Obstructions to bounded cutwidth

Jean-Florent Raymond

LIRMM and University of Warsaw

Thursday 13th April, 2017

Joint work with:

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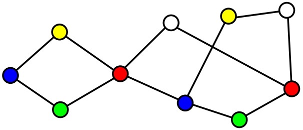
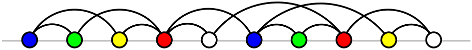
- Michał Pilipczuk (University of Warsaw);
- Dimitrios M. Thilikos (LIRMM–CNRS); and
- Marcin Wrochna (University of Warsaw).

Cutwidth

- *layout* of G: ordering of $V(G)$.





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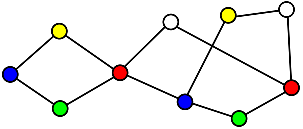

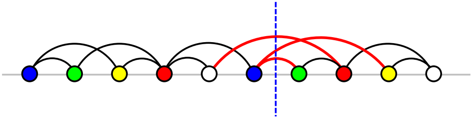

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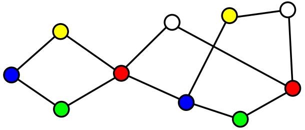

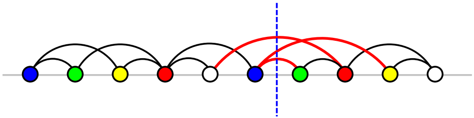

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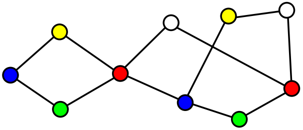
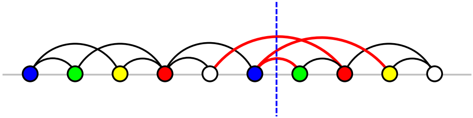
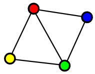
Class considered of this talk: graphs of cutwidth $\leqslant k$.





Immersions



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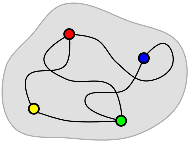

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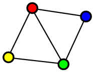

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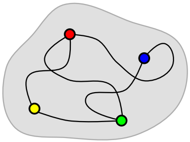

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“Monotonicity” of cutwidth:

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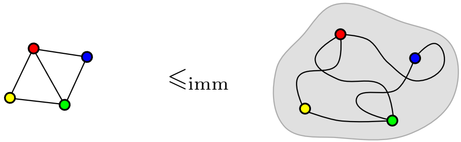

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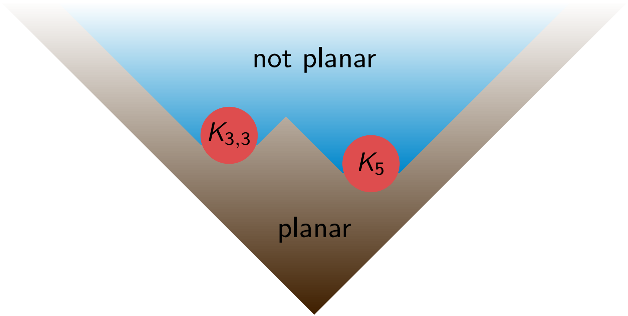
$\{G,\ \textbf{ctw}(G) \leqslant k\}$ is immersion-closed.

Obstructions

K_5 and $K_{3,3}$ are the *minor-obstructions* of planar graphs



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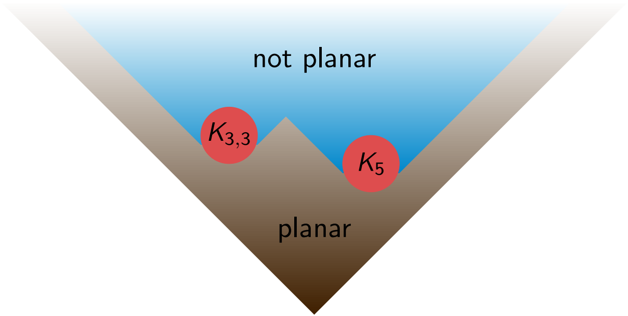

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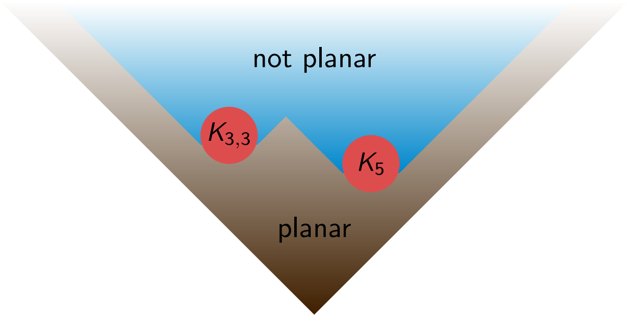

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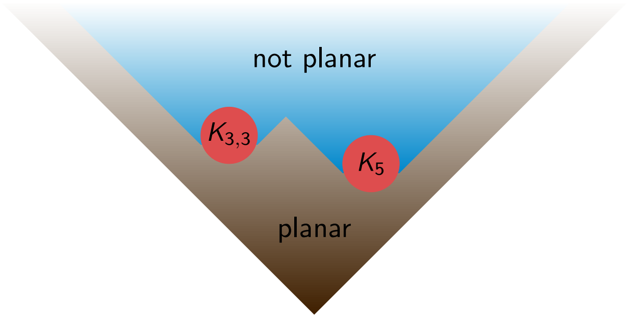

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The set of obstructions may be infinite!

Obstructions for bounded cutwidth

What about obstructions of $\{G,\ \mathbf{ctw}(G) \leqslant k\}$?

Theorem (Robertson and Seymour, consequence of GM.XXII)

For every $k \in \mathbb{N}$, $\{G,\ \mathbf{ctw}(G) \leqslant k\}$, has finitely many obstructions.

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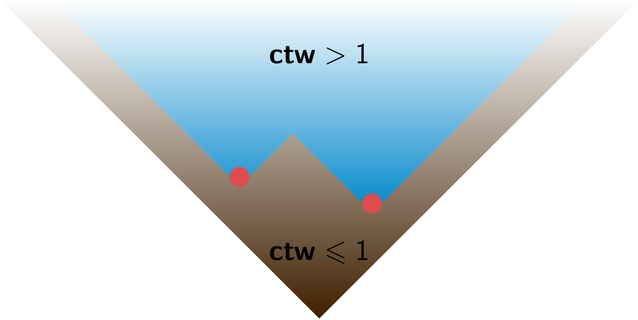

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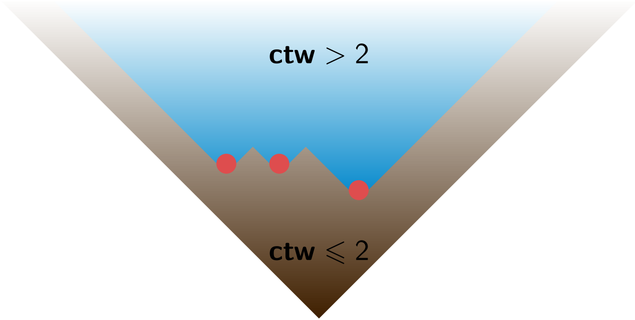

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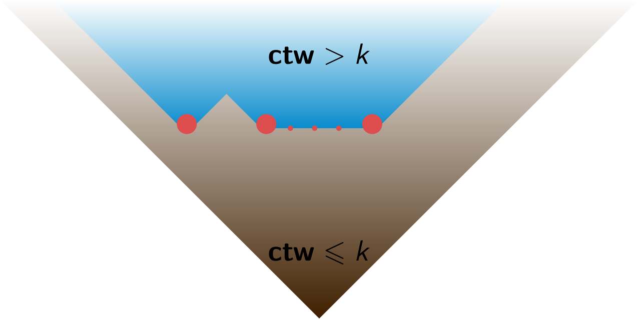

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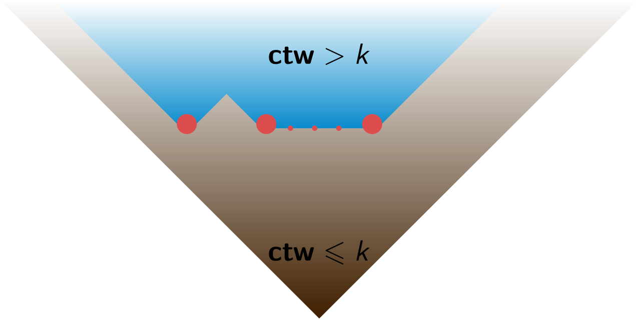

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Results of Lagergren (1998):

- G minor-obstruction for pathwidth $\leqslant k \Rightarrow |G| = 2^{O(k^4)}$;
- G minor-obstruction for treewidth $\leqslant k \Rightarrow |G| = 2^{2^{O(k^5)}}$.

How to show that obstructions are small?

General idea

If an obstruction is too large, some part of it is *redundant*.

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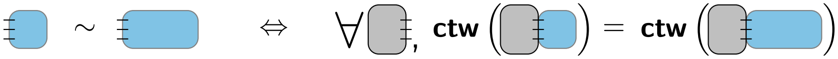

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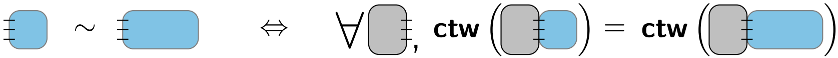

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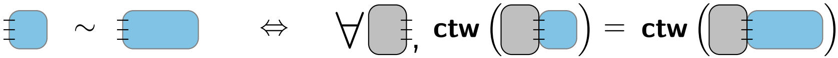

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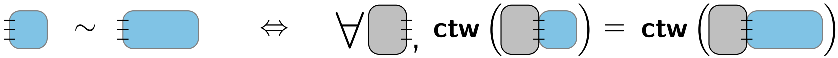

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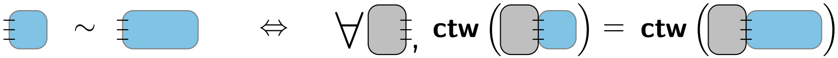

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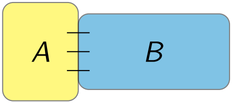
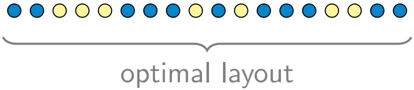
- we prove that the obtained graph is an immersion of G;
- contradicts the minimality of G!

Bounding the number of equivalence classes





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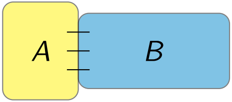

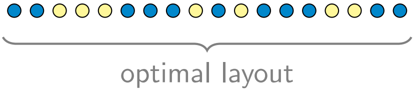

Key Lemma

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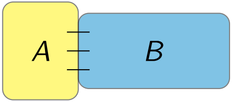

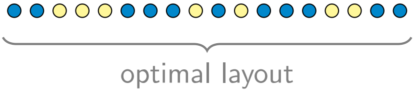

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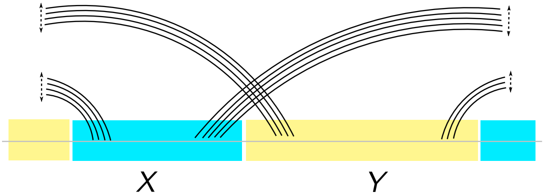

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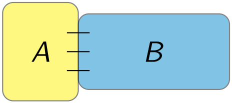

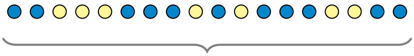

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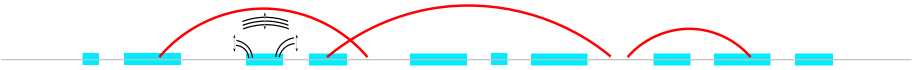

Relevant information about B: $O(k\ell)$ numbers up to k.

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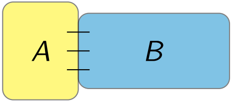

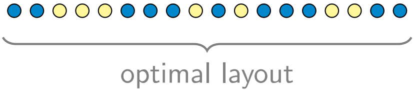

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$\rightarrow$ finite number of equivalence classes for fixed k, ℓ.

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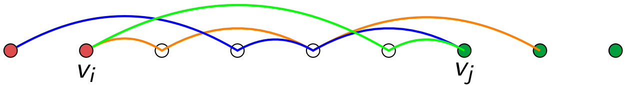

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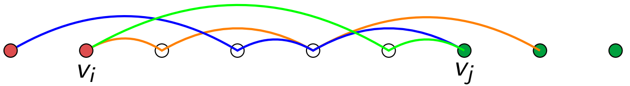

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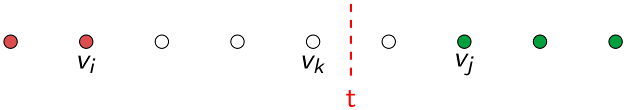

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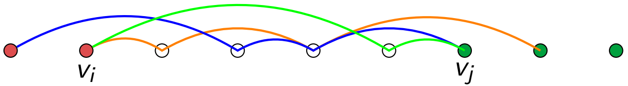

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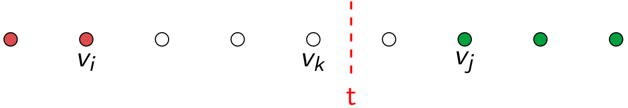

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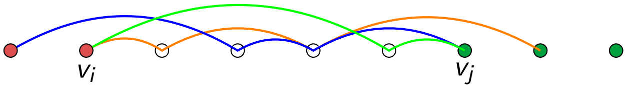

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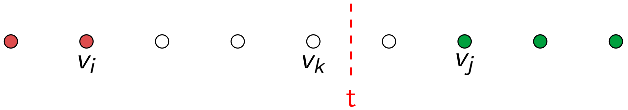

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Proof: non-linked orderings can be *improved* without increasing the width.

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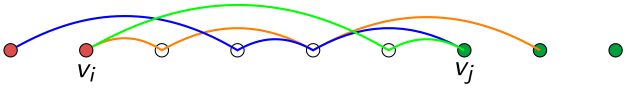

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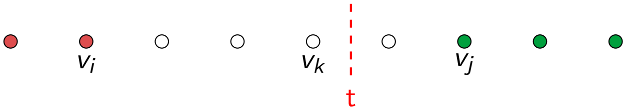

Lemma

Every graph has a linked ordering of optimal width.

Proof: non-linked orderings can be *improved* without increasing the width.

Similar notions: linked path decompositions, linked tree decompostions.

On words with repeating patterns

Lemma

If w is a word of length N^r over $[r]$, there is a $p \in [r]$ s.t.:

- *some subword u contains numbers $\geqslant p$;*
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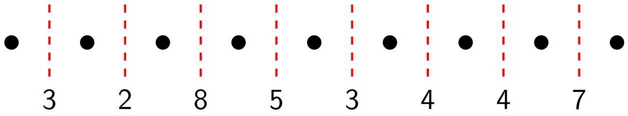

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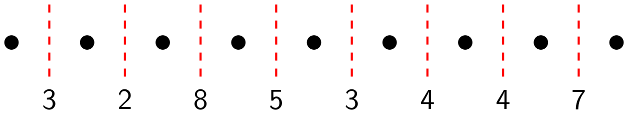

If $|G| > N^r$, some contiguous subsequence of $v_1, \ldots, v_n$ has cuts $\geqslant p$ and $\geqslant N$ cuts of size p.

Bounding the size of obstructions

Goal: show that obstructions for **ctw** $\leqslant k$ are small.

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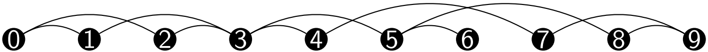

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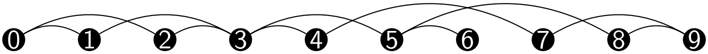

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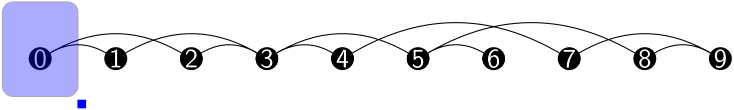

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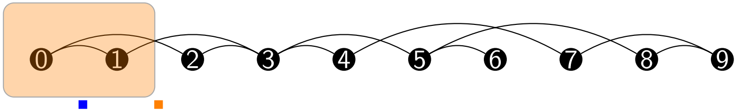

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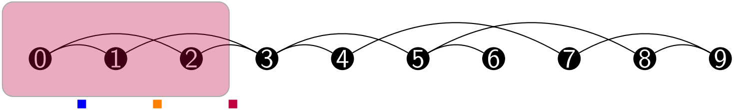

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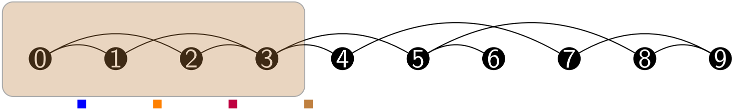

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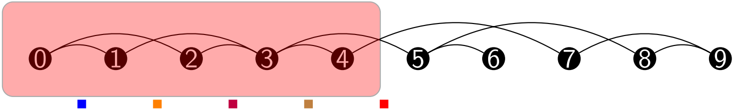

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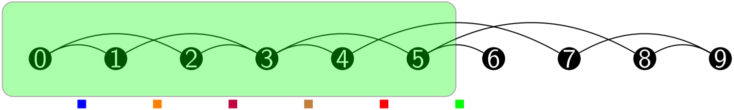

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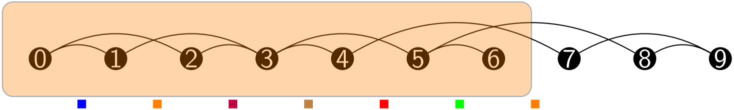

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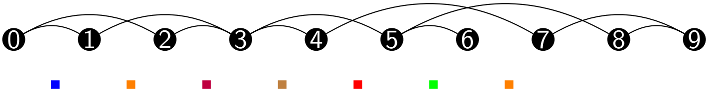

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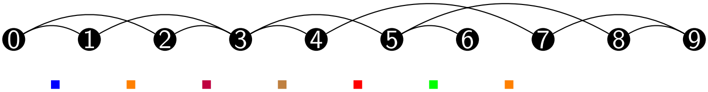

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- this immersion of G has cutwidth $k+1$: contradiction.

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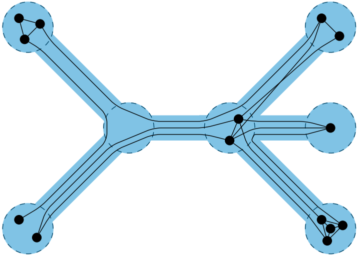
Slightly slower... but much easier!

Extension to tree-like parameters

Tree-cut decomposition:



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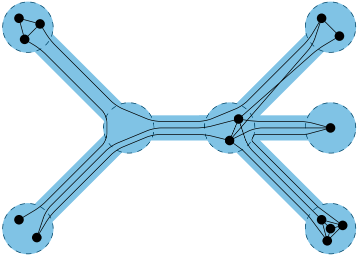

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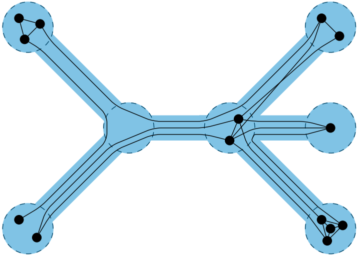

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Hope for similar results in this context (work in progress).

The end

Our contribution:

- a single exponential upper-bound on the size of the obstructions for **ctw** $\leqslant k$;
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Thank you!