A completeness result for finite bisimulations up-to congruence*

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Abstract

In this short paper, we will present conditions, in the context of λ-bialgebras for distributive laws between monads and endofunctors on **Set**, under which finite bisimulations up-to congruence are *complete* with respect to behavioural equivalence on freely and finitely generated λ-bialgebras. This completeness result can be seen as a continuation of the work in [14], in which a similar completeness result for λ-bisimulations (or bisimulations up-to context) was presented.

1998 ACM Subject Classification F.4.3 Formal Languages

Keywords and phrases decidability, bisimulation up-to congruence, coalgebra, bialgebra, weighted automata

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

Bisimulation up-to techniques have been extensively studied in recent years, for example in [6], [12], [4], [11], [5], and many other sources. A large benefit of using up-to techniques is that we can often suffice by using smaller relations than we would need if we would use ordinary bisimulations: in some cases, states of coalgebras or bialgebras can be linked by finite bisimulations up-to, but only by infinite ordinary bisimulations. So far, most of the work on bisimulation up-to has focused on *soundness*, however in this paper, continuing the work in [14], the focus is completeness.

Both the present work and the work in [14] require a setting of an endofunctor F on **Set** preserving weak pullbacks, a monad (T, η, μ) on **Set**, and a distributive law of the monad (T, η, μ) over F. In [14], the main result showed that if finitely generated (T, η, μ)-algebras are preserved under taking kernel pairs, then behaviourally equivalent states in freely and finitely generated λ-bialgebras are linked by finite λ-bisimulations. In this paper, we show that if every finitely generated (T, η, μ)-algebra is finitely presented[1], then behaviourally equivalent states in freely and finitely generated λ-bialgebras are linked by finite bisimulations up-to congruence (on a potentially larger, but still freely and finitely generated, λ-bialgebra). We note that there are known cases where the latter precondition holds, but the former does not.

A comprehensive and general introduction to distributive laws and λ-bialgebras can be found in [2], however, all of the concepts used here can also be found, presented more concisely and geared towards a result similar to the one in this paper, in [14]. Some of the applications presented in this paper apply to weighted automata, a general introduction to which can be found in [3]. Various techniques for bisimulation up-to, including bisimulations up-to congrence and up-to context, are extensively treated in e.g. [12], [4], [11], and [5], and bisimulation up-to congruence also plays a big role in [6]. Finally, the work in this paper

* This work was supported by the Polish National Science Centre (NCN) grant 2012/07/E/ST6/03026.

[1] We presently also assume that F has a final coalgebra, but the author conjectures that this condition can be removed.

42nd Conference on Very Important Topics (CVIT 2016).
Editors: John Q. Open and Joan R. Acces; Article No. 23; pp. 23:1–23:9
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

can most probably be related to recent work [9], in which the notion of a proper semiring has been extended categorically to that of a proper functor in a category of algebras.

We define a *congruence* on a category $\mathbf{C}$ as an internal equivalence relation on $\mathbf{C}$: see e.g. [1, Definition 3.9] (there simply called *equivalence relation*) for a definition.

▶ **Definition 1.** Let F be an endofunctor on $\mathbf{Set}$, let (T, η, μ) be a monad on $\mathbf{Set}$, and let λ be a distributive law of the monad (T, η, μ) over the endofunctor F. Given a λ-bialgebra (X, α, δ), a relation R on X is a *bisimulation up-to congruence* on (X, α, δ) if and only if the least congruence containing R is a bisimulation on the coalgebra (X, δ).

Note that, as a result of the soundness of bisimulation, from this definition it *directly* follows that bisimulations up-to congruence are sound. We observe that our definition in fact coincides with the definition given in [11] whenever F preserves weak pullbacks:

▶ **Proposition 2.** *If F preserves weak pullbacks, this definition of bisimulation up-to congruence coincides with that in [11].*

We can now state the main result, which can be seen as a completeness result for finite bisimulations up-to congruence (in a similar manner to how the main result of [14] was a completeness result for finite λ-bisimulations):

▶ **Theorem 3.** *Let F be a $\mathbf{Set}$-endofunctor preserving weak pullbacks, such that a final F-coalgebra exists, let (T, η, μ) be a monad on $\mathbf{Set}$ such that finitely generated and finitely presented (T, η, μ)-algebras coincide, and let λ be a distributive law of the monad (T, η, μ) over the endofunctor F.*

Given a finite FT-coalgebra (X, δ), if states $x, y \in TX$ are behaviourally equivalent with respect to the λ-bialgebra $(TX, \mu_X, \hat{\delta})$ (with $\hat{\delta} = F\mu_X \circ \lambda_{TX} \circ T\delta$), then there is a finite FT-coalgebra (Y, γ) such that (X, δ) is a sub-FT-coalgebra of (Y, γ), and a finite bisimulation up-to congruence R on the λ-bialgebra $(TY, \mu_Y, \hat{\gamma})$ with $(x, y) \in R$.

This gives us the following variant of the main result in [8]. This is a proper extension of the decidability result for Noetherian semirings because $\mathbb{N}$ is an example of a semiring that is not Noetherian but that has the property of finitely generated semimodules being finitely presented, as a result of the fact that $\mathbb{N}$-semimodules are precisely commutative monoids and Rèdei's Theorem, see [10]. For a definition of an effectively presentable semiring, see [8].

▶ **Corollary 4.** *Let S be a semiring such that every left-S-semimodule that is finitely generated is also finitely presented, and let S be moreover effectively presentable. Then equivalence of S-weighted automata is decidable.*

For the tropical semiring $\mathbb{T}$, with carrier $\mathbb{N} \cup \infty$, addition given by min and multiplication given by $+$, we obtain the following corollary, using the undecidability of equivalence $\mathbb{T}$-weighted automata:

▶ **Corollary 5.** *Not all finitely generated $\mathbb{T}$-semimodules are finitely presented.*

Similarly, using the undecidability of equivalence of context-free languages (and their bialgebraic presentation in [7]), we obtain:

▶ **Corollary 6.** *Not all finitely generated idempotent semirings are finitely presented.*

We leave as future work establishing more precise relationships between the notions of 'finitely generated algebras are finitely presented', 'completeness of finite bisimulation up-to congruence', and the notion of proper semirings (and the more general notion of proper functors, recently introduced in [9]).

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A Proofs and additional lemmata

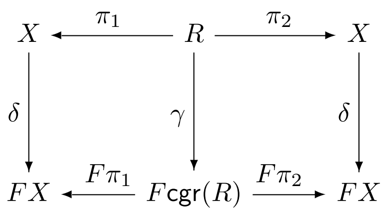
Proof of Proposition 2. In [11], the congruence closure of a relation R on a T-algebra (X, α) is given as $\mathsf{cgr}(R)$, where the function $\mathsf{cgr}(R) : \mathcal{P}(X \times X) \to \mathcal{P}(X \times X)$ is given as

$$\mathsf{cgr} = \bigcup_{i=0}^{\infty} (\mathsf{tra} \cup \mathsf{sym} \cup \mathsf{ctx} \cup \mathsf{rfl})^i$$

where tra is the function mapping a relation to its transitive closure, sym maps a relation to its symmetric closure, rfl maps a relation to its reflexive closure, and ctx maps a relation to its contextual closure, i.e. the least subalgebra of $X \times X$ containing R.

It is clear than any congruence (note that a congruence on an algebra (X, α) is precisely a subalgebra of $X \times X$ which is, taken as a relation on **Set**, an equivalence relation, see [1, Remark 3.10]) on (X, α) is closed under the operations tra, sym, rfl, and ctx, so $\mathsf{cgr}(R)$ is contained in the smallest congruence containing R. However, as $\mathsf{cgr}(R)$ is itself closed under all these operations, it follows that it is a congruence.

Hence $\mathsf{cgr}(R)$ is the smallest congruence containing R. In [11], a bisimulation up-to congruence on λ-bialgebra (X, α, δ) is defined as a relation R such that both sides of the following diagram commute:



We now show that R is a bisimulation up-to congruence if and only if $\mathsf{cgr}(R)$ is a bisimulation.

If $\mathsf{cgr}(R)$ is a bisimulation, then it follows directly that R is a bisimulation up-to congruence, because $R \subseteq \mathsf{cgr}(R)$. For the converse, assume that R is a bisimulation up-to congruence. We will show that $\mathsf{cgr}(R)$ is a bisimulation using induction, first showing that, if R is a bisimulation up-to congruence on a λ-bigebra (X, α, δ), then

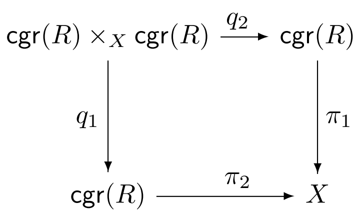
$$(\mathsf{tra} \cup \mathsf{sym} \cup \mathsf{ctx} \cup \mathsf{rfl})(R)$$

is again a bisimulation up-to congruence:

1. If $(x, x) \in \mathsf{rfl}(R)$, let $\gamma'((x, x)) = F\mathsf{diag} \circ \delta(x)$, where $\mathsf{diag}(x) = (x, x)$, and it is clear that $\gamma'((x, x)) \in F\mathsf{cgr}(R)$.

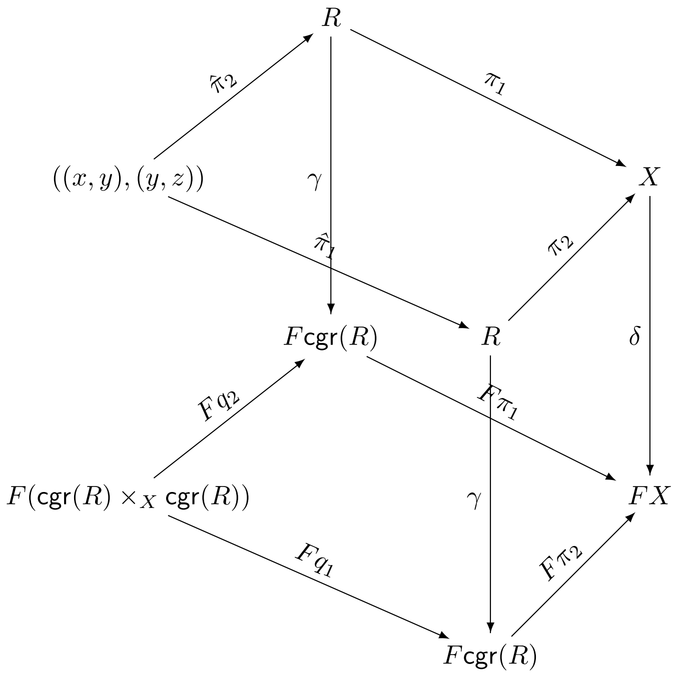
2. If $(x, y) \in \mathsf{sym}R$, let $\gamma'((x, y)) = F\mathsf{swap} \circ \gamma((y, x))$, where $\mathsf{swap}((x, y)) = (y, x)$, and it is clear that $\gamma'((x, y)) \in F\mathsf{cgr}(R)$.

3. If $(x, y), (y, z) \in \mathsf{tra}(R)$, we make use of the internal transitivity of the congruence $\mathsf{cgr}(R)$, i.e., the existence a morphism $t : \mathsf{cgr}(R) \times_X \mathsf{cgr}(R) \to \mathsf{cgr}(R)$, such that, given the pullback diagram



   we have $\pi_1 \circ q_1 = \pi_1 \circ t$ and $\pi_2 \circ q_2 = \pi_2 \circ t$. Because F preserves weak pullbacks, we can apply F to this diagram and obtain a weak pullback diagram.

Now observe the following diagram:



It is clear that $F\pi_1 \circ \gamma \circ \hat{\pi}_2 = F\pi_2 \circ \gamma \circ \hat{\pi}_1$, so there must be some mapping $h : ((x,y),(y,z)) \to F(\mathsf{cgr}(R) \times_X \mathsf{cgr}(R))$ with:

$$F\pi_1 \circ Fq_2 \circ h = F\pi_2 \circ Fq_1 \circ h$$

Now let $\gamma'((x,z)) = (Ft \circ h)((x,y),(y,z))$.
Observe that

$$\begin{aligned} & (F\pi_1 \circ \gamma')(x,z) \\ =\ & (F\pi_1 \circ Ft \circ h)((x,y),(y,z)) \\ =\ & (F\pi_1 \circ Fq_1 \circ h)((x,y),(y,z)) \\ =\ & (F\pi_1 \circ \gamma \circ \hat{\pi}_1)((x,y),(y,z)) \\ =\ & (F\pi_1 \circ \gamma)(x,y) \\ =\ & (\delta \circ \pi_1)(x,y) \\ =\ & (\delta \circ \pi_1)(x,z) \end{aligned}$$

and similarly for π_2, so $R \cup (x,z)$ is again a bisimulation up-to congruence. It follows by induction (and the fact that bisimulations up-to congruence are closed under union) that $\mathsf{tra}(R)$ is again a bisimulation up-to congruence.

4. Recall that the contextual closure of a relation R is defined as the relation

$$\mathsf{ctx}(R) = \{(\alpha \circ T\pi_1(q), \alpha \circ T\pi_2(q)) \mid q \in TR\}$$

Take any element $(x,y) \in \mathsf{ctx}(R)$, and assume $x = \alpha \circ T\pi_1(q)$ and $y = \alpha \circ T\pi_2(q)$ for some $q \in TR$. Now consider the mapping

$$TR \xrightarrow{T\gamma} TF\mathsf{cgr}(R) \xrightarrow{\lambda_{\mathsf{cgr}(R)}} FT\mathsf{cgr}(R) \xrightarrow{F\rho} F\mathsf{cgr}(R)$$

where $\rho : T\mathsf{cgr}(R) \to \mathsf{cgr}(R)$ is the T-algebra structure on $\mathsf{cgr}(R)$.
Now let $\gamma'((x,y)) = (F\rho \circ \lambda_{\mathsf{cgr}(R)} \circ T\gamma)(q)$.
We have

$$\begin{aligned}
&(\delta \circ \pi_1)((x,y)) \\
&= \delta(x) \\
&= (\delta \circ \alpha \circ T\pi_1)(q) \\
&= (F\alpha \circ \lambda_X \circ T\delta \circ T\pi_1)(q) \\
&= (F\alpha \circ \lambda_X \circ TF\pi_1 \circ T\gamma)(q) \\
&= (F\alpha \circ FT\pi_1 \circ \lambda_{\mathsf{cgr}(R)} \circ T\gamma)(q) \\
&= (F\pi_1 \circ F\rho \circ \lambda_{\mathsf{cgr}(R)} \circ T\gamma)(q) \\
&= (F\pi_1 \circ \gamma')(x,y)
\end{aligned}$$

and now it follows that $\mathsf{ctx}(R)$ is again a bisimulation up-to congruence.

We now have established that $(\mathsf{tra} \cup \mathsf{sym} \cup \mathsf{ctx} \cup \mathsf{rfl})(R)$ is a bisimulation up-to congruence, and iterating this process, it follows that $(\mathsf{tra} \cup \mathsf{sym} \cup \mathsf{ctx} \cup \mathsf{rfl})^n(R)$ is a bisimulation up-to congruence for any $n \in \mathbb{N}$.

It now follows that $\mathsf{cgr}(R)$ is a bisimulation up-to congruence, but because $\mathsf{cgr}(\mathsf{cgr}(R)) = \mathsf{cgr}(R)$, this implies that $\mathsf{cgr}(R)$ is in fact a bisimulation. This completes the proof. ◀

▶ **Lemma 7.** *Let F be an endofunctor on a category $\mathbf{C}$, let (T, η, μ) be a monad on $\mathbf{C}$ such that T preserves epimorphisms, and let λ be a distributive law of the monad (T, η, μ) over the endofunctor F.*

Let (A, α, γ) be a λ-bialgebra, and let (B, β) be a T-algebra and (B, δ) be a F-coalgebra. If a $\mathbf{C}$-epimorphism $f : A \to B$ is both a T-algebra morphism from (A, α) to (B, β) and an F-coalgebra morphism from (A, γ) to (B, δ), then (B, β, δ) is a λ-bialgebra.

Proof. We only have to check that:

$$\delta \circ \beta = F\beta \circ \lambda_B \circ T\delta \tag{1}$$

We first show that

$$\delta \circ \beta \circ Tf = F\beta \circ \lambda_B \circ T\delta \circ Tf$$

which holds because (note that $\delta \circ f = Ff \circ \gamma$ because f is a coalgebra morphism, and $\beta \circ Tf = f \circ \alpha$ because f is an algebra morphism):

$$\begin{aligned}
&\delta \circ \beta \circ Tf \\
&= \delta \circ f \circ \alpha \\
&= Ff \circ \gamma \circ \alpha \\
&= Ff \circ F\alpha \circ \lambda_A \circ T\gamma \\
&= F\beta \circ FTf \circ \lambda_A \circ T\gamma \\
&= F\beta \circ \lambda_B \circ TFf \circ T\gamma \\
&= F\beta \circ \lambda_B \circ T\delta \circ Tf
\end{aligned}$$

Now (1) follows because f is an epimorphism and T preserves epimorphisms. ◀

▶ **Lemma 8.** *Let F be a* **Set***-endofunctor preserving weak pullbacks, let (T, η, μ) be a monad on* **Set***, and let λ be a distributive law of the monad (T, η, μ) over the endofunctor F.*

Let (A, α, γ) and (B, β, δ) be λ-bialgebras, and let $f : (A, \alpha, \gamma) \to (B, \beta, \delta)$ be a morphism of λ-bialgebras. Then f factorizes uniquely as $f = m \circ e$, where e is an epimorphism in **Set***, m is a monomorphism in* **Set***, and e and m are both λ-bialgebra morphisms.*

Proof. Let $m \circ e$ be a factorization of f into a surjective function $e : A \to f[A]$ and an injective function $m : f[A] \to B$ in **Set**. Because the category of F-coalgebras has epi-mono factorizations (which correspond to surjective-injective factorizations in **Set**, see [13, Theorem 7.1]), there exists a unique F-colgebra structure ζ on $f[A]$ making m and e F-coalgebra morphisms. Because the category of T-algebras has regular epi-mono factorizations (which correspond to surjective-injective factorizations in **Set**), there exists a unique T-algebra structure ξ on $f[A]$ making m and e T-algebra morphisms.

Now, by Lemma 8, it follows that $(f[A], \xi, \zeta)$ is a λ-bialgebra. Because a λ-bialgebra morphism is simply a morphism which is both a T-algebra morphism, and a F-coalgebra morphism, it follows that m and e are λ-bialgebra morphisms. ◀

Proof of Theorem 3. Assume that F, T, and λ satisfy the conditions of the proposition, let (X, δ) be a FT-coalgebra, and let $x, y \in TX$ be behaviourally equivalent states in the λ-bialgebra $(TX, \mu_X, \hat{\delta})$.

Let (Ω, α, ω) be the final λ-bialgebra, and $[\![-]\!]_{TX}$ be the unique mapping from $(TX, \mu_X, \hat{\delta})$ to this final λ-bialgebra. By Lemma 8, we can factorize $[\![-]\!]_{TX}$ uniquely as

$$(TX, \mu_X, \hat{\delta}) \overset{e}{\twoheadrightarrow} (W, \beta, \theta) \overset{m}{\hookrightarrow} (\Omega, \alpha, \omega)$$

Note that (W, β) is a finitely generated T-algebra, and thus also finitely presented. Hence, we have a coequalizer diagram like the following in the category of T-algebras:

$$(TR, \mu_R) \underset{p_2}{\overset{p_1}{\rightrightarrows}} (TY, \mu_Y) \overset{f}{\twoheadrightarrow} (W, \beta)$$

We first show that we can obtain a new, larger, coequalizer diagram as follows:

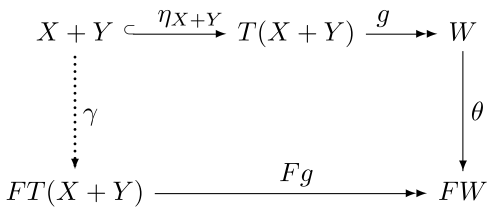
$$(T\hat{R}, \mu_{\hat{R}}) \underset{q_2}{\overset{q_1}{\rightrightarrows}} (T(X+Y), \mu_{X+Y}) \overset{g}{\twoheadrightarrow} (W, \beta) \tag{2}$$

To see this, first notice that we have a diagram (in the category of T-algebras)

$$\begin{array}{ccc} (TY, \mu_Y) & \overset{f}{\twoheadrightarrow} & (W, \beta) \\ & & \uparrow e \\ & & (TX, \mu_X) \end{array}$$

and thus, by the axiom of choice, for every $x \in X$, there is a $y \in TY$ such that $f(y) = e(\eta_X(x))$. Now let $\hat{R}$ be defined by augmenting R with all such pairs (x, y), and it follows that any element $z \in T(X + Y)$ is related by the least congruence on $\hat{R}$ to some element involving only elements of TY and none of TX. From here it follows that g is again a coequalizer of q_1 and q_2, as in (2).

The next step consists of giving a FT-coalgebra structure γ on $X + Y$, such that g is a morphism of bialgebras between $(T(X + Y), \mu_{X+Y}, \hat{\gamma})$ and (W, β, θ). For this, we need to define a γ making the following diagram commute in such a way that (X, δ) is a subcoalgebra of $(X + Y, \gamma)$:

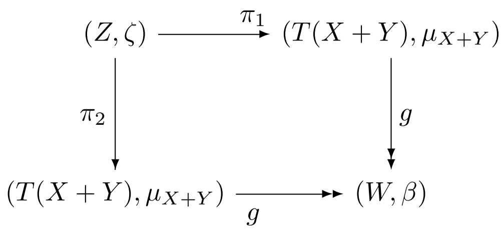


Note that such a γ exists because, by the axiom of choice, F preserves epimorphisms, so Fg is a surjective function. Moreover, because g extends e on elements of TX, we can construct γ in such a way that (X, δ) is a subcoalgebra of $(X + Y, \gamma)$.

At this point, we have established a coequalizer diagram as in (2), and also a bialgebra morphism (using [2, Lemma 4.3.4]):

$$(T(X + Y), \mu_{X+Y}, \hat{\gamma}) \overset{g}{\twoheadrightarrow} (W, \beta, \theta)$$

Finally, we will show that $\hat{R}$ is a bisimulation up to congruence, by showing that the least congruence containing $\hat{R}$ is a bisimulation. To see this, we take the kernel pair of g, in the category of T-algebras. Let (Z, ζ) together with the morphsms s_1 and s_2 be this kernel pair.



Because g identifies behaviourally equivalent states, and F preserves weak pullbacks, it moreover follows that Z is the largest bisimulation on the coalgebra $(T(X + Y), \hat{\gamma})$.

Finally, we note that Z must be the smallest congruence containing $\hat{R}$. This follows from the fact that g is a coequalizer of π_1 and π_2 as defined above (as any category that is algebraic over **Set** is a regular category), but as categories that are algebraic over **Set** are moreover exact, it follows that every congruence is effective, i.e. the least congruence containing $\hat{R}$ must be a kernel pair of some parallel pair.

Hence, $\hat{R}$ is a bisimulation up-to congruence on the bialgebra $(T(X+Y), \mu_{X+Y}, \hat{\gamma})$, with (X, δ) being a subcoalgebra of $(X + Y, \gamma)$, and the proof is complete. ◀

Proof of Corollary 4. We first note that, as observed in [8] and (using more coalgebraic language) [14] nonequivalence of S-weighted automata is semidecidable for any effectively presentable semiring S.

We now assume that S is effectively presentable and, moreover, finitely generated left-S-semimodules are finitely presentable, and will show that equivalence is again semidecidable. We do this using techniques comparable to those given in [14, Proposition 9]. First note that, given a finite relation R, it is semidecidable whether some pair (x, y) is contained in the least congruence containing R.

Now assume we are given a S-weighted automaton, in the form of a finite coalgebra $X \to S \times \mathrm{Lin}_S(X)^A$. We can enumerate all finite coalgebras extending this automaton, and moreover for each such coalgebra we can enumerate all finite relations on it, and possible transition structures that may witness that such a relation is a bisimulation up-to congruence. Because all these enumerations are countable (also thanks to the fact that S is effectively presentable), if such a bisimulation up-to congruence exists, we will eventually find it and will be able to prove that it is indeed a bisimulation up-to congruence. However, because finitely generated left S-semimodules are finitely presented, if states in this weighted automaton are equvialent, they will be linked by some finite bisimulation up-to congruence on the extension of a larger $S \times \mathrm{Lin}_S(X)^A$-coalgebra. Hence, equivalence of S-weighted automata is semidecidable. ◀

Proof of Corollary 5. Assume that every $\mathbb{T}$-semimodule (notice that left- and right-semimodules coincide in the case of $\mathbb{T}$ as $\mathbb{T}$ is commutative) that is finitely generated is finitely presented. Because $\mathbb{T}$ is effectively presentable, it would follow that equivalence of $\mathbb{T}$-weighted automata is decidable, which is known to be false. ◀

Proof of Corollary 6. We can again use the same techniques as in the previous two corollaries, now making use of the indecidability of equivalence of context-free languages. We remark that there are various, similar but slightly different, ways of treating the context-free languages coalgebraically. One of them, presented in [7], gives the context-free languages as final coalgebra mappings of bialgebras for the monad $\mathcal{P}_\omega(- + A)$, whose algebras are idempotent semirings with constants in A, over the endofunctor $2 \times -^A$. (Other presentations make use of e.g. copointed functors.)

First observe that an idempotent semiring with constants in A is finitely generated if and only if it is finitely generated as an ordinary idempotent semiring, and the same holds for the condition of being finitely presented. The next step conists of showing that if finitely generated $\mathcal{P}_\omega(- + A)$-algebras are finitely presented, then it would follow that equivalence of context-free languages is decidable (which is false). Now it follows that there must exist an idempotent semiring that is finitely generated, but not finitely presented.

To establish the equivalence of context-free languages from the assumption that finitely generated $\mathcal{P}_\omega(- + A)$-algebras are finitely presented, first note that non-equivalence is again semidecidable, and using a construction similar to the one given in Corollary 4, we can again enumerate finite $2 \times \mathcal{P}_\omega(- + A)^A$-coalgebras, and eventually identify an finite bisimulations up-to congruence, if one exists. Hence it would follow that equivalence, too, is semidecidable, which is a contradiction. ◀