A Yang-Baxter-like condition for distributive laws over endofunctors and co-pointed endofunctors

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In [2], it is shown that given two monads (S, η^S, μ^S) and (T, η^T, μ^T), in the presence of a distributive law $\lambda : TS \Rightarrow ST$, the composite functor ST can again be assigned the structure of a monad. This result is generalized in [4] for longer chains of monads $T_1, \ldots, T_n$ and distributive laws between them, stated in the case $n = 3$ as follows:

▶ **Proposition 1.** *Given monads A, B, C, and distributive laws between monads $\tau : CB \Rightarrow BC$, $\sigma : CA \Rightarrow AC$, and $\lambda : BA \Rightarrow AB$, such that the axiom*

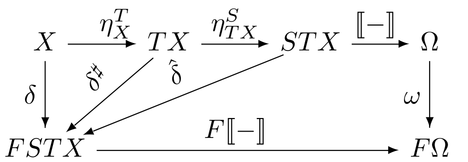
$$\lambda C \circ B\sigma \circ \tau A = A\tau \circ \sigma B \circ C\lambda$$

holds (the Yang-Baxter condition), there are distributive laws of the composite monad BC over A, and C over AB, both giving rise to the same composite monad on ABC.

The general case is then derived using an inductive argument from the case $n = 3$.

Distributive laws between monads and endofunctors and copointed endofunctors have also been widely studied, and are of importance in coalgebraic and bialgebraic methods: see e.g. [1] for a comprehensive discussion of these. (For the definitions of distributive laws between monads, and distributive laws of a monad over a (possibly copointed) endofunctor, we refer to the literature.)

We are interested in results similar to the above proposition, involving distributive laws of monads over (copointed) endofunctors. The motivation for this research arises from the diagram



(1)

an instance of which occurs in [5] with the instantiation of F as the functor for deterministic automata, S the monad for K-semimodules, and T the monad for monoids. This instantiation plays a role in the coalgebraic presentation of context-free languages and algebraic power series. It is well-known (see e.g. [3], where this is shown using quotients of distributive laws) that there is, in this instance, a distributive law of the composite monad ST over the endofunctor F, but we'd like to have a categorical formulation of the two-step determinization.

One way in which this can be done is as follows. Consider two monads (S, η^S, μ^S) and (T, η^T, μ^T) and an endofunctor F (on any category), together with:

- $\lambda^0 : TS \Rightarrow ST$, a distributive law of the monad T over the monad S.
- $\lambda^1 : TF \Rightarrow FST$, a natural transformation such that the equalities $\lambda^1 \circ \eta^T F = F\eta^{ST}$ and $\lambda^1 \circ \mu^T F = F\mu^{ST} \circ \lambda^1 T \circ T\lambda^1$ hold.
  We may regard this transformation as a (new?) type of distributive law, of a monad T over an endofunctor F *into* the composite monad ST.
- $\lambda^2 : SF \Rightarrow FS$, a distributive law of the monad S over the endofunctor F.

Such a triple of distributive laws $(\lambda^0, \lambda^1, \lambda^2)$ as above is said to satisfy the following *Yang-Baxter-like condition* whenever the following equality holds:

$$F\mu^S T \circ FS\lambda^0 \circ \lambda^1 S \circ T\lambda^2 = F\mu^S T \circ \lambda^2 ST \circ S\lambda^1 \circ \lambda^0 F$$

Conference title on which this volume is based on.
Editors: Billy Editor and Bill Editors; pp. 1–8
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Note here that the condition can be now represented as an octagonal diagram, rather than the hexagonal representation of the Yang-Baxter condition in Cheng's result.

Again, like in the case of Cheng's result, this gives rise to a composite distributive law:

▶ **Theorem 2.** *Given three distributive laws as above for monads S and T and an endofunctor F, the natural transformation $F\mu^S T \circ \lambda^2 ST \circ S\lambda^1$ is a distributive law of the composite monad ST over the endofunctor F if and only if the Yang-Baxter-like condition holds.*

When F has a final coalgebra, the resulting two-step determinization can be given as in (1), with $\delta^\sharp = F\mu^{ST}_X \circ \lambda^1_{STX} \circ T\delta$ and $\hat{\delta} = F\mu^S_{TX} \circ \lambda^2_{STX} \circ S\delta^\sharp$.

The proof (which can be simply generalized to laws over copointed functors) of this theorem is more involved than that of Cheng's result: it was shown by a Prolog program[1] that, using only naturality squares, the pentagonal diagrams for the various distributive laws, and the multiplication of the monad, a minimum amount of 23 elementary faces is required for the proof of commutativity of the most complicated diagram involved in the result. A modification of the Prolog program[2] also showed that, in the case of one of the more complicated diagrams in Cheng's result (requiring the Yang-Baxter condition), this minimum was 8, a somewhat surprising reduction from the 11 faces used in the proof in [4].[3]

Moreover, this proposition is noteworthy in the sense that the statement of the theorem gives a sufficient and necessary condition (unlike Cheng's result) for the existence of a composite distributive law.

The proof of the theorem is rather involved (and indeed, was originally proven by the aforementioned Prolog program). What thus remains to be seen, and what may be subject for further research, is the precise relationship between this result and Cheng's result, and whether this can be used for a simplification of the proof of our result (although this appears unlikely, considering the minimality found by the Prolog program). Furthermore, it may also be interesting to investigate whether, in the case of Cheng's result, a converse can be formulated similar to the case of our result.

The author would like to thank the reviewers of the (rejected) CALCO paper for various suggestions, notably the pointer to the paper [4], as well as to Henning Basold, Marcello Bonsangue, Helle Hansen, Bart Jacobs, Bartek Klin, Charles Paperman, Jurriaan Rot, Jan Rutten, and Alexandra Silva for various types of comments, suggestions, and criticism.

References

1 Falk Bartels. *On Generalized Coinduction and Probabilistic Specification Formats.* PhD thesis, Vrije Universiteit Amsterdam, 2004.

2 Jon Beck. Distributive laws. In *Seminar on triples and categorical homology theory*, pages 119–140. Springer, 1969.

3 Marcello M. Bonsangue, Helle H. Hansen, Alexander Kurz, and Jurriaan Rot. Presenting distributive laws. In Reiko Heckel and Stefan Milius, editors, *CALCO*, volume 8089 of *Lecture Notes in Computer Science*, pages 95–109. Springer, 2013.

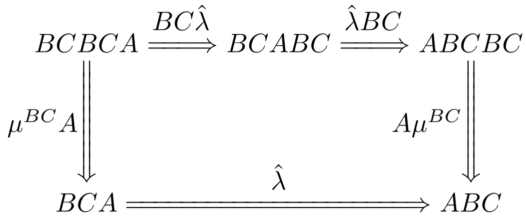
4 Eugenia Cheng. Iterated distributive laws. `http://arxiv.org/pdf/0710.1120v1.pdf`, 2007.

5 Joost Winter. *Coalgebraic Characterizations of Automata-Theoretic Classes.* PhD thesis, Radboud Universiteit Nijmegen, 2014.

[1] available at `http://www.mimuw.edu.pl/~jwinter/distlaws.prolog`
[2] also available at `http://www.mimuw.edu.pl/~jwinter/distlaws2.prolog`
[3] Both proofs are provided in full in the appendix of this note in human-readable form.

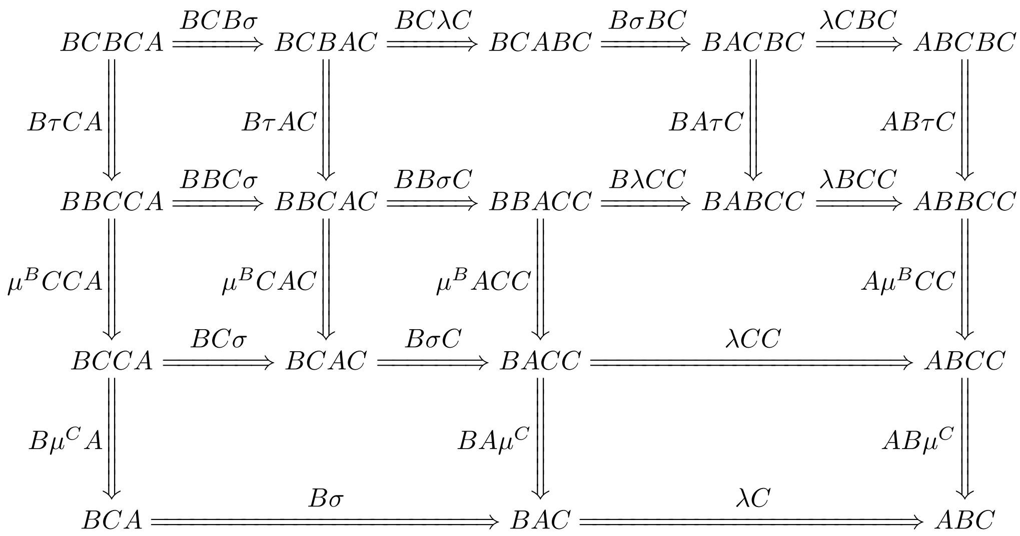
A Minimal form of one of the diagrams for the proof of Proposition 1 ([4, Theorem 2.1])

We give a minimal form of the proof of the commutativity of the diagram



where $\hat{\lambda} = \lambda C \circ B\sigma$ and $\mu^{BC} = B\mu^C \circ \mu^B CC \circ B\tau C$, the multiplication of the composite monad BC.

The Prolog program found the following diagram as a proof of this equality:



Note that, although this proof is shorter (and, indeed minimal among diagrammatic proofs involving only naturality squares, the multiplication of the monads, the pentagonal diagrams for the distributive laws, and the Yang-Baxter condition), it is not necessarily 'better' (or even significantly different) from the earlier proof, because the proof given in [4] is more symmetric. It is still fascinating that the Prolog program somehow found a shorter version.

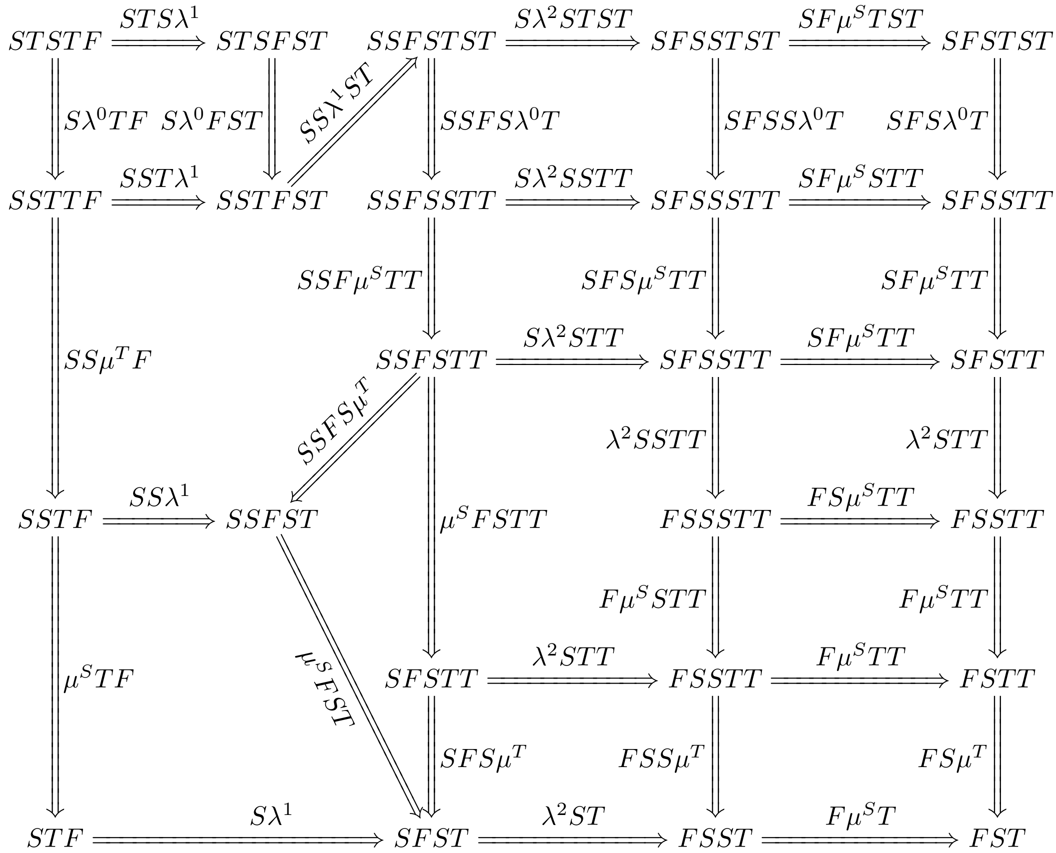
We count the proof in [4] as having 11 faces (instead of the 12 which it appears to have) because, on the left, the decomposition $\mu^B\mu^C = B\mu^C \circ \mu^B CC$ is used, whereas on the right, the decomposition $\mu^B\mu^C = \mu^B C \circ BB\mu^C$ is used. We implicitly assume the decomposition $\mu^B\mu^C = B\mu^C \circ \mu^B CC$—making this decomposition explicit in both cases would still mean a reduction from 12 to 10.

B Proof of Theorem 2

Note that one direction of the proof can also be given as a large commuting diagram involving 23 different faces, however, for the sake of presentation we split this diagram up into two diagrams which are then 'glued together'.

Proof. *(of Theorem 2)*

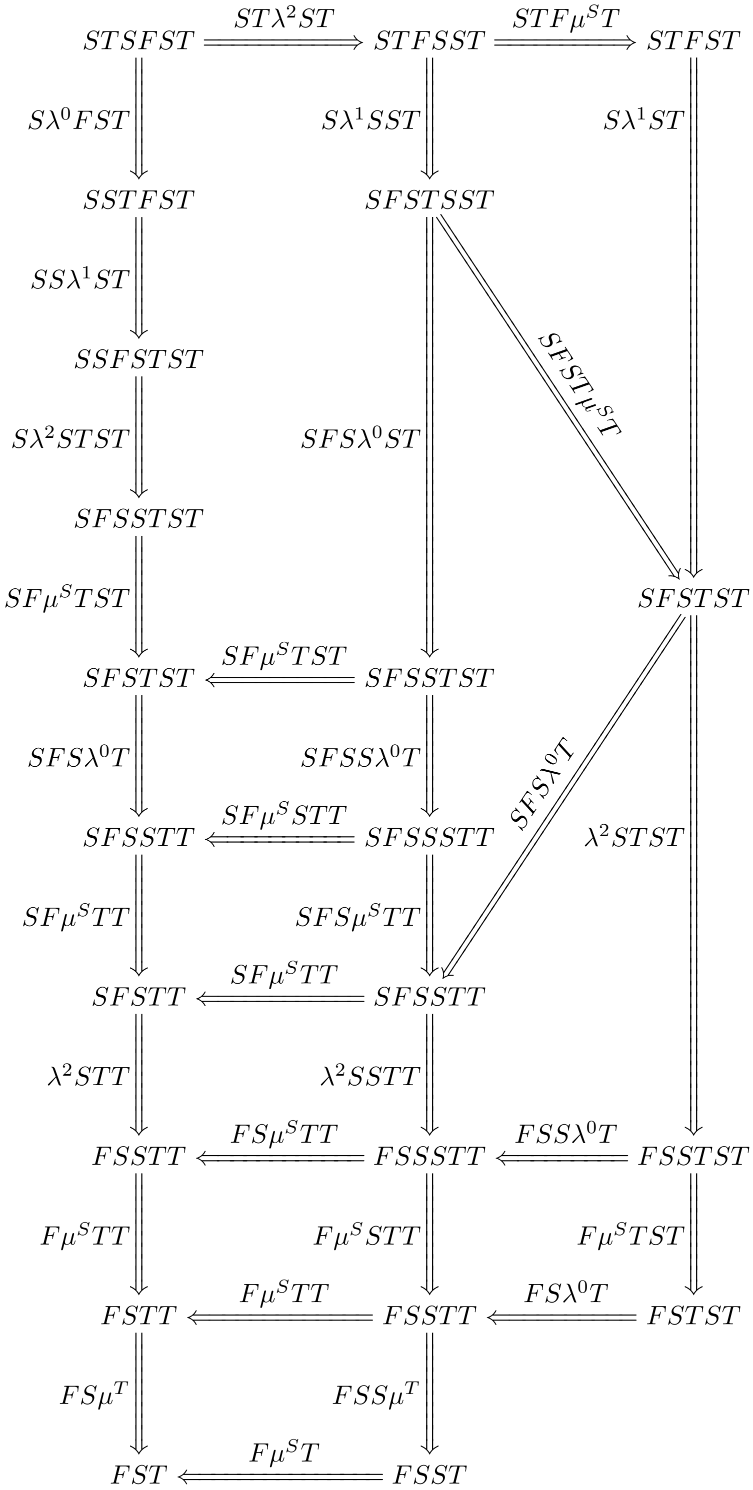
First, assume that λ^1, λ^2 and λ^0 satisfy the Yang-Baxter-like condition. Then the following diagram commutes (the commutativity of the combination of this and the next diagram was proved by the Prolog program):



and thus we have:

$$
\begin{aligned}
& FS\mu^T \circ F\mu^S TT \circ \lambda^2 STT \circ SF\mu^S TT \circ SFS\lambda^0 T \circ SF\mu^S TST \circ \\
& \qquad S\lambda^2 STST \circ SS\lambda^1 ST \circ S\lambda^0 FST \circ STS\lambda^1 \\
= {} & F\mu^S T \circ \lambda^2 ST \circ S\lambda^1 \circ \mu^S TF \circ SS\mu^T F \circ S\lambda^0 TF
\end{aligned}
$$

Next, observe the following diagram also commutes,



giving:

$$FS\mu^T \circ F\mu^S TT \circ \lambda^2 STT \circ SF\mu^S TT \circ SFS\lambda^0 T \circ SF\mu^S TST \circ$$
$$S\lambda^2 STST \circ SS\lambda^1 ST \circ S\lambda^0 FST$$

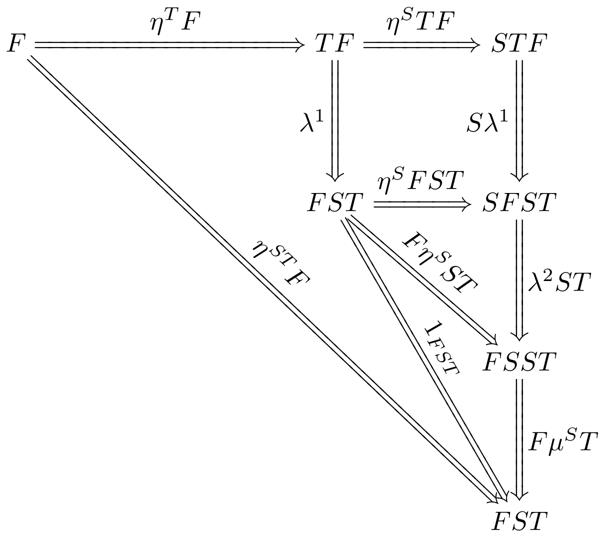
$$= F\mu^S T \circ FSS\mu^T \circ FS\lambda^0 T \circ F\mu^S TST \circ \lambda^2 STST \circ S\lambda^1 ST \circ STF\mu^S T \circ ST\lambda^2 ST$$

Combining this, we get

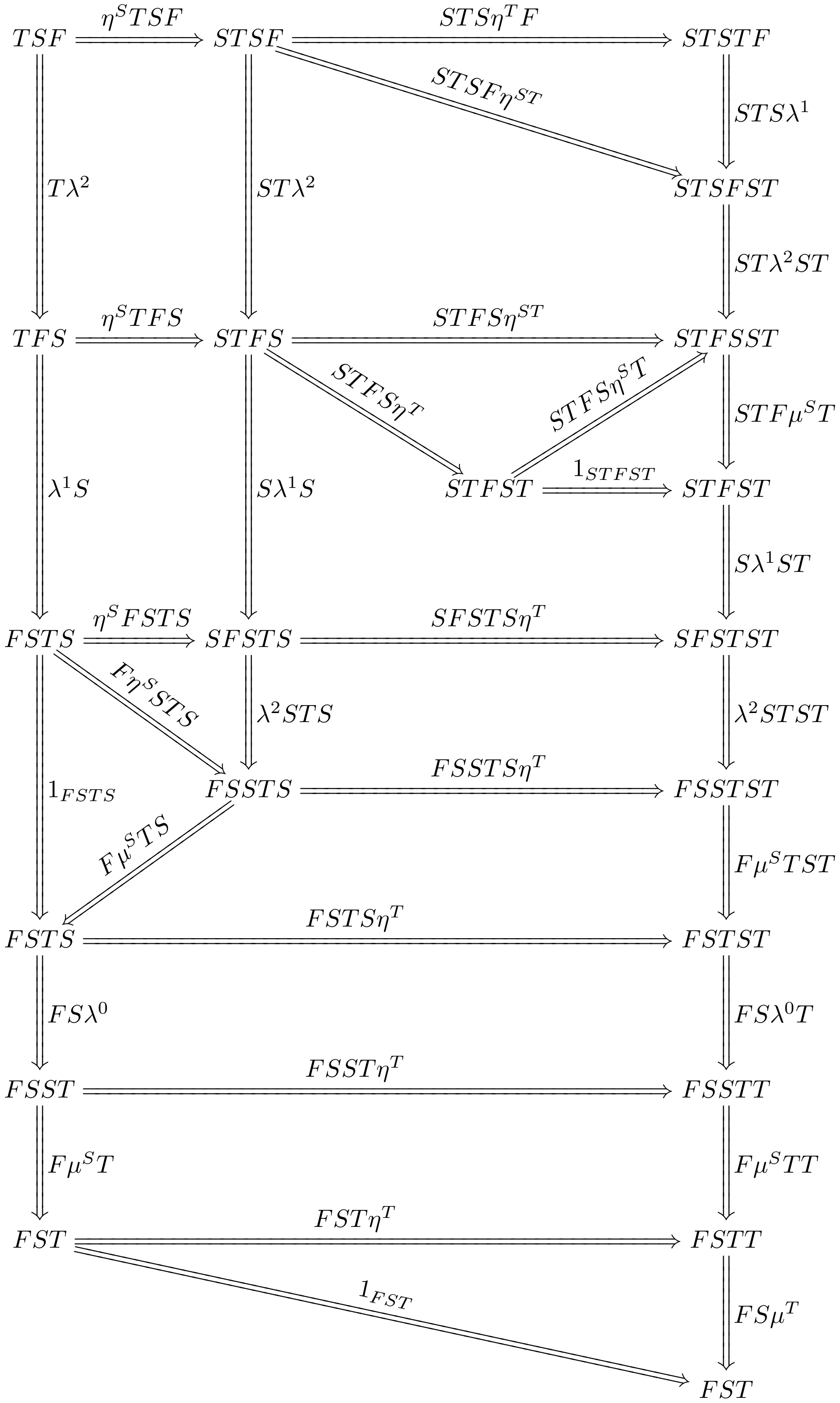
$$\begin{aligned}
&\hat{\lambda} \circ \mu^{ST} F \\
&= F\mu^S T \circ \lambda^2 ST \circ S\lambda^1 \circ \mu^S TF \circ SS\mu^T F \circ S\lambda^0 TF \\
&= FS\mu^T \circ F\mu^S TT \circ \lambda^2 STT \circ SF\mu^S TT \circ SFS\lambda^0 T \circ SF\mu^S TST \circ \\
&\qquad\quad S\lambda^2 STST \circ SS\lambda^1 ST \circ S\lambda^0 FST \circ STS\lambda^1 \\
&= F\mu^S T \circ FSS\mu^T \circ FS\lambda^0 T \circ F\mu^S TST \circ \lambda^2 STST \circ S\lambda^1 ST \circ STF\mu^S T \circ ST\lambda^2 ST \\
&= F\mu^{ST} \circ \hat{\lambda} ST \circ ST\hat{\lambda}
\end{aligned}$$

so the main diagram for the distributive law $\hat{\lambda}$ commutes.

Next, we check that the unit diagram commutes. This follows from the following diagram:



This concludes one direction of the proof. For the converse, consider the following diagram,

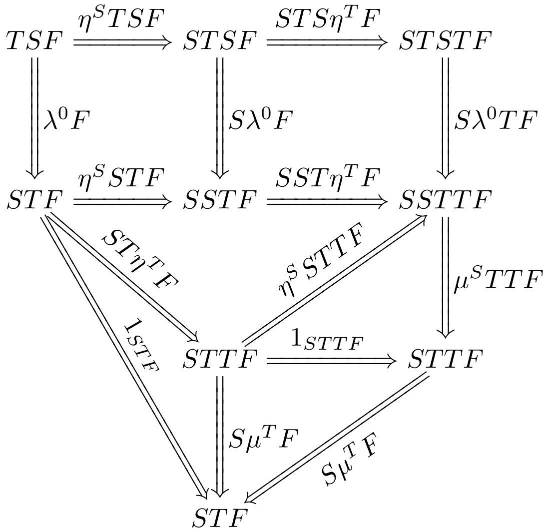


which establishes that

$$
\begin{aligned}
& FS\mu^{ST} \circ \hat{\lambda}ST \circ ST\hat{\lambda} \circ STS\eta^T F \circ \eta^S TSF \\
= & FS\mu^T \circ F\mu^S TT \circ FS\lambda^0 T \circ F\mu^S TST \circ \lambda^2 STST \circ S\lambda^1 ST \circ
\end{aligned}
$$

$$
\begin{aligned}
&STF\mu^S T \circ ST\lambda^2 ST \circ STS\lambda^1 \circ STS\eta^T F \circ \eta^S TSF \\
&= F\mu^S T \circ FS\lambda^0 \circ \lambda^1 S \circ T\lambda^2
\end{aligned}
$$

and the following diagram,



which establishes that:

$$
\begin{aligned}
&S\mu^{ST} F \circ STS\eta^T F \circ \eta^S TSF \\
&= S\mu^T F \circ \mu^S TTF \circ S\lambda^0 TF \circ STS\eta^T F \circ \eta^S TSF \\
&= \lambda^0 F
\end{aligned}
$$

Combining this, we obtain

$$
\begin{aligned}
&F\mu^S T \circ FS\lambda^0 \circ \lambda^1 S \circ T\lambda^2 \\
&= FS\mu^{ST} \circ \hat{\lambda} ST \circ ST\hat{\lambda} \circ STS\eta^T F \circ \eta^S TSF \\
&= \hat{\lambda} \circ FS\mu^{ST} \circ STS\eta^T F \circ \eta^S TSF \\
&= \hat{\lambda} \circ \lambda^0 F
\end{aligned}
$$

completing the proof. ◀