Expansion & separators

Jean-Florent Raymond

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Join work with Louis Esperet (G-SCOP, Grenoble, France).

Strongly sublinear separators and polynomial expansion

Theorem (Plotkin, Rao, and Smith, SODA 1994)

polynomial expansion ⇒ *sublinear separators*

Theorem (Dvořák and Norin, SIDMA 2016)

sublinear separators ⇒ *polynomial expansion*.

(Bounded-depth) minors

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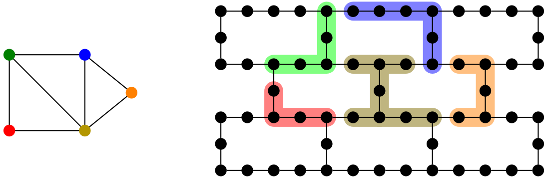

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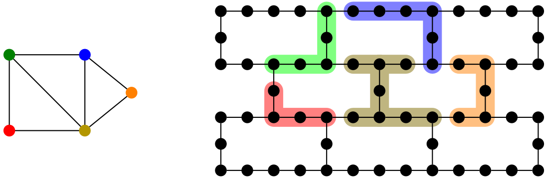

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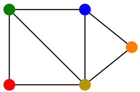

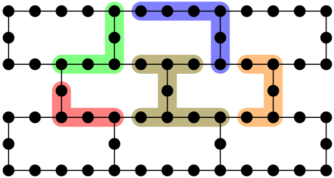

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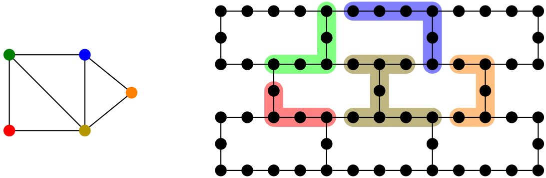

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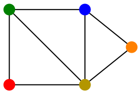

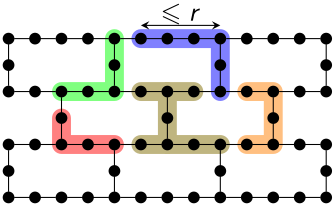

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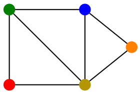

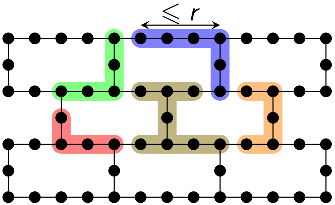

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Ex: a 0-minor is a subgraph.

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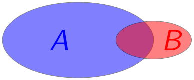
- separation (A, B): 

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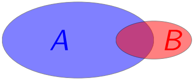
- separation (A, B): 

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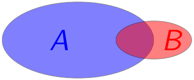
- separation (A, B): 

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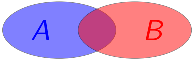
- balanced separation (A, B): 

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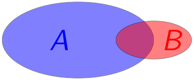
- separation (A, B): 

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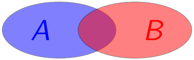
- balanced separation (A, B): 

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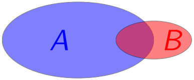
- separation (A, B): 

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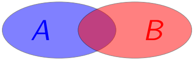
- balanced separation (A, B): 

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Ex: planar graphs have balanced separators of order $O(n^{1/2})$.

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Any graph G has a balanced separator of order at most $\mathbf{tw}(G) + 1$.

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Any graph G has a balanced separator of order at most $\mathbf{tw}(G)+1$.

Theorem (Dvořák and Norin, 2014)

Let $\beta \in [0,1)$.

$$\forall H \subseteq G,\ H \text{ has a balanced separator of order} \leqslant c|H|^{\beta}$$
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$$\forall H \subseteq G,\ \mathbf{tw}(H) \leqslant 105c|H|^{\beta}.$$

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Separators & expansion

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Expander: small subsets of V have linearly many neighbors at least.

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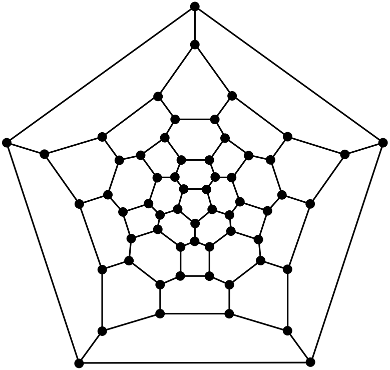

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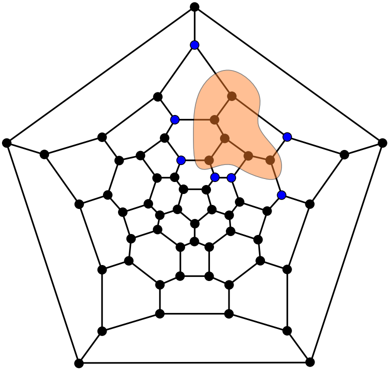

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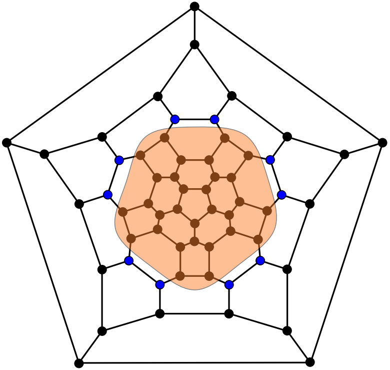

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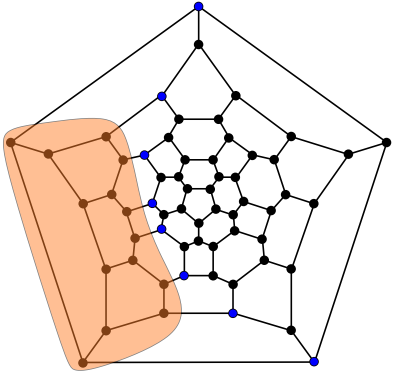

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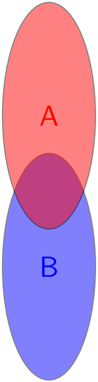

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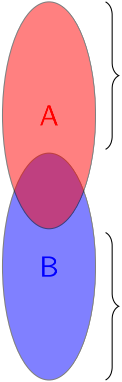

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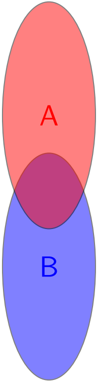

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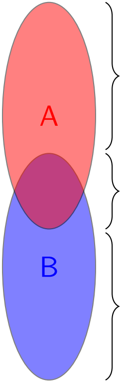

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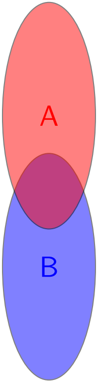

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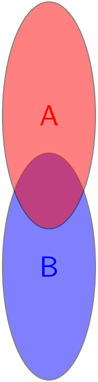

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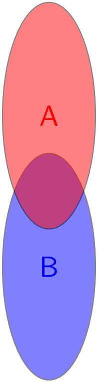

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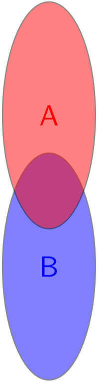

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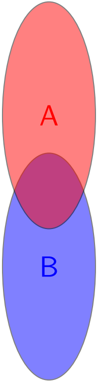

$$\left.\begin{array}{l} |A \setminus B| \leqslant n/2 \\ |A \cap B| \leqslant \mathbf{tw} + 1 \\ |A \cap B| \geqslant \alpha|A \setminus B| \\ |B \setminus A| \leqslant 2n/3 \end{array}\right\}$$

$$\begin{aligned} |A \setminus B| + |A \cap B| &\geqslant n/3 \\ \frac{|A \cap B|}{\alpha} + |A \cap B| &\geqslant n/3 \\ |A \cap B| &\geqslant \frac{\alpha}{\alpha+1} \cdot \frac{n}{3} \\ \mathbf{tw} + 1 &\geqslant \frac{\alpha}{\alpha+1} \cdot \frac{n}{3} \end{aligned}$$

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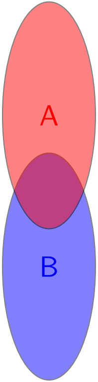

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$$\mathbf{tw} + 1 \geqslant \frac{\alpha}{\alpha+1} \cdot \frac{n}{3}$$

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Finding subgraphs of linear treewidth

Theorem (Shapira and Sudakov, Combinatorica 2015)

Any graph G contains a subgraph H s.t.:

1. $d(H) \geqslant 0.9 \cdot d(G)$;
2. *H is an $\frac{1}{\operatorname{polylog}|H|}$-expander.*

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1. $d(H) \geqslant 0.9 \cdot d(G)$;
2. H *is an* $\frac{1}{\operatorname{polylog}|H|}$*-expander.*

Corollary

Any graph G contains a subgraph H with

1. $d(H) \geqslant 0.9 \cdot d(G)$;
2. $\mathbf{tw}(H) = \Omega\left(\frac{|H|}{\operatorname{polylog}|H|}\right)$.

Sketch of the proof

Current goal

$\mathcal{C}$ has balanced sep. of order $O(n^{1-\delta}) \Rightarrow \mathcal{C}$ has expansion $O(r^{c/\delta})$

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Lemma (expander lemma)

Every F has a sgr. H with $d \geqslant 0.9d(F)$ and $\mathbf{tw} = \Omega\left(\frac{|H|}{\text{polylog}\,|H|}\right)$.

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Theorem (Chekuri and Chuzhoy, SODA 2015)

Every G with $\mathbf{tw} = k$ has a sgr. with $\Delta \leqslant 3$ and $\mathbf{tw} \geqslant \frac{k}{\text{polylog}\,k}$.

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graphs of $\mathcal{C}$ *have balanced separators of order* $O\left(n^{1-\delta}\right)$.

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Thank you!