A final coalgebra for the k-regular and k-automatic sequences

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Introduction

- *k*-automatic and *k*-regular sequences: classes defined by Allouche/Shallit
- A sequence $\sigma \in \mathbb{Z}^\omega$ is *k*-automatic if generated by a deterministic automaton with output in $\{0, \ldots, k-1\}$...
- ... where $\sigma(n)$ is output after reading *n* in base *k*.

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- ... where $\sigma(n)$ is output after reading n in base k.
- *k*-regular sequences generalize this:

$$\frac{\text{k-regular}}{\text{k-automatic}} = \frac{\text{weighted automata}}{\text{deterministic automata}}$$

- This talk: connecting *k*-regular sequences to (abstract) coalgebra and (concrete) behavioural differential equations.

k-regular sequences: a definition (for $k = 2$)

We call a sequence (or stream) σ 2-regular when there is a finite family of sequences

$$\Sigma = (\sigma_i) \quad i \leq n \in \mathbb{N}$$

with $\sigma_0 = \sigma$, s.t. for all $i \leq n$ the sequences $\mathbf{even}(\sigma_i)$ and $\mathbf{odd}(\sigma_i)$ are linear combinations of sequences from Σ.
Here **even** and **odd** are defined by

$$\mathbf{even}(\tau)(n) = \tau(2n)$$

and

$$\mathbf{odd}(\tau)(n) = \tau(2n + 1)$$

Derivative and **zip**

We will reason with the stream derivative from the coinductive stream calculus. Definition:

$$\sigma'(n) = \sigma(n+1)$$

We can define streams and operators coinductively by giving the first element and the derivative, e.g.

$$\begin{aligned} \mathbf{zip}(\sigma,\tau)(0) &= \sigma(0) \\ \mathbf{zip}(\sigma,\tau)' &= \mathbf{zip}(\tau,\sigma') \end{aligned}$$

gives

$$\begin{aligned} \mathbf{zip}(\sigma,\tau)(2k) &= \sigma(k) \\ \mathbf{zip}(\sigma,\tau)(2k+1) &= \tau(k) \end{aligned}$$

and thus

$$\mathbf{zip}(\mathbf{even}(\sigma),\mathbf{odd}(\sigma)) = \sigma$$

Systems of **zip**-equations

k-regular sequences can be seen as solutions to finite systems of equations.

$$\begin{aligned} \tau_1 &= \mathbf{zip}(\tau_1^e, \tau_1^o) \\ &\vdots \\ \tau_n &= \mathbf{zip}(\tau_n^e, \tau_n^o) \end{aligned}$$

Example: the sequence of numbers whose base 3 representation does not contain the digit '2'

$$0, 1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, \ldots$$

is a solution to

$$\begin{aligned} \sigma &= \mathbf{zip}(3\sigma, 3\sigma + \mathbf{ones}) \\ \mathbf{ones} &= \mathbf{zip}(\mathbf{ones}, \mathbf{ones}) \end{aligned}$$

(with $\mathbf{ones}(0) = 1$, $\sigma(0) = 0$)

Automata as coalgebras

- Automaton (with output in S, input in A) is coalgebra for the functor $S \times -^A$.
- Semantics $[\![-]\!]$ given by unique morphism into final automaton:

$$\begin{array}{ccc} X & \overset{\exists! [\![-]\!]}{\dashrightarrow} & S^{A^*} \\ {\scriptstyle (o,\delta)}\downarrow & & \downarrow{\scriptstyle (O,\Delta)} \\ S \times X^A & \underset{1_S \times [\![-]\!]}{\dashrightarrow} & S \times (S^{A^*})^A \end{array}$$

Fact: $[\![x]\!](w) = o(x_w)$

Streams are an instance of this

If $|A| = 1$, note that $S^{A^*} \cong S^{\mathbb{N}}$ and we get

$$\begin{array}{ccc} X & \overset{\exists! [\![-]\!]}{\dashrightarrow} & S^{\mathbb{N}} \\ \downarrow {\scriptstyle (o,\delta)} & & \downarrow {\scriptstyle (O,\Delta)} \\ S \times X & \dashrightarrow & S \times (S^{\mathbb{N}}) \end{array}$$

$$\begin{array}{rcl} O(\sigma) & = & \sigma(0) \\ \Delta(\sigma) & = & \sigma' \end{array}$$

Main result (for case $k = 2$)

Theorem

A sequence σ is 2-regular if and only if it is the unique solution to a system of stream differential equations

$$o(x) = k \qquad x' = \mathbf{zip}(x_e, x_o)$$

for a finite set X, where $k \in \mathbb{Z}$, and for each $x \in X$, x_e and x_o are given as a linear combination of elements from X.

(also found by Endrullis/Moss/Silva)

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Idea: transform flat systems into guarded systems.

or: move from standard base k numeration to bijective base k numeration

Construct a system of stream differential equations from the earlier system:

$$
\begin{aligned}
\sigma' &= \mathbf{zip}(3\sigma + \mathbf{ones}, 3\sigma') \\
\sigma'' &= \mathbf{zip}(3\sigma', 3\sigma' + \mathbf{ones}') \\
\mathbf{ones}' &= \mathbf{zip}(\mathbf{ones}', \mathbf{ones}) \\
\mathbf{ones}'' &= \mathbf{zip}(\mathbf{ones}', \mathbf{ones}')
\end{aligned}
$$

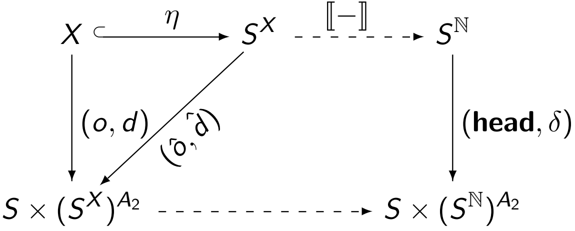
or

$$
\begin{aligned}
w' &= \mathbf{zip}(3w + y, 3x) \\
x' &= \mathbf{zip}(3x, 3x + z) \\
y' &= \mathbf{zip}(y, z) \\
z' &= \mathbf{zip}(z, z)
\end{aligned}
$$

Add output values to specification and you're done!

A final coalgebra diagram

Semantics can be given by the following diagram (initiality + finality):



with

$$\begin{aligned} \delta(\sigma)(1) &= \mathbf{even}(\sigma') \\ \delta(\sigma)(2) &= \mathbf{odd}(\sigma') \end{aligned}$$

An isomorphism of final coalgebras

$$\begin{array}{ccc} S^{\mathbb{N}} & \overset{\cong}{=\!=\!=\!=} & S\langle\!\langle A_2\rangle\!\rangle \\ (\mathbf{head}, \delta)\downarrow & & \downarrow(O, \Delta) \\ S \times (S^{\mathbb{N}})^{A_2} & = & S \times S\langle\!\langle A_2\rangle\!\rangle^{A} \end{array}$$

Can be proven using the bijective base k numeration between $\mathbb{N}$ and $(A_k)^*$.

Gives correspondence with weighted automata (over any semiring S).

Application: divide and conquer recurrences

On the Online Encyclopedia of Integer Sequences, some formats for divide and conquer recurrences are given. E.g.

$$\begin{aligned} a(2n) &= Ca(n) + Ca(n-1) + P(n) \\ a(2n+1) &= 2Ca(n) + Q(n) \end{aligned}$$

where P and Q are expressible by a rational g.f.

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Q (asked on `oeis.org/somedcgf.html`): 'An open question would be whether all sequences here discussed are 2-regular.'

A: if you replace the condition 'expressible by a rational g.f.' by '2-regular' yes (includes all their examples), otherwise no.

Generalizations, conclusions and future work

- Everything told here about 2 works for any $k \geq 2$.
- We established a correspondence between rational power series in k (noncomm.) variables and k-regular sequences over arbitrary semirings.
- . . . allowing us to translate back and forth between recurrences and systems of stream differential equations.

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- ...allowing us to translate back and forth between recurrences and systems of stream differential equations.
- Future work: how about k-algebraic sequences?
- ...further investigate the connections with recurrences.