_0 \\
o(r_{k+2}) & = & 0 & & (r_{k+2})_a & = & r_0 & & (r_{k+2})_b & = & r_{k+1}
\end{array}
$$

A very large automaton

Consider the automaton $(\mathcal{L}, o, \delta)$:

- $\mathcal{L} = \mathcal{P}(\mathcal{P}(A^*))$, the set of all languages over A
- $o(L)$ is 1 if $1 \in L$, and 0 otherwise
- $L_a = \{w \mid aw \in L\}$ for all alphabet symbols a

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Proposition

For every automaton (Q, o', δ'), there is exactly one mapping $[\![\ ]\!] : Q \to \mathcal{L}$, such that

1. $o'(q) = o([\![q]\!])$ *for all states q, and*
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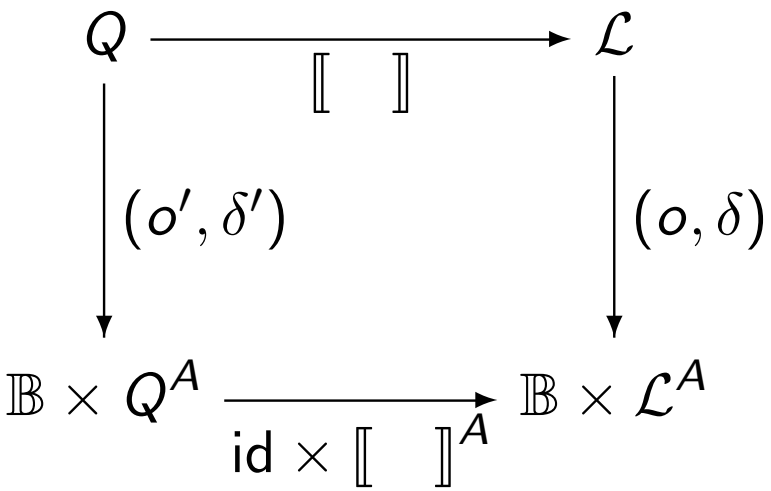
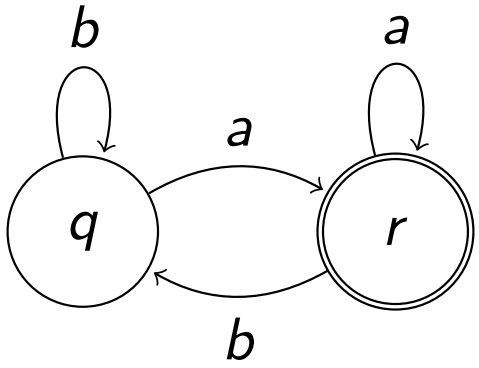
In other words: $\mathcal{L}$ is a *final automaton* or *final coalgebra*.

This is not a categorical diagram (1)



The final homomorphism



$[\![\ ]\!]$ assigns to states the language accepted by those states:

$$\begin{aligned} [\![q]\!] &= \{w \mid w \text{ ends in } a\} \\ [\![r]\!] &= \{w \mid w \text{ ends in } a \text{ or is empty}\} \end{aligned}$$

Bisimulation (1)

A bisimulation between two automata with state spaces Q_0 and $Q_1 \ldots$

Bisimulation (1)

A bisimulation between two automata with state spaces Q_0 and Q_1...

...is a relation $R \subset Q_0 \times Q_1$ such that, whenever $(q, r) \in R$:

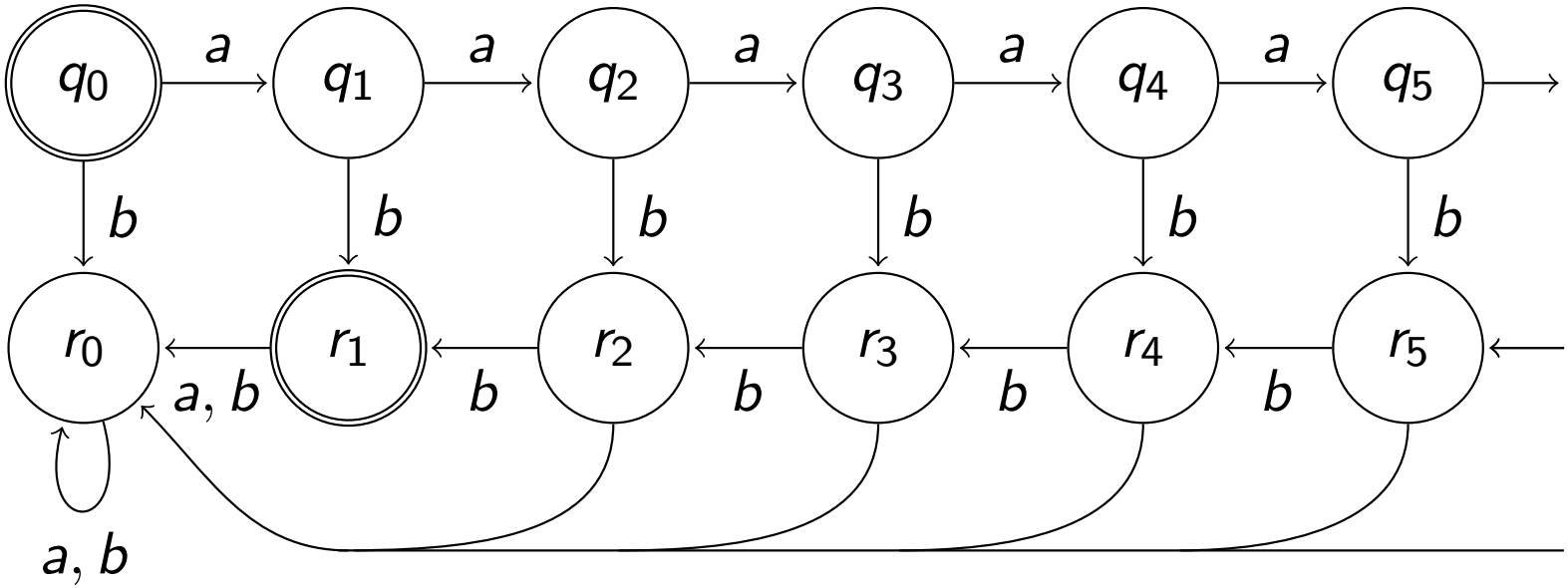
1. $o(q) = o(r)$, and ...
2. $(q_a, r_a) \in R$ for all alphabet symbols a.

Bisimulation (2)

Fact

If $(q, r) \in R$ *for some bisimulation* R*, then* $[\![q]\!] = [\![r]\!]$.

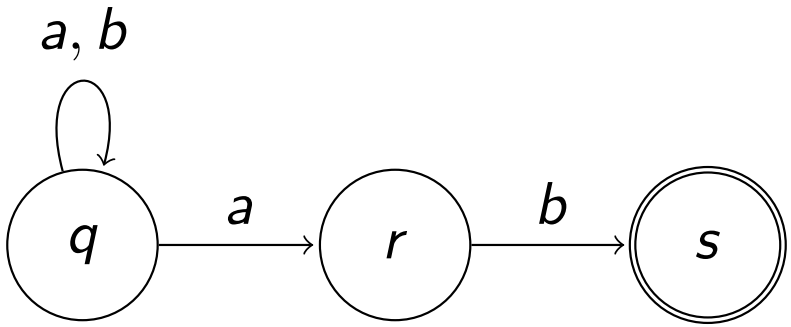
Bisimulation (3)



$$R = \{(q_i, a^n b^{n+i}) \mid i \in \mathbb{N}\} \cup \{(r_0, 0), (r_1, 1)\} \cup \{(r_{i+2}, b^{i+1}) \mid i \in \mathbb{N}\}$$

is a bisimulation. Hence, $[\![q_0]\!] = a^n b^n$, etc. . .

A nondeterministic automaton



$$\begin{array}{lllllllll} o(q) & = & 0 & & q_a & = & q+r & & q_b & = & q \\ o(r) & = & 0 & & r_a & = & 0 & & r_b & = & s \\ o(s) & = & 1 & & s_a & = & 0 & & s_b & = & 0 \end{array}$$

Right hand sides of the equations: sums of states.

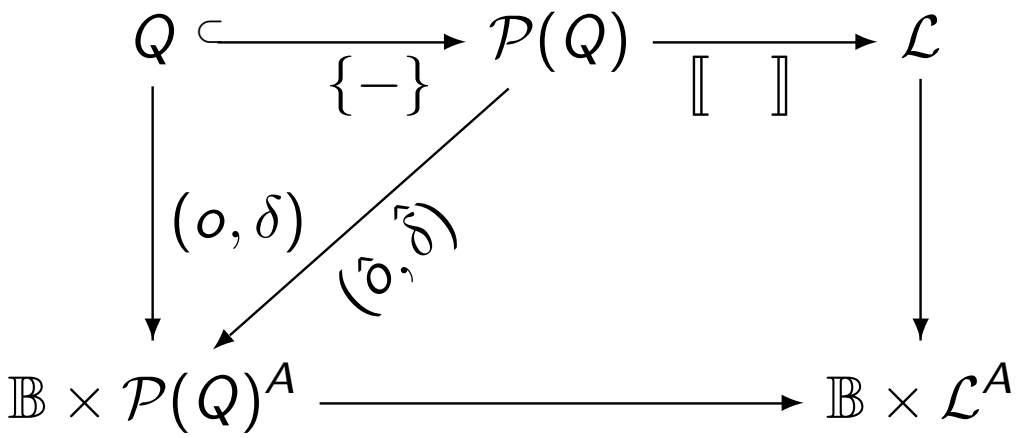
Determinization

We can transform a nondeterministic automaton (Q, o, δ) into a deterministic automaton $(\mathcal{P}(Q), \hat{o}, \hat{\delta})$, by enforcing the rules

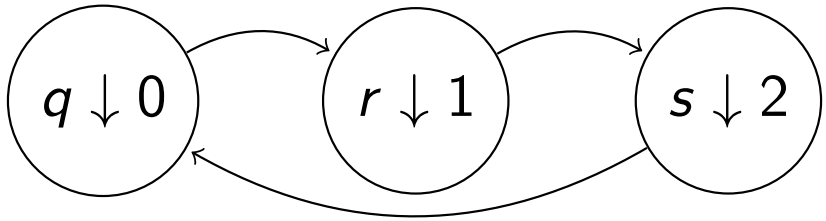
$$\begin{array}{rcl@{\qquad}rcl} \hat{o}(0) & = & 0 & 0_a & = & 0 \\ \hat{o}(s+t) & = & \max(o(s), o(t)) & (s+t)_a & = & s_a + t_a \end{array}$$

(Representing elements $\{q_1, \ldots q_n\} \in \mathcal{P}(Q)$ as sums $\sum_{1 \leq i \leq n} q_i$).

This is not a categorical diagram (2)



A simple stream automaton

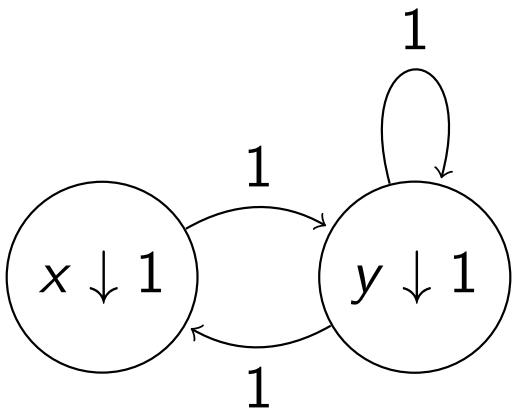


$$\begin{array}{lllllllll} o(q) & = & 0 & q' & = & r & [\![q]\!] & = & (0,1,2,0,1,2,\ldots) \\ o(r) & = & 1 & r' & = & s & [\![r]\!] & = & (1,2,0,1,2,0,\ldots) \\ o(s) & = & 2 & s' & = & q & [\![s]\!] & = & (2,0,1,2,0,1,\ldots) \end{array}$$

(Finite stream automata are not very spectacular)

A weighted automaton (1)

A weighted automaton...



...and the corresponding system of equations ...

$$o(x) = 1 \qquad x' = y$$
$$o(y) = 1 \quad y' = y + x$$

Right hand side: linear combination of states

Determinization (2)

We can transform a weighted automaton (Q, o, δ) into a stream automaton $(\mathrm{span}(Q), \hat{o}, \hat{\delta})$, by enforcing the rule

$$\hat{o}\left(\sum_{0\leq i<k} r_i q_i\right) = \sum_{0\leq i<k} r_i o(q_i) \qquad \left(\sum_{0\leq i<k} r_i q_i\right)' = \sum_{0\leq i<k} r_i q_i'$$

for all linear combinations of elements $q \in Q$.

The Fibonacci sequence

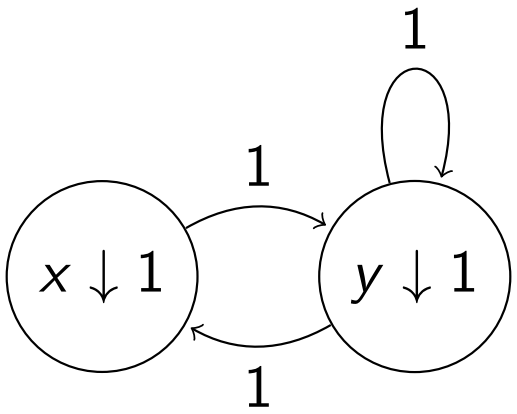
$$\mathbf{fib} = (1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots)$$

Inductive definition:

$$\mathbf{fib}(0) = \mathbf{fib}(1) = 1 \qquad \mathbf{fib}(n+2) = \mathbf{fib}(n+1) + \mathbf{fib}(n)$$

A weighted automaton (2)

A weighted automaton...



...and the corresponding system of equations ...

$$o(x) = 1 \qquad x' = y$$
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Proposition

$[\![x]\!] = \mathbf{fib}$

A proof

The relation

$$R = \{(\mathbf{fib}, x), (\mathbf{fib}', y)\} \cup \{(\mathbf{fib}^{(n+2)}, \mathbf{fib}(n)x + \mathbf{fib}(n+1)y) \mid n \in \mathbb{N}\}$$

is a bisimulation w.r.t. the function **fib** and the system

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E.g.:

$$\begin{aligned} \mathbf{fib}^{(n+2)\prime} &= \mathbf{fib}^{(n+3)} \\ &\overset{R}{} \quad \mathbf{fib}(n+1)x + \mathbf{fib}(n+2)y \\ &= \mathbf{fib}(n+1)x + (\mathbf{fib}(n+1) + \mathbf{fib}(n))y \\ &= (\mathbf{fib}(n)x + \mathbf{fib}(n+1)y)' \end{aligned}$$

More complex systems

We now look at systems where the right hand side of any derivative can be a polynomial over the set X of variables, e.g.

$$x^3y^3 + xy^2 + y$$

We determinize using the product rule

$$o(st) = o(s)o(t) \qquad (st)_a = s_a t + o(s)t_a$$

(in addition to the earlier linearity rules)
Resulting class of languages: the context-free languages!

A context-free system

An example of such a system:

$$\begin{array}{llllll} o(x) = 1 & x_a = xy & x_b = 0 \\ o(y) = 0 & y_a = 0 & y_b = 1 \end{array}$$

A context-free system

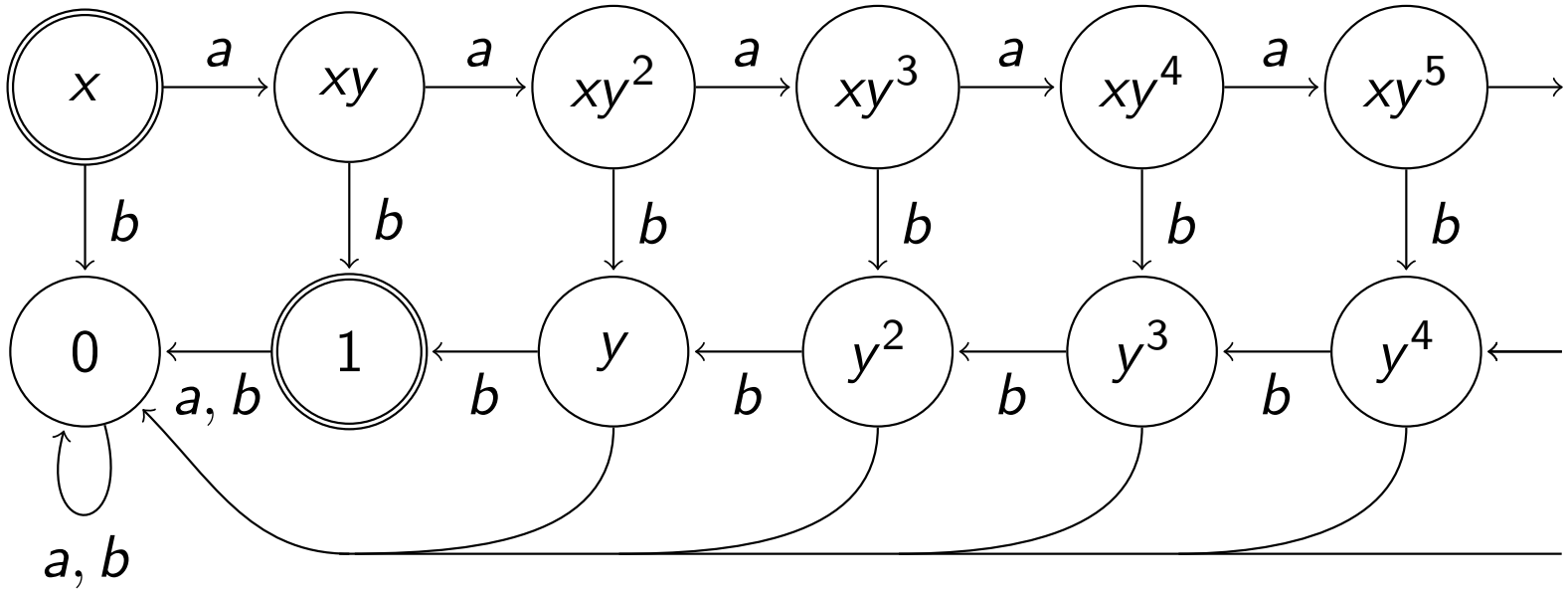
An example of such a system:

$$\begin{array}{lllllllll} o(x) & = & 1 & \quad x_a & = & xy & \quad x_b & = & 0 \\ o(y) & = & 0 & \quad y_a & = & 0 & \quad y_b & = & 1 \end{array}$$

We get

$$[\![x]\!] = a^n b^n$$

Determinization of a context-free system



Classes of b.d.e., their languages and streams

rhs	Class of languages	Class of streams
Single element	Regular	Finitary
Linear combination	(Nondeterministic) regular	Rational
Polynomial	Context-free	Algebraic

Conclusions

- Bisimulation is a very nice coinductive proof technique!
- Behavioural differential equations are an effective way of defining automata-like structures.
- Casting a new light on the existing theory of (weighted) automata.