Erratum to various proofs of Christol's theorem

Joost Winter
University of Warsaw
email: joostwinter@gmail.com

May 19, 2015

Abstract

This note presents an erratum to the proofs of Christol's theorem found in [Ber02], [AS03], and [BR11].

1 The problem

In the proofs of Christol's theorem in [Ber02], [AS03], [BR11], a problem occurs revolving around a division of two formal power series, which is not guaranteed to result in an (ordinary) formal power series, but only a Laurent series.

- In the proof of [BR11, Theorem 5.4.1], the division is made as 'Set $v = u/c_0$'. Although we are guaranteed that $c_0 \neq 0$, we do not have the guarantee that $c_0(0) \neq 0$, so it is possible that v only exists as a Laurent series.

  Later in the proof, it is implicitly assumed that $v = u/c_0$ is an ordinary power series, and the proof relies on two earlier results (Lemma 5.4.2 and Corollary 5.4.3) that are only given for ordinary power series, as well as the operator $\circ$ which is only defined for ordinary power series.

- In the proof of [AS03, Theorem 12.2.5], the problem is essentially the same. Here the division resulting in a Laurent series is given as 'Put $G = A(X)/B_0(X)$', and the underlying definitions and lemmas that need to be adjusted are Definition 12.2.1 and Lemma 12.2.2.

- In the proof of [Ber02, Theorem 3.2.1], the problem and the solution are, again, essentially the same, and centers around the usage of formal power series in a context where Laurent series are needed.

This problem, however, does not occur in the original papers [Chr79] and [CKFR80]. However, unlike the above books, neither of these papers gives an explicit formulation of the underlying lemma, which this note tries to make explicit.

2 The solution

We will use a representation of Laurent series as (equivalence classes) of pairs consisting of a power series and an integer, the offset. The Laurent series

$$\sum_{i\geq k} a_i x^i$$

is represented by the equivalence class of pairs of the form:

$$\left[\sum_{i\geq 0} a_{k+i} x^i, \quad k\right]$$

Note that:

1. The equivalence relation can be given by

   $$[x^j S, n-j] \sim [x^k S, n-k]$$

   for all $j, k \in \mathbb{N}$, $n \in \mathbb{Z}$, and all power series S.

2. Addition and multiplication of Laurent series can be given in terms of (in the case of addition, suitably chosen) representatives of the equivalence classes as

   $$[S, n] + [T, n] = [S + T, n]$$

   and

   $$[S, n][T, m] = [ST, n + m] \tag{1}$$

   and these operations respect the equivalence relation and define the usual operations on Laurent series. The definition of the product gives rise to the exponentiation rule:

   $$[S, n]^k = [S^k, nk] \tag{2}$$

We identify a Laurent series with any of its representations.

The operator $\mathcal{A}_{i,q}$, corresponding to Λ_i in [AS03] and (with some conceptual differences) the operator/monoid action $\circ$ in [BR11] is defined (first on power series) by

$$\mathcal{A}_{i,q}\left(\sum_{j\geq 0} a_j x^j\right) = \sum_{j\geq 0} a_{qj+i} x^j$$

We first state the power series version of the lemma, which is part b) of [AS03, Lemma 12.2.2] and has an easy correspondence to [BR11, Lemma 5.4.2, Corollary 5.4.3]:

Lemma 1. *For all power series S and T, all $q, i \in \mathbb{N}$ with $q \geq 1$ and $0 \leq i < q$, we have:*

$$\mathcal{A}_{i,q}(ST^q) = \mathcal{A}_{i,q}(S)T$$

We now define the operation $\mathcal{B}_{i,q}$ extending $\mathcal{A}_{i,q}$ to Laurent series as follows:

$$\mathcal{B}_{i,q}([S, aq]) = [\mathcal{A}_{i,q}(S), a]$$

(Note that $\mathcal{B}$ is not defined for *all* representatives of a Laurent series, but always for some representatives, and where it is, it again respects the equivalence relation.)

Lemma 2. *For all Laurent series V and W, all $q, i \in \mathbb{N}$ with $q \geq 1$ and $0 \leq i < q$, we have:*

$$\mathcal{B}_{i,q}(VW^q) = \mathcal{B}_{i,q}(V)W$$

Proof. Note that V must have some representation as a pair $V = [S, aq]$ for some power series S and natural number a. Furthermore, let $W = [T, b]$. We now have:

$$\begin{aligned}
&\mathcal{B}_{i,q}([S, aq][T, b]^q) \\
&= \mathcal{B}_{i,q}([S, aq][T^q, bq]) \qquad \textit{by (2)} \\
&= \mathcal{B}_{i,q}([ST^q, (a+b)q]) \qquad \textit{by (1)} \\
&= [\mathcal{A}_{i,q}(ST^q), a+b] \qquad \textit{definition of } \mathcal{B}_{i,q} \\
&= [\mathcal{A}_{i,q}(S)T, a+b] \qquad \textit{by Lemma 1} \\
&= [\mathcal{A}_{i,q}(S), a][T, b] \qquad \textit{by (1)} \\
&= \mathcal{B}_{i,q}([S, aq])[T, b] \qquad \textit{definition of } \mathcal{B}_{i,q}
\end{aligned}$$

□

With this extended lemma, we can fix the problem in the proofs as follows:

- In [AS03], note that (given q), we have

  $$\Lambda_r(S) = \mathcal{A}_{r,q}(S)$$

  and thus, the definition of $\mathcal{B}_{i,q}$ and Lemma 2 give the extension of the definition of Λ and of Lemma 12.2.2 (b) to Laurent series.

  The proof of (the right to left direction of) Christol's theorem (Theorem 12.2.5) now is fixed by

  - Observing that in 'Put $G = A(X)/B_0(X)$', G is defined as a Laurent series.
  - Replacing '$H \in GF(q)[[X]]$' by '$H \in GF(q)((X))$' in the definition of $\mathcal{H}$. (Note that $\mathcal{H}$ still is finite.)
  - Appealing to the extended Lemma 12.2.2 (b) and the extended version of Λ_r in the chain of equations starting with $\Lambda_r(H)$ near the end of the proof.

  Because Λ_r is consistent with its extension on power series, it follows that the restriction of $\mathcal{H}$ to power series is closed under the (ordinary) Λ_r operators.

- In [BR11], note that the operation

$$(r \circ u)$$

on power series, in the case where $|r| = 1$, or equivalently $r \in \mathbf{q}$, is the same as

$$\mathcal{A}_{r,q}(u)$$

as defined in this note. More generally, for words $w \in \mathbf{q}^*$, with $|w| = n$, there is always some i with $0 \leq i < q^n$ such that:

$$(w \circ u) = \mathcal{A}_{i,q^n}(u)$$

If we extend this operation to Laurent series u by defining

$$(r \circ u) = \mathcal{B}_{r,q}(u)$$

for elements $r \in \mathbf{q}$, the extension to words of arbitrary length again holds. Thus it follows from Lemma 2 that the extended versions of Lemma 5.4.2 and Corollary 5.4.3 again hold.

In the proof of Theorem 4.4.1, the problem can now be fixed by making the following observations/modifications:

  - Note that $v = u/c_0$ defines a Laurent series instead of a power series.
  - F should now be defined as an (again finite) set of Laurent series.
  - The appeal to Corollary 5.4.3 should be replaced by an appeal to its extension to Laurent series.

Finally observe that, because u is in fact a power series, so is $(r \circ u)$ for every $r \in \mathbf{q}^*$.

References

[AS03] Jean-Paul Allouche and Jeffrey O. Shallit. *Automatic Sequences – Theory, Applications, Generalizations.* Cambridge University Press, 2003.

[Ber02] Valérie Berthé. Automatic sequences and transcendence. In N. Pytheas Fogg, editor, *Substitutions in Dynamics, Combinatorics and Arithmetics*, volume 1794 of *Lecture Notes in Mathematics*, pages 53–80. Springer, 2002.

[BR11] Jean Berstel and Christophe Reutenauer. *Noncommutative Rational Series with Applications.* Cambridge University Press, 2011.

[Chr79] Gilles Christol. Ensembles presque périodques k-recoinnassables. *Theoretical Computer Science*, 9:141–145, 1979.

[CKFR80] Gilles Christol, T. Kamae, Michel Mèndes France, and Gérard Rauzy. Suites algébriques, automates et substitutions. *Bull. Soc. Math. France*, 108:401–419, 1980.