Polynomial expansion and sublinear separators

Jean-Florent Raymond

LIRMM, University of Montpellier, France, and
MIMUW, University of Warsaw, Poland

Join work with Louis Esperet (G-SCOP, Grenoble).

Strongly sublinear separators and polynomial expansion

Theorem (Plotkin, Rao, and Smith, SODA 1994)

polynomial expansion ⇒ *sublinear separators*

Theorem (Dvořák and Norin, SIDMA 2016)

sublinear separators ⇒ *polynomial expansion*.

(Bounded-depth) minors

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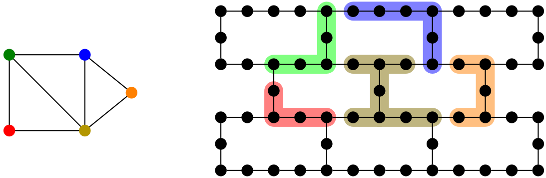

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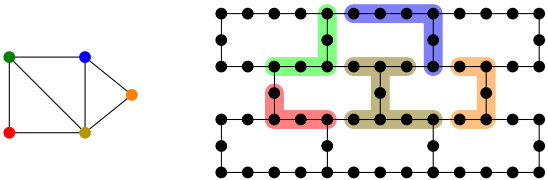

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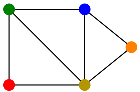

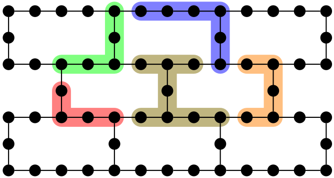

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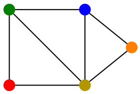

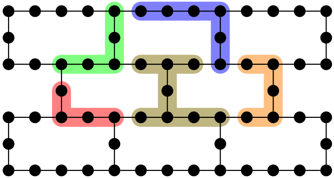

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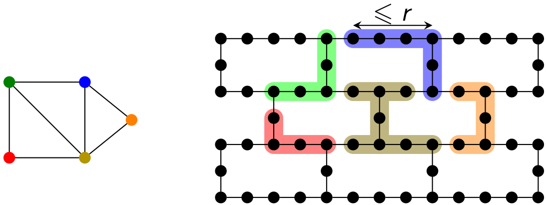

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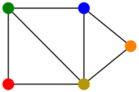

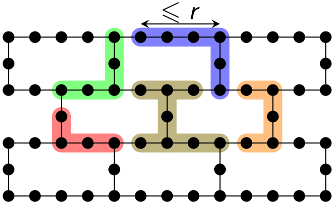

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$\mathrm{radius}(G) \leqslant r$: G has a vertex at distance $\leqslant r$ of the others.

Ex: a 0-minor is a subgraph.

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- $\mathcal{C}$ has polynomial expansion if f is polynomial.

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- many known sparse classes have bounded expansion: H-minor free, bounded degree, random graphs of bounded average degree, etc.;

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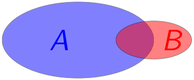
- separation (A, B): 

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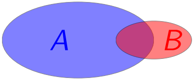
- separation (A, B): 

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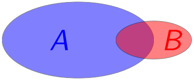
- separation (A, B): 

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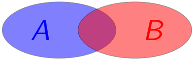
- balanced separation (A, B): 

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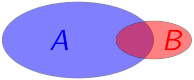
- separation (A, B): 

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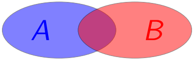
- balanced separation (A, B): 

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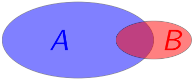
- separation (A, B): 

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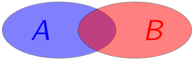
- balanced separation (A, B): 

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Ex: planar graphs have balanced separators of order $O(n^{1/2})$.

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Any graph G has a balanced separator of order at most $\mathbf{tw}(G) + 1$.

Theorem (Dvořák and Norin, 2014)

Let $\beta \in [0, 1)$.

$$\forall H \subseteq G,\ H \text{ has a balanced separator of order} \leqslant c|H|^{\beta}$$
$$\Downarrow$$
$$\forall H \subseteq G,\ \mathbf{tw}(H) \leqslant 105c|H|^{\beta}.$$

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Separators & expansion

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Theorem (Plotkin, Rao, and Smith, SODA 1994)

$\mathcal{C}$ has expansion $O\left(r^{1/\delta}\right)$

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Expander: small subsets of V have linearly many neighbors at least.

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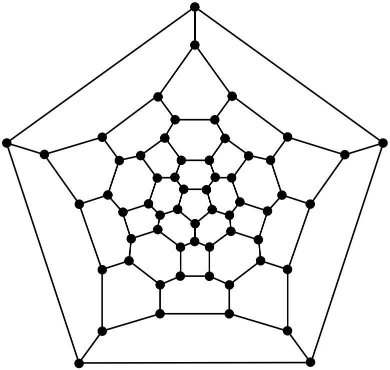

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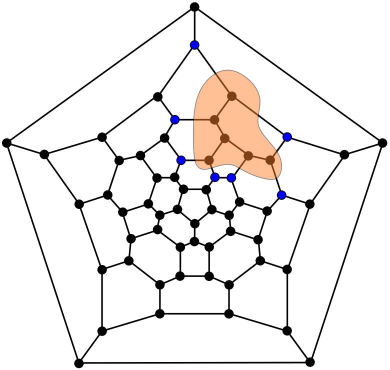

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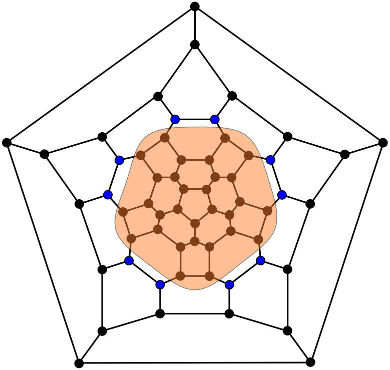

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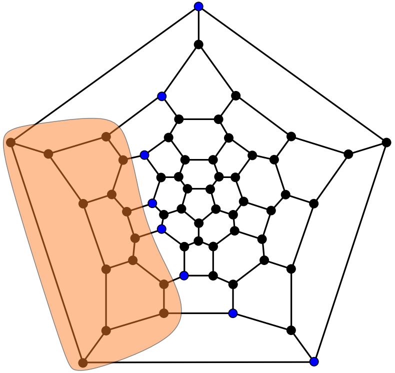

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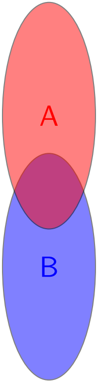

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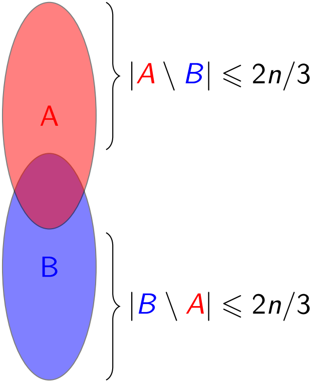

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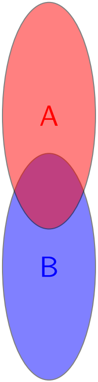

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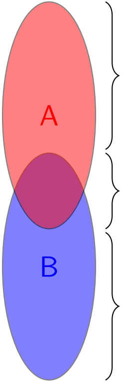

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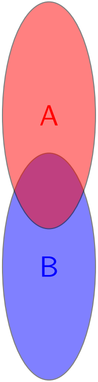

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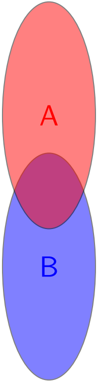

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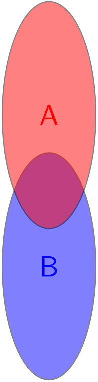

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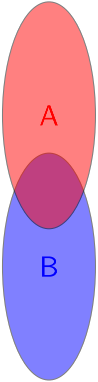

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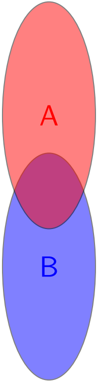

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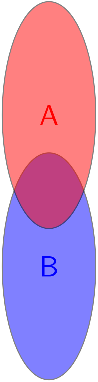

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Finding subgraphs of linear treewidth

Theorem (Shapira and Sudakov, Combinatorica 2015)

Any graph G contains a subgraph H s.t.:

1. $d(H) \geqslant 0.9 \cdot d(G)$;
2. *H is an $\frac{1}{\operatorname{polylog}|H|}$-expander.*

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Corollary

Any graph G contains a subgraph H with

1. $d(H) \geqslant 0.9 \cdot d(G)$;
2. $\mathbf{tw}(H) = \Omega\left(\frac{|H|}{\operatorname{polylog}|H|}\right)$.

Sketch of the proof

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Lemma (expander lemma)

Every F has a sgr. H with $d \geqslant 0.9d(F)$ and $\mathbf{tw} = \Omega\left(\frac{|H|}{\operatorname{polylog}|H|}\right)$.

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Theorem (Chekuri and Chuzhoy, SODA 2015)

Every G with $\mathbf{tw} = k$ has a sgr. with $\Delta \leqslant 3$ and $\mathbf{tw} \geqslant \frac{k}{\operatorname{polylog} k}$.

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Thank you!