A Coalgebraic View on Context-Free Languages and Streams

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Overview

- In a paper published at CALCO 2011, we presented a coalgebraic treatment of context-free languages.
- Formal languages can be seen as a specific instance of *formal power series* over a semiring, here over the Boolean semiring $\mathbb{B}$.
- Two different definitions of *algebraic* power series exist. We can easily generalize our approach to context-free languages to obtain these algebraic power series.

Formal power series, formal languages, and streams

- Given a finite set A called the *alphabet*, and a semiring R, a *formal power series* on A with coefficients in R is a function

$$A^* \to R.$$

- When R is the Boolean semiring $\{0, 1\}$ (with $1 + 1 = 1$), formal power series on A with coefficients in R correspond to formal languages over the alphabet A.
- When A is a singleton set, formal power series on A with coefficients on R correspond to *streams* over R.

Coalgebras representing formal power series (1)

In the this talk, we will represent these power series with coalgebras over functors of the type $R \times (-)^A$.

Coalgebras representing formal power series (2)

Given a coalgebra (X, f) over the functor $R \times (-)^A$, we will use the following notation:

- Given $x \in X$, $o(x)$ (called the *output value of* x) will be the first component of $f(x)$.
- Given $x \in X$, x_a (called the *a-derivative of* x) will be the second component of $f(x)$, applied to a.

So: for every x, $o(x)$ is an element of the semiring R, and for every x and a, x_a is an element of X again.
When (in the case of streams) the alphabet is a singleton set $\{a\}$, we usually write x' instead of x_a.

Coalgebras representing formal power series (3)

We can extend the notion of derivatives from alphabet symbols (i.e. elements of A) to words (i.e. elements of A^*) inductively:

- $x_\lambda = x$
- $x_{a \cdot w} = (x_a)_w$.

Homomorphisms and bisimulations (1)

Given two $R \times (-)^A$-coalgebras (X, f) and (Y, g), a function $h : X \to Y$ is a *homomorphism* if the following hold:

1. For every $x \in X$, $o(x) = o(h(x))$.
2. For every $x \in X$ and $a \in A$, $h(x_a) = (h(x))_a$.

Homomorphisms and bisimulations (2)

Given two $R \times (-)^A$-coalgebras (X, f) and (Y, g), a relation $R \subseteq X \times Y$ is a *bisimulation* if the following hold:

1. If $(x, y) \in R$, then $o(x) = o(y)$.
2. If $(x, y) \in R$, then for all $a \in A$, $(x_a, y_a) \in R$.

Final coalgebras (1)

A final coalgebra for the functor $2 \times (-)^A$ is the coalgebra $(\mathcal{L}, l)$, defined as follows:

- $\mathcal{L}$ is the set of all languages on the alphabet A.
- For any $L \in \mathcal{L}$:
  - $o(L)$ is 1 iff the empty word is in L.
  - $F_a = \{w \mid a \cdot w \in L\}$.

Given a $2 \times (-)^A$-coalgebra (X, f), and an element $x \in X$, we let $[\![x]\!]$ denote the value of x under the unique homomorphism from (X, f) to the final coalgebra.

Final coalgebras (2)

Generalizing the picture from the previous slide to arbitrary semirings R, the final coalgebras will sets of formal power series:

- $\mathcal{S}$ is the set of all formal power series on A with coefficients in R, i.e. the function space from A^* to R.
- For any $F \in \mathcal{S}$:
  - $o(F) = F(\lambda)$.
  - $F_a = G : G(x) = F(a \cdot x)$.

Again, we let $[\![x]\!]$ denote the value of x under the final homomorphism.

Languages and streams

The presentation above is a generalization of both languages and streams:

- When we choose the Boolean semiring $\{0, 1\}$, we get coalgebras representing languages over the alphabet A.
- When we choose a singleton alphabet A, we get coalgebras representing streams over the semiring R.

The coalgebra of rational series (1)

The set $\mathcal{E}$ of *rational series* over a finite alphabet A and a semiring $(R, +, \cdot, 0, 1)$ can be defined as follows:

$$t ::= a \in A \mid r \in R \mid t + t \mid t \cdot t \mid t^*$$

We can assign a $\mathcal{D}$-coalgebra structure to this set of regular expressions by specifying the output values and derivatives for each expression, giving us a $\mathcal{D}$-coalgebra $(\mathcal{E}, e)$:

Coalgebras of rational series (2)

t	$o(t)$	t_a
$r \in R$	r	0
$b \in A$	0	if $b = a$ then 1 else 0
$u + v$	$o(u) + o(v)$	$u_a + v_a$
$u \cdot v$	$o(u) \cdot o(v)$	$u_a \cdot v + o(u) \cdot v_a$
u^*	1	$u_a \cdot u^*$

When the underlying semiring R is the Boolean semiring 2, these rational series are exactly the regular languages.

Introduction: context-free languages and systems

- The 'next step up' from regular expressions and languages, and finite automata, in the Chomsky hierarchy, are the context-free languages and grammars, and pushdown automata.
- We will present a format of coinductively defined systems of equations.
- When using $\mathbb{B}$, it turns out that these systems of equations chracterize precisely the context-free languages.
- For arbitrary semirings, these systems of equations correspond to so-called *algebraic power series*

Systems of equations (1)

We will use terms t specified as follows:

$$t ::= a \in A \,|\, x \in X \,|\, r \in R \,|\, t + t \,|\, t \cdot t$$

where X is a finite set of variables, R is the carrier of the underlying semiring, and A, as before, is a finite alphabet. Given X, we let TX denote the set of terms over X.
A well-formed system of equations, for a set of variables X, consists of:

1. For every $x \in X$, exactly one equation of the form $o(x) = v$, where $v \in \{0, 1\}$.
2. For every $x \in X$ and $a \in A$, exactly one equation of the form $x_a = t$, where $t \in TX$.

Systems of equations (2)

Alternatively, we can consider a well-formed system of equations as a mapping

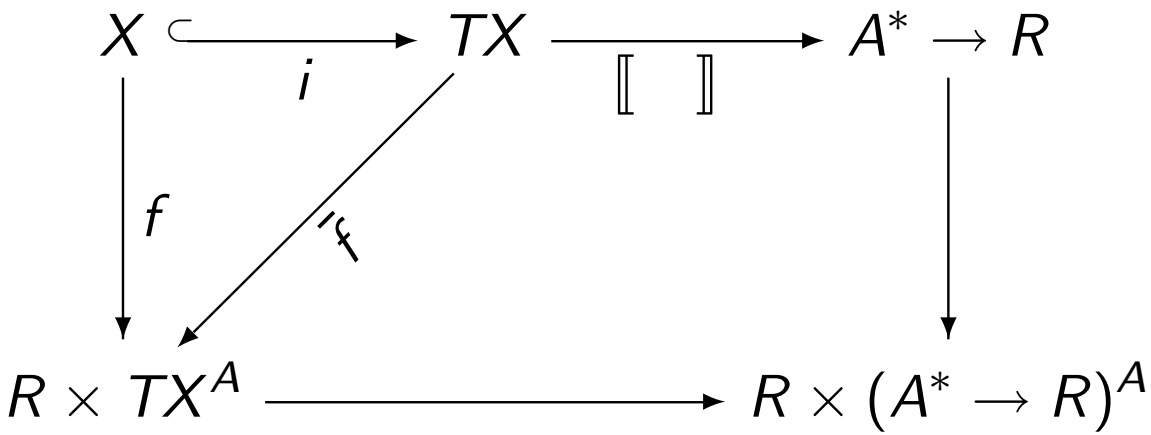
$$f : X \to R \times TX^A$$

We can extend such a mapping f to the $\mathcal{D}$-coalgebra $(TX, \bar{f})$ *generated by* (X, f) as follows:

t	$o(t)$	t_a
$x \in X$	$o(x)$	x_a (as specified by f)
$r \in R$	r	0
$b \in A$	0	if $b = a$ then 1 else 0
$u + v$	$o(u) + o(v)$	$u_a + v_a$
$u \cdot v$	$o(u) \cdot o(v)$	$u_a \cdot v + o(u) \cdot v_a$

Systems of equations (3)

The construction can be summarized as follows:



Systems of equations (3)

When taking $\mathbb{B}$ as underlying semiring, we obtain:
Proposition: *A language L is context-free iff there is a well-formed system of equations (X, f) and an $x \in X$, such that $[\![x]\!] = L$ w.r.t. the coalgebra $(TX, \bar{f})$ generated by it.*

First definition (1)

(The following definition can be found in work by Salomaa, Droste, Kuich et al., and seems to originate with Michel Fliess in the 70s.) An *R-algebraic system* is a set of equations of the form

$$x_i = p_i \qquad (i \in \{1, \ldots, n\})$$

where $p_i \in R\langle A \cup X \rangle$. Such a system is called *proper* if, for all i, $(p_i, \lambda) = 0$, and for all i, j, $(p_i, y_j) = 0$. Furthermore, such a system is said to be in *Greibach normal* form if its support is contained in the set $A \cup AX \cup AXX$.

First definition (2)

A *strong solution to an R-algebraic system* is an assignment of power series to variables x_i, such that for each x_i, $o(x_i) = 0$. We call a formal power series s *constructively R-algebraic* if

$$s = r + \bar{s},$$

where $r \in R$ and $\bar{s}$ is a strong solution to an R-algebraic system.

Results w.r.t. the first definition

- Every proper R-algebraic system has exactly one strong solution.
- Any component of a strong solution to a proper R-algebraic system also occurs as a strong solution to such a system in Greibach normal form.
- A formal power series is R-algebraic iff it occurs as a solution to a context-free system of equations.

Second definition

For a field F, a stream σ over F is F-algebraic if there are polynomial streams $a_0, \ldots a_n$ such that

$$\sum_{0 \leq i \leq n} a_i \sigma^i = 0$$

This definition occurs in the world of *automatic sequences*: it is well-known that, over a finite field $\mathbb{F}_p$, a sequence is p-automatic iff it is $\mathbb{F}_p$-algebraic.

Correspondence between the two definitions

- Proposition: *Given a stream σ over a perfect field F, σ is F-algebraic iff it is constructively F-algebraic* (Fliess 1971)
- Proposition: *A power series s is constructively R-algebraic iff there is a well-formed system of equations (X, f) and an $x \in X$, such that $[\![x]\!] = s$ w.r.t. the coalgebra $(TX, \bar{f})$ generated by it.*

Examples: context-free streams (1)

When we take the semiring of natural numbers as underlying semiring, the Catalan numbers

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots$$

are context-free, and are generated by the following system of equations:

$$o(x) = 1 \quad x' = x \cdot x$$

Examples: context-free streams (2)

When we take the Boolean *field* $\mathbb{F}_2$ (with $1 + 1 = 0$) as underlying semiring, the Prouhet-Thue-Morse sequence

$$10\,01\,0110\,01101001\ldots$$

is context-free, and is generated by the following system of equations:

$$\begin{aligned} o(x) &= 1 & x' &= y \\ o(y) &= 0 & y' &= z \\ o(z) &= 0 & z' &= x + y + z + z \cdot w \\ o(w) &= 1 & w' &= y \cdot w \end{aligned}$$

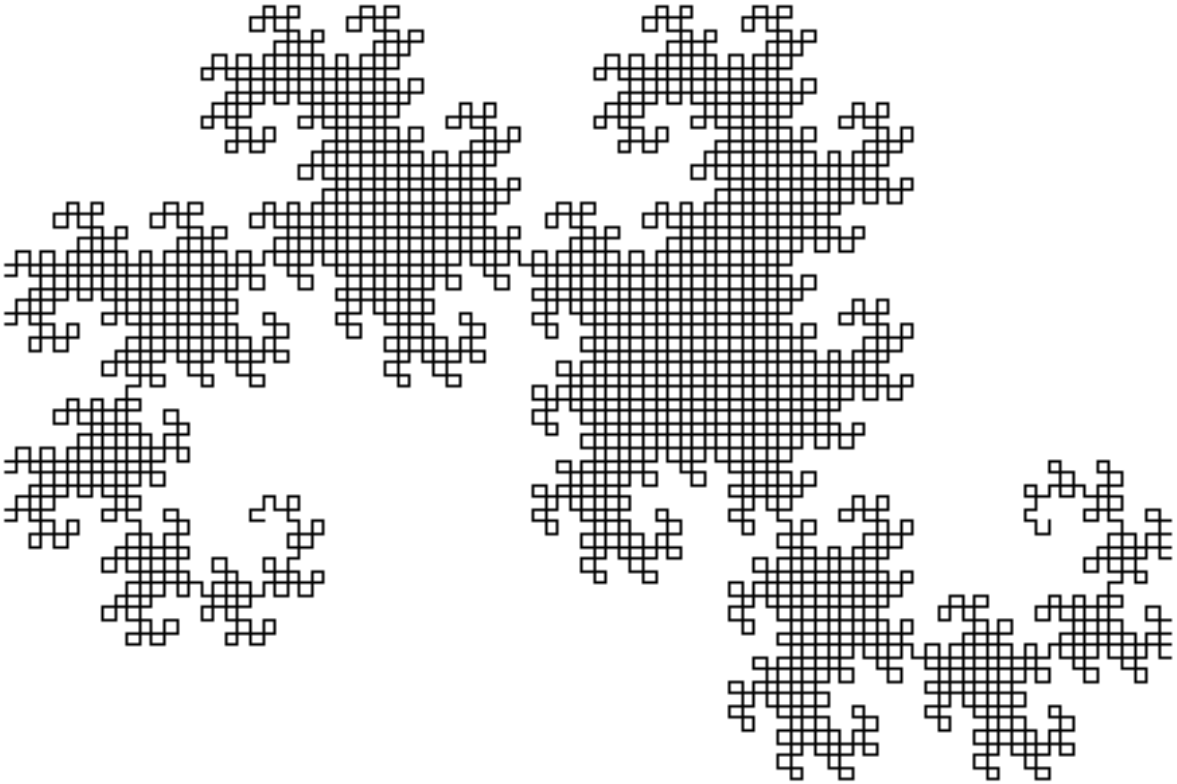
Examples: context-free streams (3)

Still using $\mathbb{F}_2$ as underlying semiring, the context-free system of equations

$$\begin{aligned} o(x) &= 1 & x' &= y \\ o(y) &= 1 & y' &= z \\ o(z) &= 0 & z' &= w \\ o(w) &= 1 & w' &= v \\ o(v) &= 1 & v' &= v + w + v \cdot x + x \cdot x \end{aligned}$$

gives us the paperfolding sequence, or Dragon curve sequence...

Examples: context-free streams (4)



Conclusions and further work

- We have a neat correspondence...
- ...but still want to understand Fliess' 1971 proof...
- ... and find a constructive way of transforming the two definitions of algebraicity into each other.
- Can we say anything about algebraic reals? (See: coalgebraic work on exact real arithmetic.)