Hitting minors, subdivisions, and immersions in tournaments

Jean-Florent Raymond*[a]

[a] *Institute of Computer Science, University of Warsaw, Poland, and LIRMM, University of Montpellier, France.*

Abstract

The Erdős–Pósa property usually relates parameters of covering and packing of combinatorial structures, and has been mostly studied in the setting of undirected graphs. In this note, we use results of Chudnovsky, Fradkin, Kim, and Seymour to show that for every directed graph H, the class of all minor-expansions (resp. subdivisions) of H has the vertex-Erdős–Pósa property in the class of tournaments. We also prove that if H is a strongly connected directed graph, the class of all immersion-expansions of H has the edge-Erdős–Pósa property in the class of tournaments. Our results are orthogonal to the recent results of Amiri et al. [arXiv:1603.02504, March 2016] in the sense that we restrict the class of "host graphs", whereas they restrict the class of "guest graphs".

Keywords: directed Erdős–Pósa property, packing and covering, topological minors, immersions, tournaments.

1 Introduction

In this note we are concerned with the Erdős–Pósa property in the setting of directed graphs. This property, which has mostly been studied on undirected graphs, is originated from the following classic result.

Theorem 1 (Erdős–Pósa Theorem, [EP65])**.** *There is a function $f\colon \mathbb{N} \to \mathbb{N}$, $f(k) = O(k \log k)$ such that for every (undirected) graph G and every positive integer k, one of the following holds:*

- *G contains k vertex-disjoint cycles; or*

*Supported by the grant PRELUDIUM 2013/11/N/ST6/02706 of the Polish National Science Center (NCN). Email: jean-florent.raymond@mimuw.edu.pl

- *there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ and such that $G \setminus X$ has no cycle.*

This theorem expresses a duality between a parameter of *packing*, the maximum number of vertex-disjoint cycles in a graph, and a parameter of *covering*, the minimum number of vertices, the removal of which yields a forest. Informally, we say that a class has the *Erdős–Pósa property* when such a result holds for graphs in this class. The Erdős–Pósa Theorem states that the class of cycles has this property. Since then, several results appeared, stating that the same type of relation holds for various classes of graphs (see surveys [Ree97, RT16]). In particular, attention has been drawn by classes defined using containment relations, like the minor relation. A notable extension of the Erdős–Pósa Theorem has been obtained as a byproduct of Graph Minors in [RS86].

Theorem 2 ([RS86]). *Let H be an undirected planar graph. There is a function $f\colon \mathbb{N} \to \mathbb{N}$, such that for every undirected graph G and every positive integer k, one of the following holds:*

- *G has k vertex-disjoint subgraphs, each having H as a minor; or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ and such that $G \setminus X$ does not have H as a minor.*

In the setting of directed graphs however, a few results are known. Until recently, the largest class of digraphs that has been studied under the prism of the Erdős–Pósa property is the class of directed cycles [RRST96, RS96, GT10, Sey96, HM13]. We should also mention that the directed Erdős–Pósa property, besides its combinatorial interest, has applications in bioinformatics and in the study of Boolean networks [ARS16, ADG04]. The most general result about the directed Erdős–Pósa property is certainly the following extension of Theorem 2 to directed graphs, that appeared recently.

Theorem 3 ([AKKW16]). *Let H be a strongly connected digraph that is a butterfly-minor[1] (resp. topological minor[2]) of a cylindrical grid. There is a function $f\colon \mathbb{N} \to \mathbb{N}$, such that for every digraph G and every positive integer k, one of the following holds:*

- *G has k vertex-disjoint subdigraphs, each having H as a butterfly-minor (resp. topological minor); or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ does not have H as a butterfly-minor (resp. topological minor).*

The purpose of this note is to shed some light on the symmetries between directed and undirected graphs wrt. the Erdős–Pósa property by obtaining new Erdős–Pósa type results on directed graphs, using techniques that have been successfully applied

[1]Butterfly-minors are one of the possible extensions of the concept of minors to directed graphs; we omit the precise definition as we will not use it.

[2]This notion will be defined in the forthcoming paragraph.

to undirected graphs. Following [KS15], we say that a digraph H is a *minor* of a graph G if a digraph isomorphic to H can be obtained form G by repeatedly contracting a strongly connected component to a single vertex (this definition differs from the one of butterfly-minor mentioned above). Unlike minors, immersions and topological minors are concepts that are easily extended to the setting of directed graphs as they can be defined in terms of paths. We say that a digraph H is a *topological minor* of a digraph G if there is a subdigraph of G that can be obtained from a digraph isomorphic to H by replacing arcs by vertex-disjoint directed paths. If we allow these paths to share internal vertices but not arcs, then we say that H is an *immersion* of G.

Our results hold on superclasses of the well-studied class of tournaments, which consist of all orientations of undirected complete graphs. For $s \in \mathbb{N}$, a digraph is said to be *s-semicomplete* if for every vertex v there are at most s vertices that are not connected to v by an arc (in either direction). A *semicomplete* digraph is a 0-semicomplete digraph. It is not hard to see that these classes generalize the class of tournaments. Our contributions are the following two theorems.

Theorem 4. *For every family $\mathcal{H}$ of strongly connected digraphs not containing K_1, there is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every semicomplete G and every positive integer k, one of the following holds:*

- *G contains k arc-disjoint subdigraphs, each having a digraph of $\mathcal{H}$ as an immersion; or*
- *there is a set $X \subseteq E(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ does not have a digraph of $\mathcal{H}$ as an immersion.*

Theorem 5. *For every finite family $\mathcal{H}$ of digraphs and every $s \in \mathbb{N}$, there is a function $f: \mathbb{N} \to \mathbb{N}$ such that for every s-semicomplete digraph G and every positive integer k, one of the following holds:*

- *G contains k vertex-disjoint subdigraphs, each having a digraph of $\mathcal{H}$ as a minor (resp. topological minor); or*
- *there is a set $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $G \setminus X$ does not have a digraph of $\mathcal{H}$ as a minor (resp. topological minor).*

These theorems deal with two different versions of the Erdős–Pósa property: the first one is related to arc-disjoint subdigraphs and sets of arcs (arc version), whereas the second one is concerned with vertex-disjoint subdigraphs and sets of vertices (vertex version). In Theorem 4, the requirement $K_1 \notin \mathcal{H}$ is necessary as we cannot cover an edgless subgraph (as the one-vertex graph) with arcs. Observe that our results differ from the aforementioned results of [AKKW16] in the sense that we restrict the hosts class (graphs where the packings/coverings are done) whereas they restrict the guests class (graphs that are packed/covered). These two orthogonal lines of research have already been followed in the context of undirected graphs, see e.g. [FST11]. Our proofs rely on the following results (c.f. Section 2 for a definition of cutwidth and pathwidth).

Theorem 6 ([CFS12, (1.2)]). *For every digraph H, there is a positive integer $\omega_H^{\mathcal{I}}$ such that every semicomplete G that has cutwidth more than $\omega_H^{\mathcal{I}}$ contains H as an immersion.*

Theorem 7 ([KS15, (1.4)]). *For every digraph H, there is a positive integer $\omega_H^{\mathcal{M}}$ such that every semicomplete G that has pathwidth more than $\omega_H^{\mathcal{M}}$ contains H as a minor.*

Theorem 8 ([FS13, (1.1)]). *For every digraph H, there is a positive integer $\omega_H^{\mathcal{T}}$ such that every semicomplete G that has pathwidth more than $\omega_H^{\mathcal{T}}$ contains H as a topological minor.*

Theorem 9 ([KKT15, Theorem 2]). *There is a function $h\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that every s-semicomplete digraph with pathwidth at least $h(s,k)$ has a subdigraph that is semicomplete and has of pathwidth at least k.*

Theorem 6, Theorem 7, and Theorem 8 are *exclusion theorems*: they relate a graph parameter (in this case, cutwidth or pathwidth) with the absence of a digraph as a substructure. Exclusion theorems have proven useful in order to obtain Erdős–Pósa type results for undirected graphs. The upper-bound on the parameter that they provide is in general directly used to find a small hitting set.

Organization of the paper. Definitions and notation are introduced in Section 2. In Section 3 we prove two lemmas, which are then directly used in the proofs of Theorem 4 and Theorem 5 (Section 4). Section 5 contains directions for future research.

2 Preliminaries

For every $i, j \in \mathbb{N}$ we denote by $[\![i, j]\!]$ the interval of integers $\{k \in \mathbb{N},\ i \leq k \leq j\}$.

Digraphs. A digraph is a pair (V, E), where V is a set, the elements of which are called *vertices*, and E is a multiset, the elements of which (called *arcs*) belong to $V \times V$. All the digraphs we consider in this paper are finite (i.e. both V and E are finite), and may have loops or multiple arcs. We denote by $V(G)$ the set of vertices of a digraph G and by $E(G)$ its multiset of arcs. A digraph is *nontrivial* if it has more than one vertex. If $\mathcal{H}$ is a digraph class, an *$\mathcal{H}$-subdigraph* of a digraph G is a subgraph of G that is isomorphic to some digraph in $\mathcal{H}$. For every two subsets $X, Y \subseteq V(G)$, we denote by $E_G(X, Y)$ the set of arcs of G of the form (x, y) with $x \in X$ and $y \in Y$. For every digraph G we denote by $\mathbf{cc}(G)$ the number of connected components of G. A digraph is said to be strongly connected if is has at least one vertex and for every $u, v \in V(G)$ there is a directed path from u to v. A strongly connected component of a digraph is a maximal subdigraph that is strongly connected. If G and H are two graphs we denote by $H \cup G$ their union and by $H \uplus G$ their disjoint union.

Hitting sets and packings. Given a digraph class $\mathcal{H}$, an *$\mathcal{H}$-vertex-hitting set* (resp. *$\mathcal{H}$-arc-hitting set*) of a digraph G is a subset X of $V(G)$ (resp. $E(G)$) such that $G \setminus X$ has no $\mathcal{H}$-subdigraph. An *$\mathcal{H}$-vertex-packing* (resp. *$\mathcal{H}$-arc-packing*) is a collection of vertex-disjoint (resp. arc-disjoint) $\mathcal{H}$-subdigraphs of G. For a digraph G, we denote by $\tau^{\mathsf{v}}_{\mathcal{H}}(G)$ (resp. $\tau^{\mathsf{e}}_{\mathcal{H}}(G)$) the minimum size of an $\mathcal{H}$-vertex-hitting set (resp. $\mathcal{H}$-arc-hitting set) of G and by $\nu^{\mathsf{v}}_{\mathcal{H}}(G)$ (resp. $\nu^{\mathsf{e}}_{\mathcal{H}}(G)$) the maximum size of an $\mathcal{H}$-vertex-packing (resp. $\mathcal{H}$-arc-packing) of G

Models, subdivisions, and expansions. The definitions of minors, topological minors and immersions have been given in the introduction. Notice that the immersions we deal with are what is sometimes called strong immersions. For every digraph H, we denote by $\mathcal{M}(H)$ (resp. $\mathcal{T}(H)$, $\mathcal{I}(H)$) the class of all subdigraph-minimal digraphs containing H as a minor (resp. topological minor, immersion). Notice that H is a minor (resp. topological minor, immersion) of a digraph G iff G has an $\mathcal{M}(H)$-subdigraph (resp. $\mathcal{T}(H)$-subdigraph, $\mathcal{I}(H)$-subdigraph). Digraphs in $\mathcal{M}(H)$ (resp. $\mathcal{T}(H)$, $\mathcal{I}(H)$) will be called *models* (resp. *subdivisions*, *expansions*) of H. If $\mathcal{H}$ is a class, then we also set $\mathcal{M}(\mathcal{H}) = \bigcup_{H \in \mathcal{H}} \mathcal{M}(H)$ (resp. $\mathcal{T}(\mathcal{H}) = \bigcup_{H \in \mathcal{H}} \mathcal{T}(H)$).

Observe that, according to definition we use, every topological minor of G is an immersion of G. However, this simple remark does not allow us to straightforwardly extract an Erdős–Pósa-type result about the one relation from a result about the other one. Indeed, if any packing wrt. topological minors is a packing wrt. immersions, a hitting set wrt. topological minors is not necessarily a hitting set wrt. immersions, and vice-versa.

Let us now define the parameters cutwidth and pathwidth, that play main roles in the proofs of Theorem 4 and Theorem 5.

Cutwidth. Let $n \in \mathbb{N}$ and let G be a digraph on n vertices. A *enumeration* of the vertices of G is a bijection $f \colon [\![1, n]\!] \to V(G)$. The *cutwidth* of an enumeration f of the vertices of G is equal to $\max_{i \in [\![2,n]\!]} |E_G(\{f(1), \ldots, f(i-1)\}, \{f(i), \ldots, f(n)\})|$. The *cutwidth* of G, that we write $\mathbf{ctw}(G)$, is the minimum cutwidth over all enumerations of $V(G)$.

Pathwidth. A *path-decomposition* of a digraph G is a sequence $(X_1, \ldots, X_r)$ (for some $r \in \mathbb{N}$) satisfying the following properties:

(i) $V(G) = \bigcup_{i=1}^{r} X_i$

(ii) for every arc $(u, v) \in E(G)$, there are integers $i \in [\![1, r]\!]$ and $j \in [\![1, i]\!]$ such that $u \in X_i$ and $v \in X_j$;

(iii) for every $i, j \in [\![1, r]\!]$, if a vertex $u \in V(G)$ belongs to X_i and X_j, then it also belongs to X_k for every $k \in [\![i, j]\!]$.

The sets $\{X_i\}_{i\in[\![1,r]\!]}$ are called *bags* of the path-decomposition. Intuitively, item (ii) asks that every arc of G either have its endpoints in some bag, or is oriented "backwards".

The *width* of a path-decomposition $(X_1,\ldots,X_r)$ of a digraph G is defined as $\max_{i\in[\![1,r]\!]}|X_i|-1$. The *pathwidth* of G is the minimum width over all path-decompositions of G.

Separations. A $\mathcal{H}$*-arc-separation* of a digraph G is a triple (A,X,B) of disjoint subsets of $E(G)$ such that every $\mathcal{H}$-subdigraph of $G\setminus X$ has its arcs in exactly one of A and B. In particular, we do not require that (A,X,B) is a partition of $E(G)$.

3 Hitting sets in digraphs of bounded width

Let $\mathcal{H}$ be a class of strongly connected digraphs. In this section, we show that when the cutwidth (resp. pathwidth) of a digraph class is bounded, then the size of a minimum $\mathcal{H}$-hitting set is linearly bounded by the maximum size of an $\mathcal{H}$-packing. More precisely, we show the following lemmas that, together with exclusion theorems, will be used to prove the main results.

Lemma 1. *For every class $\mathcal{H}$ of nontrivial strongly connected digraphs, every digraph G has an $\mathcal{H}$-arc-hitting set of size at most $\nu^{\mathsf{e}}_{\mathcal{H}}(G)\cdot\mathbf{ctw}(G)$.*

The following notation allows us to state the forthcoming Lemma 2 in a general way so that it can be used in the proof of both variants of Theorem 5 (minors and topological minors).

If $\mathcal{H}_1,\ldots,\mathcal{H}_p$ are classes of digraphs, we denote by $\Cup_{i=1}^{p}\mathcal{H}_i$ the class of all digraphs of the form $H_1\uplus\cdots\uplus H_p$ with $H_i\in\mathcal{H}_i$, for every $i\in[\![1,p]\!]$.

Lemma 2. *Let $\mathcal{H}_1,\ldots,\mathcal{H}_p$ be classes of connected digraphs and let $\mathcal{H}=\Cup_{i=1}^{p}\mathcal{H}_i$. Then every digraph G has an $\mathcal{H}$-vertex-hitting set of size at most $p\cdot(\nu^{\mathsf{v}}_{\mathcal{H}}(G)+1)(\mathbf{pw}(G)+1)$.*

A slight extension of Lemma 2 is given by Corollary 1. Even if the statements of the aforementioned lemmas are similar, their proofs carry sensible differences. First, the technique used to prove Lemma 1 rely on the crucial concept of $\mathcal{H}$-arc-separations. Also, contrarily to Lemma 1, Lemma 2 deals with disconnected digraphs H and the technique used to handle this case is different from the one we would use in the connected case.

Proof of Lemma 1. We will prove by induction on $k\in\mathbb{N}$ the following statement: for every digraph G such that $\nu^{\mathsf{e}}_{\mathcal{H}}(G)\leq k$, it holds that $\tau^{\mathsf{e}}_{\mathcal{H}}(G)\leq k\cdot\mathbf{ctw}(G)$.

The base case $k=0$ is trivial. Let us prove the above statement for $k>0$ assuming that it holds for all lower values of k (induction step). Let G be a digraph such that $\nu^{\mathsf{e}}_{\mathcal{H}}(G)\leq k$, let $n=|V(G)|$, and let $f\colon[\![1,n]\!]\to V(G)$ be a enumeration of $V(G)$ of minimum width. We use L_i (as *left of* i) and R_i (as *right of* i) as shorthands for $G\left[\{f(j)\}_{j\in[\![1,i]\!]}\right]$ and $G\left[\{f(j)\}_{j\in[\![i+1,n]\!]}\right]$, respectively, for every $i\in[\![0,n]\!]$.

Observation 1. *For every $i \in [\![0,n]\!]$, $(E(L_i), E_G(V(L_i), V(R_i)), E(R_i))$ is an $\mathcal{H}$-arc-separation.*

Indeed, in $G \setminus E_G(V(L_i), V(R_i))$, there is no arc going from a vertex of L_i to a vertex of R_i (though there may be arcs from R_i to L_i). Therefore not strongly connected subdigraph of G, and in particular no $\mathcal{H}$-subdigraph of G, can contain arcs from both L_i and R_i. ◇

For every $i \in [\![0,n]\!]$, we set $p(i) = \nu^{\mathsf{e}}_{\mathcal{H}}(L_i)$. Observe that p is non-decreasing, from $p(0) = 0$ to $p(n) = \nu^{\mathsf{e}}_{\mathcal{H}}(G)$. Let $i \in \mathbb{N}$ be the maximum integer such that $p(i)$ is equal to zero, and let $X = E_G(L_i, R_i)$.

This definition implies that L_i has no $\mathcal{H}$-subdigraph. Therefore, as $(E(L_i), X, E(R_i))$ is an $\mathcal{H}$-arc-separation (cf. Observation 1), every $\mathcal{H}$-subdigraph of $G \setminus X$ belongs to R_i. Consequently, if Y is an $\mathcal{H}$-hitting set of R_i, then $X \cup Y$ is an $\mathcal{H}$-hitting set of G. Hence we can write:

$$\tau^{\mathsf{e}}_{\mathcal{H}}(G) \leq \tau^{\mathsf{e}}_{\mathcal{H}}(R_i) + |X|. \tag{1}$$

Since L_{i+1}, which is arc-disjoint from R_i, contains an $\mathcal{H}$-subdigraph (by definition of i), we have $\nu^{\mathsf{e}}_{\mathcal{H}}(R_i) \leq k-1$. Therefore we can apply the induction hypothesis on R_i and get $\tau^{\mathsf{e}}_{\mathcal{H}}(R_i) \leq (k-1)\,\mathbf{ctw}(R_i)$. Also we have $\mathbf{ctw}(R_i) \leq \mathbf{ctw}(G)$ and $|X| \leq \mathbf{ctw}(G)$. Together with (1) these facts imply

$$\tau^{\mathsf{e}}_{\mathcal{H}}(G) \leq k \cdot \mathbf{ctw}(G),$$

as required. This proves the induction step and concludes the proof. □

In the proof of Lemma 2 we use the two following lemmas.

Lemma 3 (see [RS86, (8.6)])**.** *Let P be a path, let $p, k \in \mathbb{N}$, and let $\{\mathcal{P}_i\}_{i \in [\![1,p]\!]}$ be families of subpaths of P. If, for every $i \in [\![1,p]\!]$, there are at least pk elements of $\mathcal{P}_i$ that are pairwise vertex-disjoint, then for every $i \in [\![1,p]\!]$ there is a collection $\mathcal{C}_i \subseteq \mathcal{P}_i$ of size k such that the elements of $\bigcup_{i=1}^{p} \mathcal{C}_i$ are pairwise vertex-disjoint.*

The following is a well-known result on subpaths of a path, that is, in some sense, an Erdős–Pósa type result.

Lemma 4 (Folklore)**.** *Let P be a path (undirected) and let $\mathcal{P}$ be a family of subpaths of P. Then for every $k \in \mathbb{N}$, one of the following holds:*

- *$\mathcal{P}$ has k vertex-disjoint elements; or*
- *there is set $X \subseteq V(P)$ with $|X| \leq k-1$ such that no subgraph of $P \setminus X$ belongs to $\mathcal{P}$.*

The proof of Lemma 2 follows the steps of [RS86, (8.8)] (see also [FJW13, Lemma 2.3]) and we include it for completeness.

Proof of Lemma 2. As in the proof of Lemma 1 we proceed by induction. Let us show by induction on $k \in \mathbb{N}$ the following statement: for every digraph G such that $\nu^{\mathsf{v}}_{\mathcal{H}}(G) \leq k$, it holds that $\tau^{\mathsf{v}}_{\mathcal{H}}(G) \leq p \cdot (\nu^{\mathsf{v}}_{\mathcal{H}}(G) + 1)(\mathbf{pw}(G) + 1)$.

The base case $k = 0$ is trivial. Let us prove the above statement for $k > 0$ assuming that it holds for all lower values of k (induction step). For this we consider a digraph G such that $\nu^{\mathsf{v}}_{\mathcal{H}}(G) \leq k$. Let $(X_1, \dots, X_l)$ be a path decomposition of G (for some $l \in \mathbb{N}$) of minimum width. Let P be the undirected path on vertices $v_1, \dots, v_l$, in this order.

For every subdigraph F of G, we set $A_F = \{i \in [\![1, l]\!],\ V(F) \cap X_i \neq \emptyset\}$ and $\mathcal{P}(F) = P[\{v_i,\ i \in A_F\}]$. In other words, A_F is the set of indices of the bags met by F and $\mathcal{P}(F)$ is the subgraph of P induced by the vertices with these indices. Observe that, due to the properties of path decompositions, $\mathcal{P}(F)$ is a path. Also, for every pair F, F' of subdigraphs of G, if $\mathcal{P}(F)$ and $\mathcal{P}(F')$ are vertex-disjoint, then so are F and F'.

For every $i \in [\![1, p]\!]$, we consider the family $\mathcal{F}_i$ of all $\mathcal{H}_i$-subdigraphs of G and we set $\mathcal{P}_i = \{\mathcal{P}(F),\ F \in \mathcal{F}_i\}$. If for every $i \in [\![1, p]\!]$, the family $\mathcal{P}_i$ contains $(k+1)p$ elements (i.e. subpaths of P) that are vertex-disjoint, then, according to Lemma 3, there are, for every $i \in [\![1, p]\!]$, some $\mathcal{H}_i$-subdigraphs $F^i_1, \dots, F^i_{k+1}$ of G such that the elements of $\{\mathcal{P}(F^i_j)\}_{i \in [\![1,p]\!], j \in [\![1,k+1]\!]}$ are pairwise vertex-disjoint. According to the observation above, the digraphs $\{F^i_j\}_{i \in [\![1,p]\!], j \in [\![1,k+1]\!]}$ are vertex-disjoint as well, hence so are the digraphs $\{F^1_i \cup \dots \cup F^p_i\}_{i \in [\![1,k+1]\!]}$. Observe that these digraphs are $\mathcal{H}$-subdigraphs of G. Consequently $\nu^{\mathsf{v}}_{\mathcal{H}}(G) \geq k + 1$, a contradiction.

Therefore we now assume that for some $i \in [\![i, p]\!]$, the family $\mathcal{P}_i$ does not contain $(k+1)p$ vertex-disjoint elements. Then, according to Lemma 4, there is a set $L \subseteq V(P)$ with $|L| \leq (k+1)p-1$ such that $P \setminus L$ does not have any subgraph in $\mathcal{P}_i$. Consequently, by removing form G the vertices of $X = \bigcup_{j \in \{j' \in [\![1,l]\!],\ v_{j'} \in L\}} X_j$ we obtain a digraph with no $\mathcal{H}_i$-subdigraph. Observe that this graph does not contain a $\mathcal{H}$-subdigraph neither since by definition, every graph of $\mathcal{H}$ has one connected component that belongs to $\mathcal{H}_i$. Thus X is a H-hitting set. It is easy to see that $|X| \leq |L| \cdot \max_{i \in [\![1,l]\!]} |X_i| \leq ((k + 1)p - 1)(\mathbf{pw}(G) + 1)$. This completes the induction step and concludes the proof. □

Corollary 1. *Let $q \in \mathbb{N}$, let $p_1, \dots, p_q \in \mathbb{N}$, let $\{\mathcal{H}^j_i\}_{i \in [\![1,q]\!], j \in [\![1,p_i]\!]}$ be classes of connected digraphs, let $\mathcal{H}_i = \Cup_{j=1}^{p_i} \mathcal{H}^j_i$, and let $\mathcal{H} = \bigcup_{i=1}^{q} \mathcal{H}_i$. Then every digraph G has an $\mathcal{H}$-vertex-hitting set of size at most $qp \cdot (\nu^{\mathsf{v}}_{\mathcal{T}(\mathcal{H})}(G) + 1)(\mathbf{pw}(G) + 1)$, where $p = \max_{i \in [\![1,q]\!]} p_i$.*

Proof. Let $k = \nu^{\mathsf{v}}_{\mathcal{H}}(G)$. Notice that for every $i \in [\![1, q]\!]$ it holds that $\nu^{\mathsf{v}}_{\mathcal{H}_i}(G) \leq k$. According to Lemma 2, $\tau^{\mathsf{v}}_{\mathcal{H}_i}(G) \leq p \cdot (\nu^{\mathsf{v}}_{\mathcal{H}_i}(G) + 1)(\mathbf{pw}(G) + 1)$. Besides, notice that $\tau^{\mathsf{v}}_{\mathcal{H}}(G) \leq \sum_{i \in [\![1,q]\!]} \tau^{\mathsf{v}}_{\mathcal{H}_i}(G)$. We deduce the desired inequality form these two facts.

□

4 Proofs of Theorem 4 and Theorem 5

Using the lemmas of Section 3, the proofs are now straightforward.

Proof of Theorem 4. Let H be some digraph of $\mathcal{H}$. Let $k \cdot H$ denote the disjoint union of k copies of H, and let $f(k) = (k-1) \cdot \omega^{\mathcal{I}}_{k \cdot H}$ for every $k \in \mathbb{N}$. Let us show that this function satisfies the statement of Theorem 4. Let G be a semicomplete and let $k \in \mathbb{N}$. Let us assume that $\nu^{\mathsf{e}}_{\mathcal{I}(\mathcal{H})}(G) < k$, otherwise we are done. Obviously $\mathbf{ctw}(G) \leq \omega^{\mathcal{I}}_{k \cdot H}$. As every digraph in $\mathcal{I}(\mathcal{H})$ is strongly connected and nontrivial, we can apply Lemma 1 and obtain an $\mathcal{I}(\mathcal{H})$-arc-hitting set of G size at most $\nu^{\mathsf{e}}_{\mathcal{I}(\mathcal{H})}(G)\, \mathbf{ctw}(G) \leq (k-1) \cdot \omega^{\mathcal{I}}_{k \cdot H} = f(k)$. □

Proof of Theorem 5. We only give the proof for the minor version as the topological minor one is identical (replace every occurrence of $\mathcal{M}$ by $\mathcal{T}$). Let $q = |\mathcal{H}|$, let $p = \max_{H \in \mathcal{H}} \mathbf{cc}(H)$, and let $H \in \mathcal{H}$. Let $k \cdot H$ denote the disjoint union of k copies of H We set $f(k) = qp \cdot (k+1) \cdot \left(h\left(s, \omega^{\mathcal{M}}_{k \cdot H}\right) + 1\right)$ for every $k \in \mathbb{N}$, where h is the function of Theorem 9. Observe that if $\mathbf{pw}(G) > h\left(s, \omega^{\mathcal{M}}_{k \cdot H}\right)$, then G contains a subdigraph G' that is semicomplete and has pathwidth at least $\omega^{\mathcal{M}}_{k \cdot H}$ (Theorem 9). By definition of $\omega^{\mathcal{M}}_{k \cdot H}$, G' contains k vertex-disjoint $\mathcal{M}(H)$-subgraphs, and then so do G and we are done. Let us therefore assume that $\nu^{\mathsf{v}}_{\mathcal{M}(H)}(G) < k$; by the above observation it also implies $\mathbf{pw}(G) \leq h\left(s, \omega^{\mathcal{M}}_{k \cdot H}\right)$.

Let us call $H_1, \ldots, H_q$ the elements of $\mathcal{H}$. For every $i \in [\![1, q]\!]$ let $p_i = \mathbf{cc}(H_i)$ and let $\{H_i^j\}_{j \in [\![1, p_i]\!]}$ be the connected components of H_i. Then observe that $\mathcal{M}(H_i) = \uplus_{j=1}^{p_i} \mathcal{M}(H_i^j)$. Besides, by definition we have $\mathcal{M}(\mathcal{H}) = \bigcup_{i=1}^{q} \mathcal{M}(H_i)$. Therefore we can apply Corollary 1 and obtain a $\mathcal{M}(H)$-vertex-hitting set of size at most $qp \cdot (\nu^{\mathsf{v}}_{\mathcal{M}(\mathcal{H})}(G) + 1)(\mathbf{pw}(G) + 1) \leq qp \cdot (k+1)(h(s, \omega_{k \cdot H}) + 1) = f(k)$. □

5 Discussion

In this note we proved that the (immersion) expansions of members of any family of strongly connected digraphs have the Erdős–Pósa property in semicomplete digraphs, and that the (minor) models and the subdivisions of members of any finite digraph family also have this property in s-semicomplete digraphs, for fixed s. A novelty is the use of $\mathcal{H}$-arc-separations, that allow us to remove less arcs that we would do with a regular separations (i.e. one that disconnected the graph when removed). This work also shows that some techniques previously used for undirected graphs can be adapted to the setting of directed graphs. Let us now highlight two directions for future research.

Optimization of the gap. The upper bound on $\omega^{\mathcal{I}}_{H}$ of Theorem 6 that can be obtained from the proof (in [CFS12]) is $\omega^{\mathcal{I}}_{H} \leq 72 \cdot 2^{2t(t+2)} + 8 \cdot 2^{t(t+2)}$, where $t = |V(H)| + 2|E(H)|$. As a consequence, we have $f(k) = 2^{O(k^2 t^2)}$ in Theorem 4. It would be interesting to know whether a gap that is polynomial in k can be obtained.

The same question can be asked for Theorem 5, however the upper bound of $\omega_H^{\mathcal{T}}$ in Theorem 8 that we can compute from the proof in [FS13] is large (triply exponential).

Generalization. The results presented in this note were related to (generalizations of) semicomplete digraphs. One direction for future research would be to extend them to wider classes of hosts. On the other hand, in Theorem 4, we require the guest digraph to be strongly connected. It is natural to ask if we can drop this condition. This would require a different proof as ours draws upon this condition.

References

[ADG04] Julio Aracena, Jacques Demongeot, and Eric Goles. Positive and negative circuits in discrete neural networks. *Neural Networks, IEEE Transactions on*, 15(1):77–83, Jan 2004.

[AKKW16] Saeed Akhoondian Amiri, Ken-Ichi Kawarabayashi, Stephan Kreutzer, and Paul Wollan. The Erdős–Pósa property for directed graphs. *ArXiv e-prints*, 2016.

[ARS16] Julio Aracena, Adrien Richard, and Lilian Salinas. Number of fixed points and disjoint cycles in monotone Boolean networks. *ArXiv e-prints*, February 2016.

[CFS12] Maria Chudnovsky, Alexandra Fradkin, and Paul Seymour. Tournament immersion and cutwidth. *Journal of Combinatorial Theory, Series B*, 102(1):93 – 101, 2012.

[EP65] Paul Erdős and Louis Pósa. On independent circuits contained in a graph. *Canadian Journal of Mathematics*, 17:347–352, 1965.

[FJW13] Samuel Fiorini, Gwenaël Joret, and David R. Wood. Excluded forest minors and the Erdős-Pósa property. *Combinatorics, Probability & Computing*, 22(5):700–721, 2013.

[FS13] Alexandra Fradkin and Paul Seymour. Tournament pathwidth and topological containment. *Journal of Combinatorial Theory, Series B*, 103(3):374 – 384, 2013.

[FST11] Fedor V. Fomin, Saket Saurabh, and Dimitrios M. Thilikos. Strengthening Erdős–Pósa property for minor-closed graph classes. *Journal of Graph Theory*, 66(3):235–240, 2011.

[GT10] Bertrand Guenin and Robin Thomas. Packing directed circuits exactly. *ArXiv e-prints*, December 2010.

[HM13] Frédéric Havet and Ana Karolinna Maia. On disjoint directed cycles with prescribed minimum lengths. Research Report RR-8286, INRIA, April 2013.

[KKT15] Kenta Kitsunai, Yasuaki Kobayashi, and Hisao Tamaki. On the pathwidth of almost semicomplete digraphs. In Nikhil Bansal and Irene Finocchi, editors, *Algorithms - ESA 2015: 23rd Annual European Symposium, Patras, Greece, September 14-16, 2015, Proceedings*, pages 816–827, Berlin, Heidelberg, 2015. Springer Berlin Heidelberg.

[KS15] Ilhee Kim and Paul Seymour. Tournament minors. *Journal of Combinatorial Theory, Series B*, 112:138 – 153, 2015.

[Ree97] Bruce A. Reed. *Tree width and tangles: A new connectivity measure and some applications*, pages 87–162. Cambridge University Press, 1997.

[RRST96] Bruce A. Reed, Neil Robertson, Paul D. Seymour, and Robin Thomas. Packing directed circuits. *Combinatorica*, 16(4):535–554, 1996.

[RS86] Neil Robertson and Paul D. Seymour. Graph Minors. V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(2):92–114, 1986.

[RS96] Bruce A. Reed and Bruce F. Shepherd. The Gallai-Younger conjecture for planar graphs. *Combinatorica*, 16(4):555–566, 1996.

[RT16] Jean-Florent Raymond and Dimitrios M. Thilikos. Recent results on the Erdős–Pósa property. *ArXiv e-prints*, 2016.

[Sey96] Paul D. Seymour. Packing circuits in eulerian digraphs. *Combinatorica*, 16(2):223–231, 1996.