POLYNOMIAL EXPANSION AND SUBLINEAR SEPARATORS

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Abstract. Let $\mathcal{C}$ be a class of graphs that is closed under taking subgraphs. We prove that if for some fixed $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $O(n^{1-\delta})$, then any depth-k minor (i.e. minor obtained by contracting disjoint subgraphs of radius at most k) of a graph in $\mathcal{C}$ has average degree $O\big((k \operatorname{polylog} k)^{1/\delta}\big)$. This confirms a conjecture of Dvořák and Norin.

1. Introduction

For an integer $k \geq 0$, a *depth-k minor* of a graph G is a subgraph of a graph that can be obtained from G by contracting pairwise vertex-disjoint subgraphs of radius at most k. Let $d(G)$ denote the average degree of a graph $G = (V, E)$, i.e. $d(G) = 2|E|/|V|$. For some function f, we say that a class $\mathcal{C}$ of graphs has *expansion bounded by f* if for any graph $G \in \mathcal{C}$ any integer k, any depth-k minor of G has average degree at most $f(k)$. We say that a class has *bounded expansion* if it has expansion bounded by some function f, and *polynomial expansion* if f can be taken to be a polynomial.

Classes of bounded expansion play a central role in the study of sparse graphs [6]. From an algorithmic point of view, a very useful property of theses classes is that when their expansion is not too large (say subexponential), graphs in the class have sublinear separators. A *separator* in a graph $G = (V, E)$ is a pair of subsets (A, B) of vertices of G such that $A \cup B = V$ and no edge of G has one endpoint in $A \setminus B$ and the other in $B \setminus A$. The separator (A, B) is said to be *balanced* if both $|A \setminus B|$ and $|B \setminus A|$ contain at most $\frac{2}{3}|V|$ vertices. The *order* of the separator (A, B) is $|A \cap B|$.

A class $\mathcal{C}$ of graphs is *monotone* if for any graph $G \in \mathcal{C}$, any subgraph of G is in $\mathcal{C}$. Dvořák and Norin [5] observed that the following can be deduced from a result of Plotkin, Rao, and Smith [7].

Theorem 1 ([5]). *Let $\mathcal{C}$ be a monotone class of graphs with expansion bounded by $k \mapsto c(k+1)^{1/4\delta - 1}$, for some constant $c > 0$ and $0 < \delta \leq 1$. Then there is a constant C such that every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$.*

Dvořák and Norin [5] also proved the following partial converse.

The first author was supported by ANR Projects STINT (anr-13-bs02-0007) and GATO (anr-16-ce40-0009-01) and LabEx PERSYVAL-Lab (anr-11-labx-0025-01). The second author was supported by the Polish National Science Centre grant PRELUDIUM DEC-2013/11/N/ST6/02706.

Theorem 2 ([5])**.** *Let $\mathcal{C}$ be a monotone class of graphs such that for some constants $C > 0$ and $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$. Then the expansion of $\mathcal{C}$ is bounded by the function $k \mapsto c \cdot k^{5/\delta^2}$, for some constant c.*

They conjectured that the exponent $5/\delta^2$ of the polynomial expansion in Theorem 2 could be improved to match (asymptotically) that of Theorem 1.

Conjecture 3 ([5])**.** *Let $\mathcal{C}$ be a monotone class of graphs such that for some constants $C > 0$ and $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$. Then the expansion of $\mathcal{C}$ is bounded by the function $k \mapsto c_1 \cdot k^{c_2/\delta}$, for some constants c_1, c_2.*

In this short note, we prove this conjecture.

Theorem 4. *For any $c > 0$ and $0 < \delta \leq 1$, if a monotone class $\mathcal{C}$ has the property that every n-vertex graph in $\mathcal{C}$ has a balanced separator of order at most $Cn^{1-\delta}$, then $\mathcal{C}$ has expansion bounded by the function $f : k \mapsto c_1 \cdot (k(\log(3k+1))^{c_2})^{1/\delta}$, for some constants c_1 and c_2.*

The proof of Theorem 4 is given in the next section, and we conclude with some open problems in Section 3.

2. Proof of Theorem 4

We need the following results. The first is a classical connection between balanced separators and tree-width (see [5]).

Lemma 5. *Any graph G has a balanced separator of order at most $\mathrm{tw}(G) + 1$.*

Dvořák and Norin [4] proved that the following partial converse holds.

Theorem 6 ([4])**.** *If every subgraph of G has a balanced separator of order at most k, then G has tree-width at most $105k$.*

Note that in our proof of Theorem 4 we could also use the weaker (and easier) result of [1] that under the same hypothesis, G has tree-width at most $1 + k \log |V(G)|$, but the computation is somewhat less cumbersome if we use Theorem 6 instead.

For a set S of vertices in a graph G, we let $N(S)$ denote the set of vertices not in S with at least one neighbor in S. We will use the following result of Shapira and Sudakov [8].

Theorem 7 ([8])**.** *Any graph G contains a subgraph H of average degree $d(H) \geq \frac{255}{256} d(G)$ such that for any set S of at most $n/2$ vertices of H (where $n = |V(H)|$), $|N(S)| \geq \frac{1}{2^8 \log n (\log \log n)^2} |S|$.*

In fact, we will only need a much weaker version, where the vertex-expansion is of order $\Omega\left(\frac{1}{\mathrm{polylog}\, n}\right)$ instead of $\Omega\left(\frac{1}{\log n (\log\log n)^2}\right)$.

Finally, we need a result of Chekuri and Chuzhoy [2] on bounded-degree subgraphs of large tree-width in a graph of large tree-width.

Theorem 8 ([2]). *There are constants α, β such that for any integer k, any graph G of tree-width at least k contains a subgraph H of tree-width at least $\alpha k/(\log k)^\beta$ and maximum degree 3.*

Observe that instead of Theorem 8, our proof of Theorem 4 could rely on an earlier result of Chekuri and Chuzhoy [3] which, under the same assumptions, guarantees the existence of a subgraph of G of treewidth $\Omega(k/(\log k)^6)$ and maximum degree $O((\log k)^3)$.

We are now ready to prove our main result.

Proof of Theorem 4. Let G be a graph of $\mathcal{C}$ and let F be a depth-k minor of G. Any subgraph of F is a depth-k minor of G so it is enough to prove that $d(F) \leq c_1 \cdot (k(\log(3k+1))^{c_2})^{1/\delta}$, for some constants c_1 and c_2. Assume without loss of generality that $d(F) \geq 10^8$ (since otherwise the result clearly holds by choosing appropriate values of c_1, c_2). By Theorem 7, F has a subgraph H of average degree $d(H) \geq \frac{255}{256} d(F)$ such that for any set S of at most $|V(H)|/2$ vertices of H,

$$|N(S)| \geq \tfrac{1}{2^8 \log |V(H)| (\log\log |V(H)|)^2} |S| \geq \tfrac{1}{2^8 (\log |V(H)|)^3} |S|.$$

It follows from Lemma 5 that H contains a balanced separator (A, B) with $|A \cap B| \leq \mathrm{tw}(H) + 1$. By symmetry, we can assume that $|A \setminus B| \leq |V(H)|/2$ and thus

$$|A \cap B| \geq \tfrac{1}{2^8 (\log |V(H)|)^3} |A \setminus B|.$$

Since (A, B) is balanced, $|A \setminus B| + |A \cap B| \geq \frac{1}{3}|V(H)|$ and so

$$\tfrac{1}{3}|V(H)| \leq |A \cap B|(1 + 2^8 (\log |V(H)|)^3).$$

As a consequence,

$$\mathrm{tw}(H) \geq \tfrac{|V(H)|}{3 \cdot 2^8 (\log |V(H)|)^3 + 3} - 1 \geq \tfrac{|V(H)|}{2^{10} (\log |V(H)|)^3},$$

using that $|V(H)| \geq d(H) \geq \frac{255}{256} \cdot 10^8$.

By Theorem 8, H has a subgraph H' of maximum degree 3 such that

$$\mathrm{tw}(H') \geq \tfrac{\alpha\, \mathrm{tw}(H)}{(\log \mathrm{tw}(H))^\beta} \geq \tfrac{\alpha\, |V(H)|}{2^{10} (\log |V(H)|)^{\beta+3}},$$

since $\mathrm{tw}(H) \leq |V(H)|$. Note that H' is a subgraph of H (and F) and therefore also a depth-k minor of G. In G, H' corresponds to a subgraph G' (before contraction of the subgraphs of radius k) with $|V(G')| \leq (3k+1)|V(H')| \leq (3k+1)|V(H)|$. Indeed, since H' has maximum degree 3, each subgraph of radius at most k in G' whose contraction corresponds to a vertex of H' contains at most $3k+1$ vertices. Since H' is a minor of G', we have

$$\mathrm{tw}(G') \geq \mathrm{tw}(H') \geq \tfrac{\alpha\, |V(H)|}{2^{10} (\log |V(H)|)^{\beta+3}}.$$

Since $\mathcal{C}$ is monotone, every subgraph of G' is in $\mathcal{C}$ and thus has a balanced separator of order at most $C|V(G')|^{1-\delta}$. Hence, by Theorem 6, $\mathrm{tw}(G') \leq 105C|V(G')|^{1-\delta} \leq 2^7 C |V(G')|^{1-\delta}$. As a consequence,

$$\tfrac{\alpha\, |V(H)|}{2^{10} (\log |V(H)|)^{\beta+3}} \leq 2^7 C\, |V(G')|^{1-\delta} \leq 2^7 C \left((3k+1)|V(H)|\right)^{1-\delta}, \text{ and thus}$$

$$\frac{|V(H)|^\delta}{(\log|V(H)|)^{\beta+3}} \le \frac{2^{17}C}{\alpha}(3k+1)^{1-\delta} \le \frac{2^{17}C}{\alpha}(3k+1), \text{ and}$$

$$|V(H)| \le \left(\frac{2^{17}C}{\alpha}(3k+1)(\log|V(H)|)^{\beta+3}\right)^{1/\delta}.$$

It follows that

$$\log|V(H)| \le \tfrac{1}{\delta}\log\left(\frac{2^{17}C}{\alpha}(3k+1)\right) + \tfrac{\beta+3}{\delta}\log\log|V(H)|.$$

Since $\log\log|V(H)| \le \frac{1}{2}\log|V(H)|$, this implies

$$\log|V(H)| \le \tfrac{1}{\delta}\log\left(\frac{2^{17}C}{\alpha}(3k+1)\right) + \tfrac{\beta+3}{2\delta}\log|V(H)|, \text{ and thus}$$

$$\log|V(H)| \le \tfrac{2}{2\delta-\beta-3}\log\left(\frac{2^{17}C}{\alpha}(3k+1)\right).$$

We conclude that

$$|V(H)| \le \left(\frac{2^{17}C}{\alpha}(3k+1)\left(\tfrac{2}{2\delta-\beta-3}\log\left(\frac{2^{17}C}{\alpha}(3k+1)\right)\right)^{\beta+3}\right)^{1/\delta} \le \tfrac{255}{256}c_1\left(k(\log(3k+1))^{c_2}\right)^{1/\delta},$$

for some constants c_1, c_2 depending only on C and the constants α, β of Theorem 8. Recall that $d(F) \le \frac{256}{255}d(H)$. Since $d(H) \le |V(H)|$, we obtain $d(F) \le c_1\left(k(\log(3k+1))^{c_2}\right)^{1/\delta}$, as desired. This concludes the proof of Theorem 4. □

3. Open problems

A natural problem is to determine the infimum real α, such that if a monotone class $\mathcal{C}$ has the property that every n-vertex graph in $\mathcal{C}$ has a balanced separator of order $O(n^{1-\delta})$, then $\mathcal{C}$ has expansion bounded by some function $k \mapsto O(k^{\alpha/\delta})$. It directly follows from Theorems 1 and 4 that $\frac{1}{4} \le \alpha \le 1$. Moreover, the proof of Theorem 1 in [5] can be slightly optimized to show that $\frac{1}{2} \le \alpha \le 1$.

One way to measure the sparsity of a class of graphs is via its expansion (as defined in Section 1). Another way (which turns out to be equivalent) is via its *generalized coloring parameters*. Given a linear order L on the vertices of a graph G, and an integer r, we say that a vertex v of G is *strongly r-reachable* from a vertex u (with respect to L) if $v \le_L u$, and there is a path P of length at most r between u and v, such that $u <_L w$ for any internal vertex w of P. If we only require that v is the minimum of the vertices of P (with respect to L), we say that v is *weakly r-reachable* from u. The *strong r-coloring number* $\mathrm{col}_r(G)$ of G is the minimum integer k such that there is a linear order L on the vertices of G such that for any vertex u of G, at most k vertices are strongly r-reachable from u (with respect to L). By replacing *strongly* by *weakly* in the previous definition, we obtain the *weak r-coloring number* $\mathrm{wcol}_r(G)$ of G. Note that for any graph G and any integer r, $\mathrm{col}_r(G) \le \mathrm{wcol}_r(G)$. For more on these parameters and their connections with the expansion of graph classes, the reader is referred to [6].

As we have seen before, it follows from [5] that a monotone class of graphs has polynomial expansion if and only if, for some fixed $0 < \delta \le 1$, each n-vertex graph in the class has a balanced separator of order $O(n^{1-\delta})$. Joret and Wood asked whether this is also equivalent to having weak and strong r-coloring numbers bounded by a polynomial function of r.

Problem 9 (Joret and Wood, 2017)**.** *Assume that $\mathcal{C}$ is a monotone class of graphs. Are the following statements equivalent?*

(1) $\mathcal{C}$ has polynomial expansion.
(2) There exists a constant c, such that for every r, every graph in $\mathcal{C}$ has strong r-coloring number at most $O(r^c)$.
(3) There exists a constant c, such that for every r, every graph in $\mathcal{C}$ has weak r-coloring number at most $O(r^c)$.

Note that clearly (3) implies (2). It was known that (3) implies (1) (this is a consequence of Lemma 7.11 in [6]), and Norin recently made the following observation, which shows that (2) implies (1).

Observation 10 (Norin, 2017)**.** *Every depth-r minor of a graph G has average degree at most $2\,\mathrm{col}_{4r}(G)$.*

Proof. Let L be a linear order on the vertices of G, such that for any vertex v of G, at most $\mathrm{col}_r(G)$ vertices are strongly r-reachable from v (with respect to L). Let H be a depth-r minor of a graph G. For any vertex u of H, let S_u be a subgraph of G of radius at most r, such that the S_u's are vertex-disjoint and for any edge uv of H, there is an edge in G between a vertex of S_u and a vertex of S_v. It is enough to prove that there is a linear order L' on the vertices of H such that any vertex u of H, at most $\mathrm{col}_{4r}(G)$ vertices of H are strongly 1-reachable from u.

We construct L' from L as follows: for u, v in H, we set $u <_{L'} v$ if and only if, with respect to L, the smallest vertex of S_u precedes the smallest vertex of S_v. This clearly defines a linear order on the vertices of H. Consider a vertex u of H and let x be the smallest vertex of S_u (with respect to L). Let v be a neighbor of u in H with $v <_{L'} u$ (i.e. v is strongly 1-reachable from u in H). Let $t \in S_u$ and $z \in S_v$ be such that tz is an edge of G. Observe that there is a path P_u from x to t in S_u (and x is the smallest vertex in this path with respect to L), and a path P_v from z to y in S_v. Let w be the first vertex of P_v such that $w <_L x$ (note that possibly $w = z$). The concatenation of P_u, zt, and the subpath of P_v between z and w has length at most $4r$ and thus shows that w is strongly $4r$-reachable from x in G. Hence, at most $\mathrm{col}_{4r}(G)$ vertices of H are strongly 1-reachable from u in H with respect to L', as desired. ∎

Acknowledgements

We thank Z. Dvořák for the discussion about [5], G. Joret and D. Wood for allowing us to mention Problem 9, and S. Norin for allowing us to mention Observation 10 and its proof.

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