Immersions and edge-deletion problems

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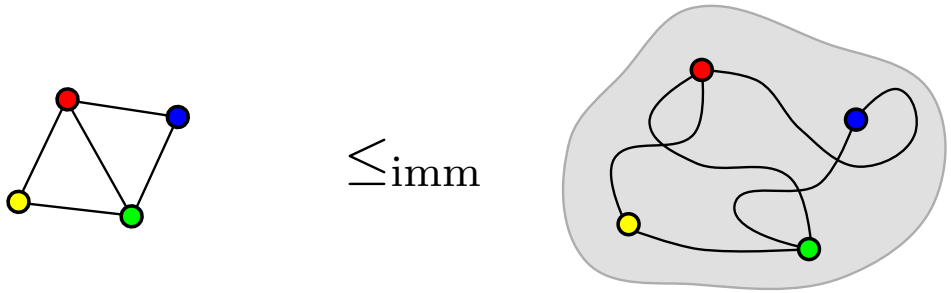
Joint work with Archontia Giannopoulou (TU Berlin), Michał Pilipczuk (University of Warsaw), Dimitrios M. Thilikos (LIRMM–CNRS), and Marcin Wrochna (University of Warsaw).

Immersions

H is an *immersion* of G if we can map

- vertices of H to distinct vertices of G;
- edges of H to edge-disjoint paths of G between the corresponding vertices.



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Input: a graph G and an integer k;
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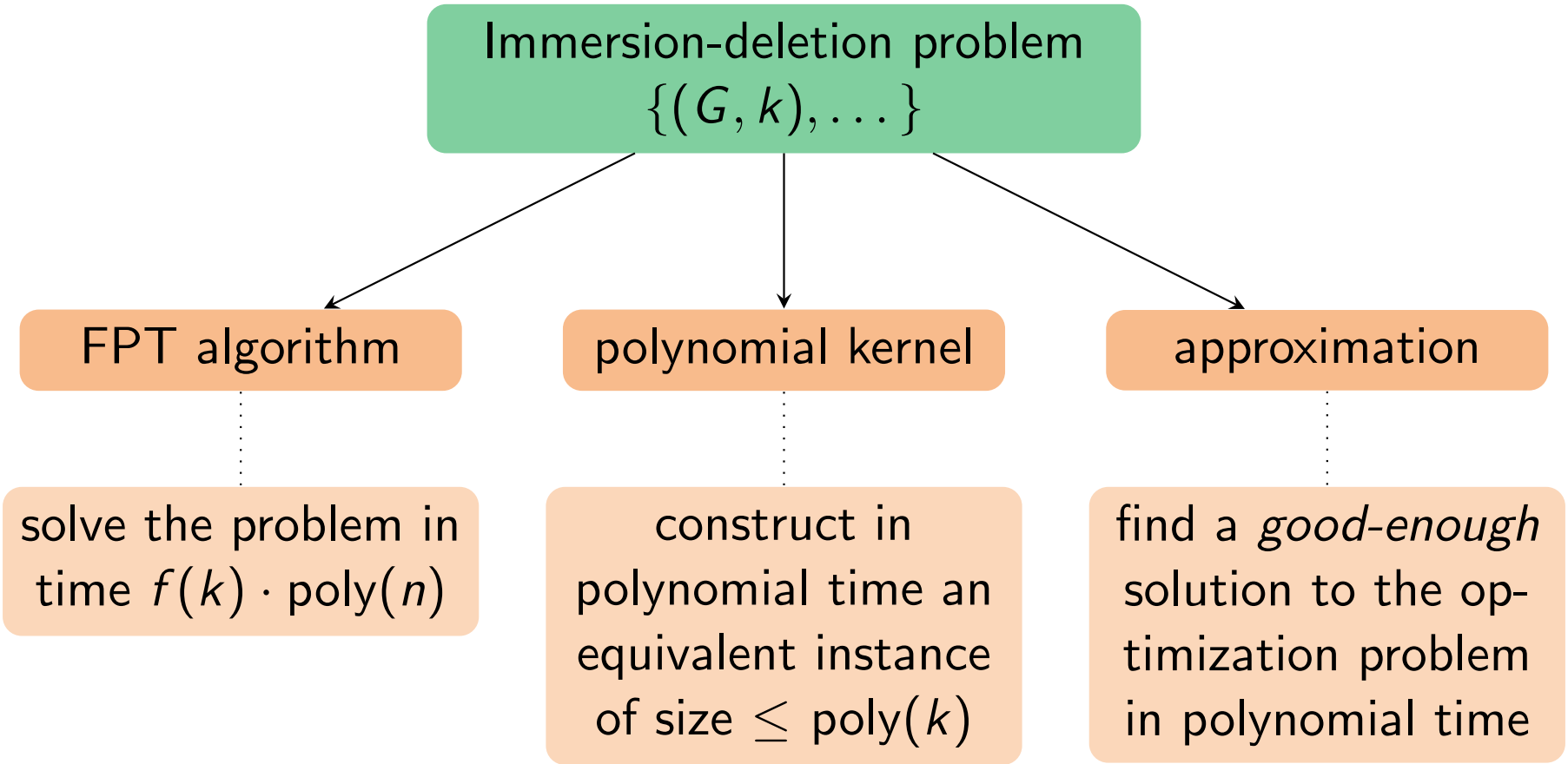
Example: Feedback Arc Set.

The targets


Immersion-deletion problem
$\{(G, k), \ldots\}$
FPT algorithm
polynomial kernel
approximation
solve the problem in time $f(k) \cdot \text{poly}(n)$
construct in polynomial time an equivalent instance of size $\leq \text{poly}(k)$
find a *good-enough* solution to the optimization problem in polynomial time

A meta-theorem on immersion-deletion problems

F: connected planar subcubic graph (in all this talk)

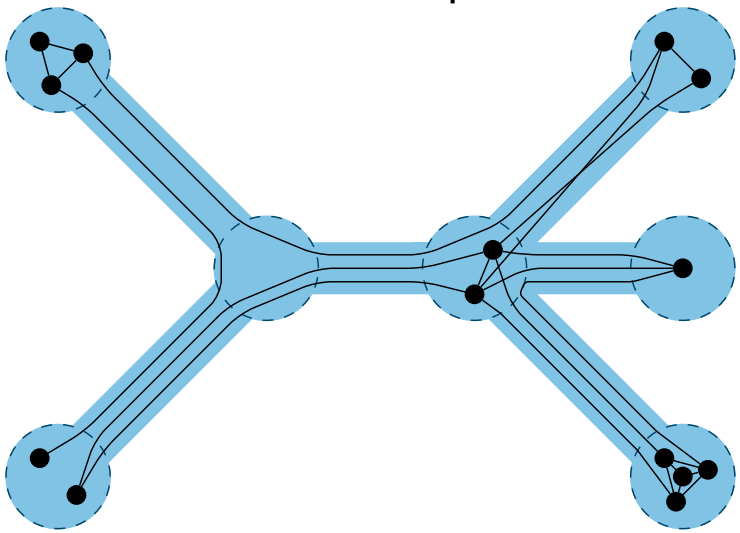
Theorem (Giannopoulou, Pilipczuk, Thilikos, R., Wrochna, 2016+)

F-Immersion-Deletion admits

- a constant factor approximation that runs in polynomial time;
- a $O(k)$-kernel
- (a single exponential FPT algorithm).

The tools for dealing with immersions

Tree-cut decomposition:



Associated parameter: **tcw**
(tree-cut width)

Small **tcw** implies:

- *small* bags;
- *thin* edges;
- small number of *thick* neighbors.

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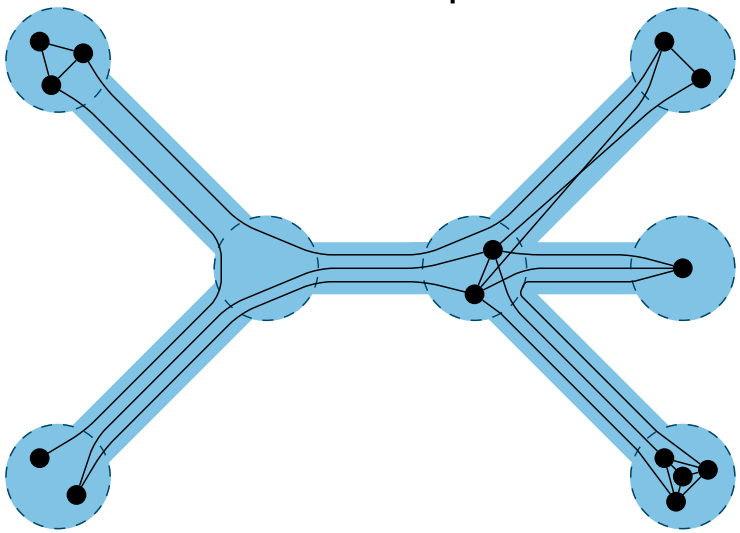

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Theorem [Wollan, JCTB 2015] + [Chuzhoy STOC 2015]

For every planar subcubic graph F,

$$F \not\leq_{\text{imm}} G \quad \Rightarrow \quad \mathbf{tcw}(G) = O(|F|^{30}).$$

Approximation algorithm: structure of an instance

OPT: min size of a set of edges meeting all F-immersions (solution)

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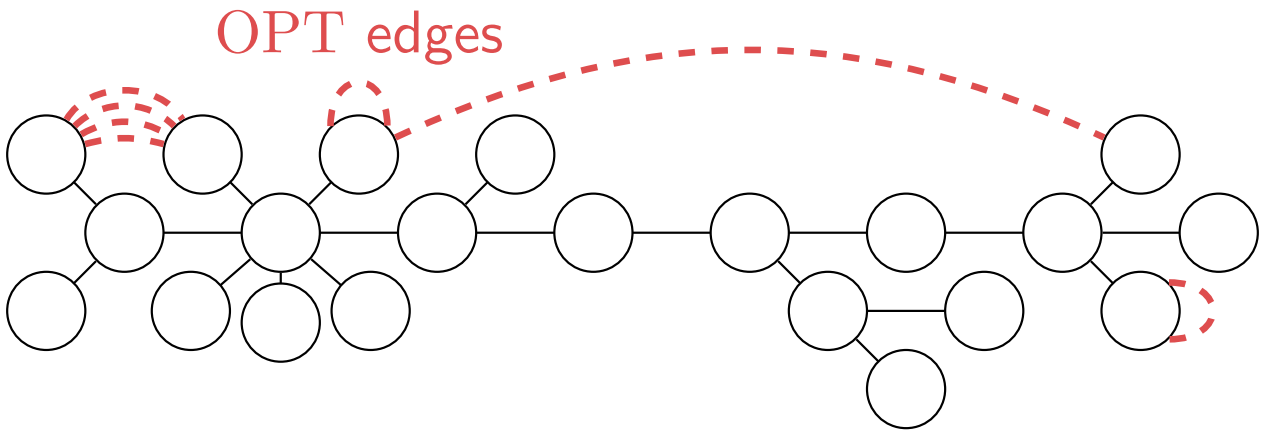

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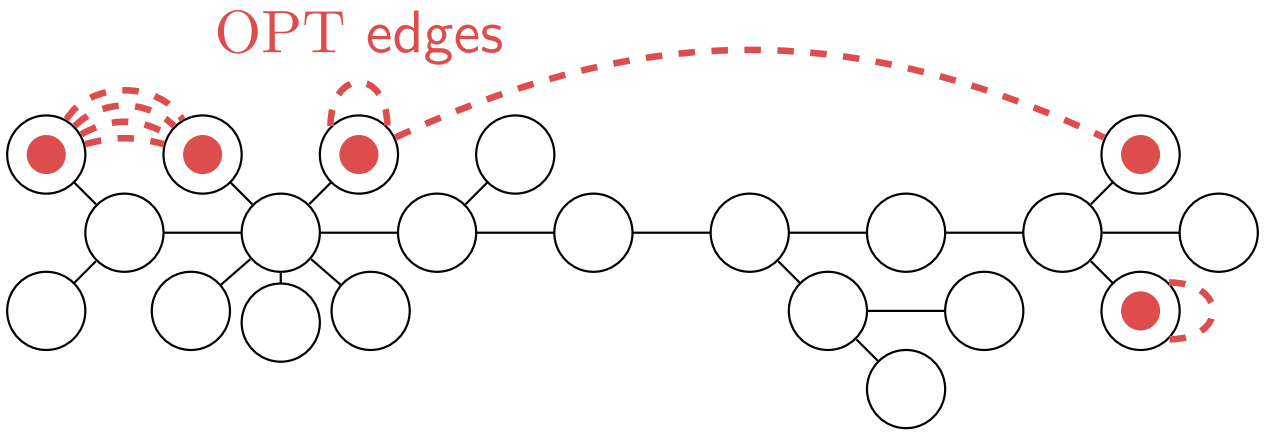

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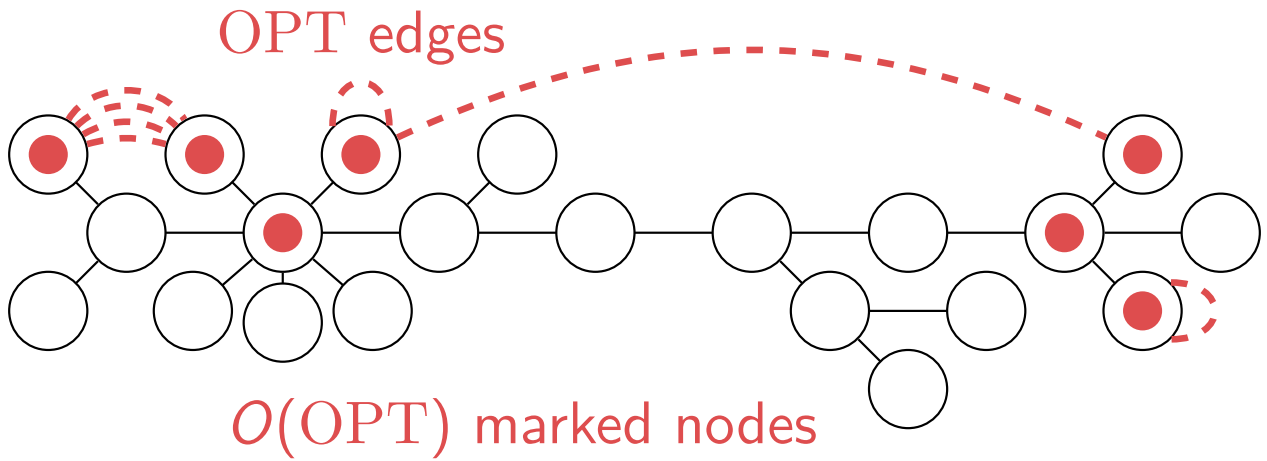

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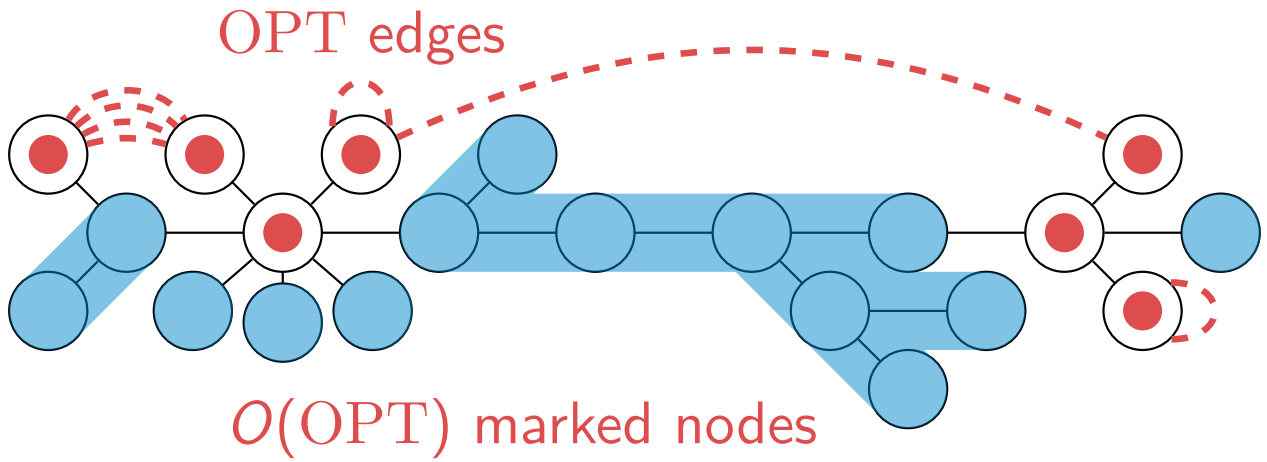

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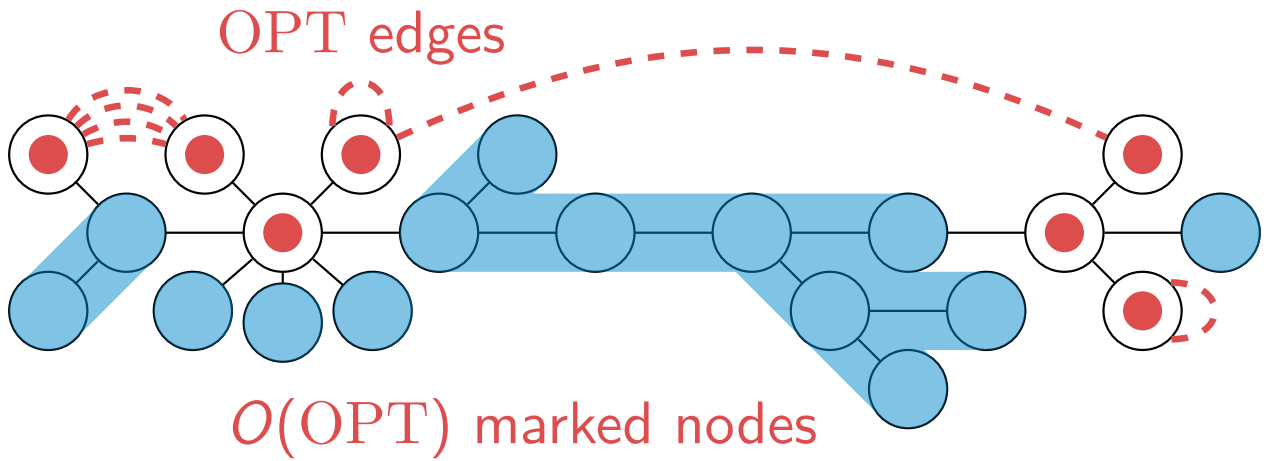

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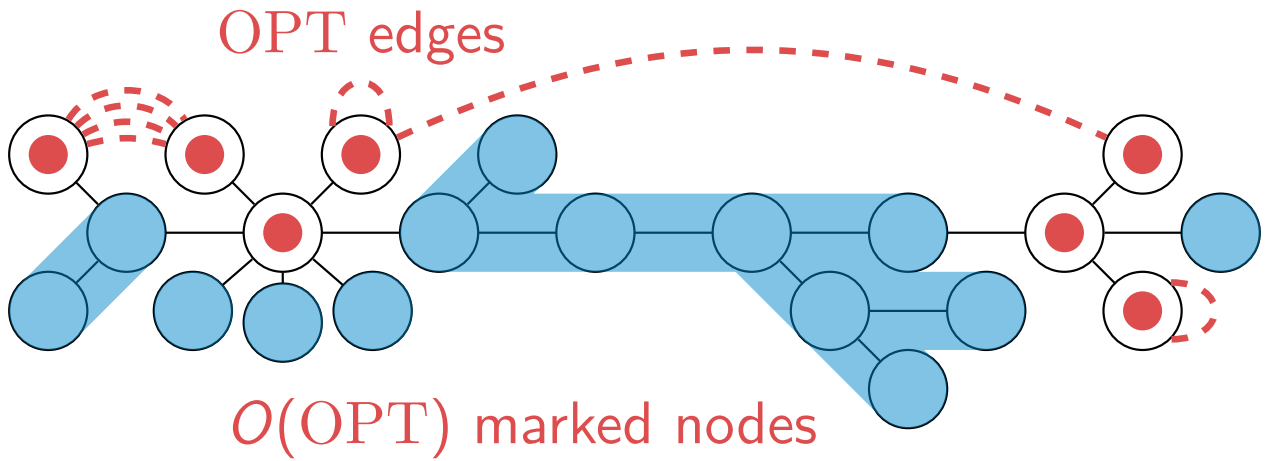

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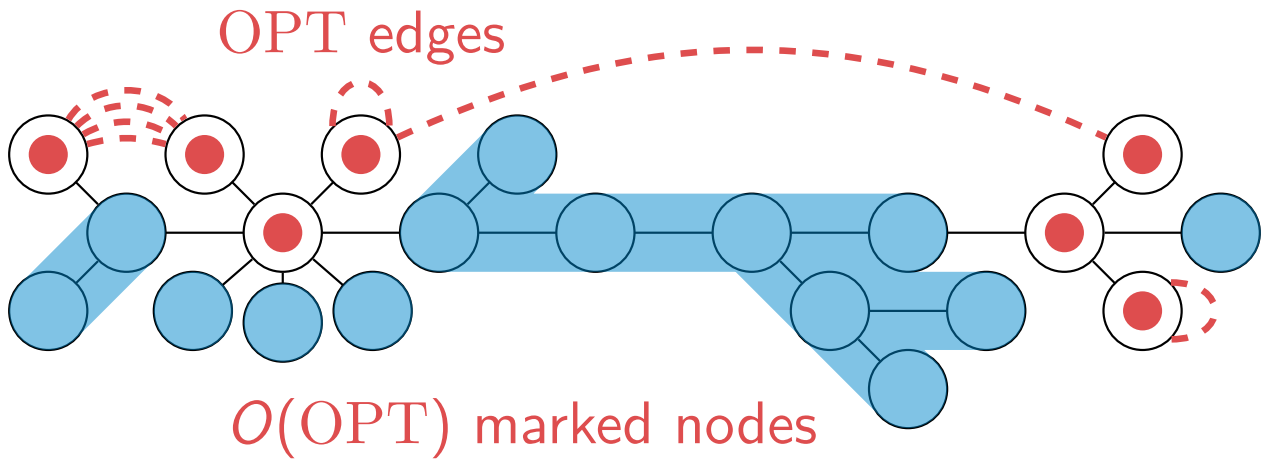

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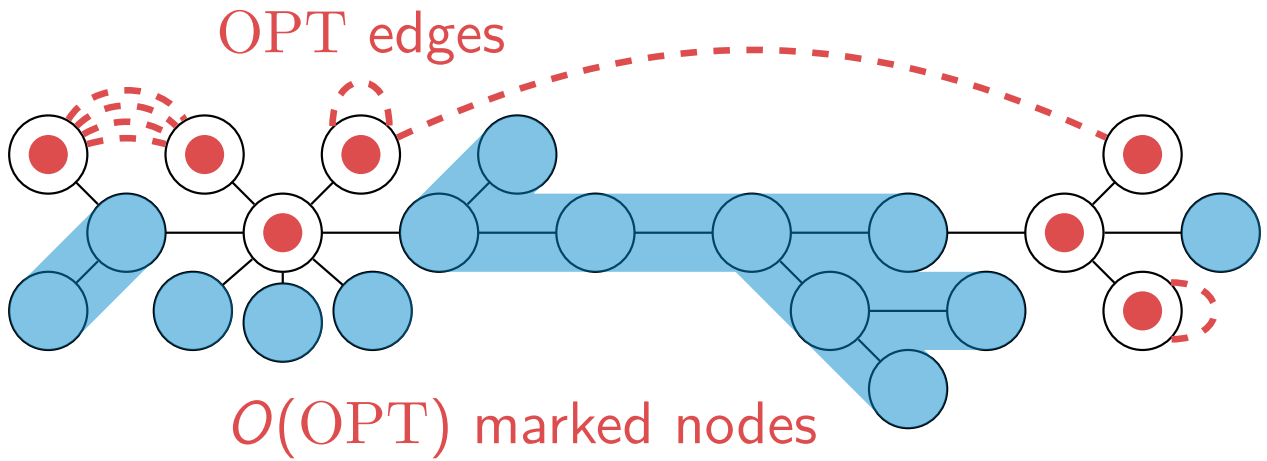

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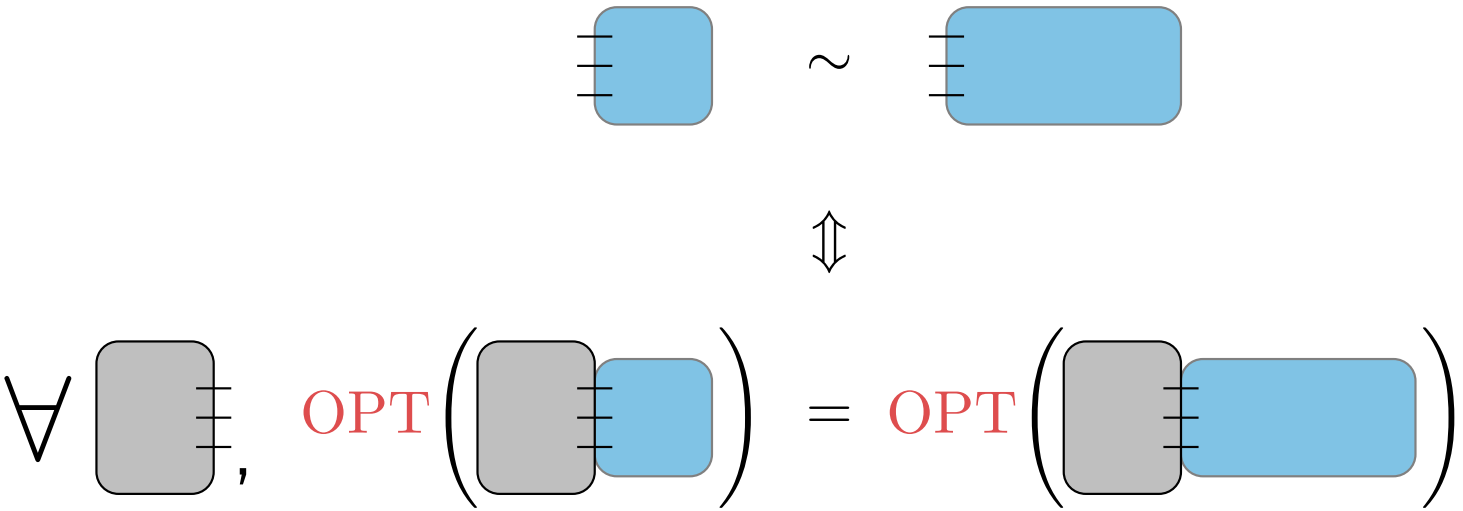
Dealing with blue parts

We define an equivalence relation on subgraphs with a small boundary:



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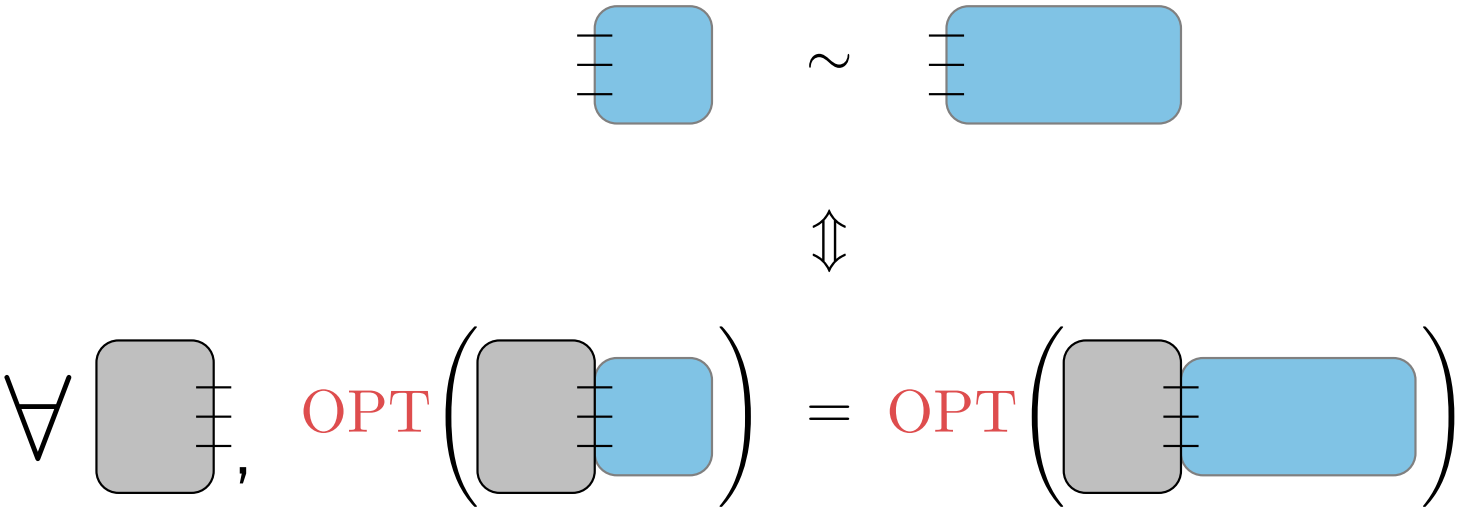

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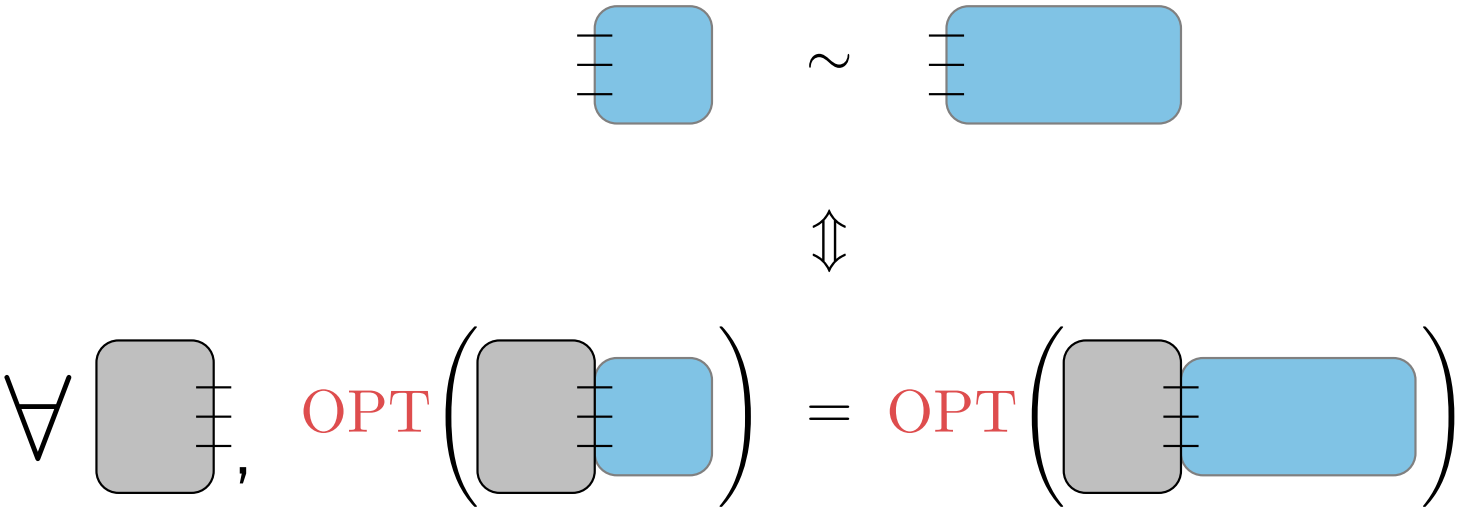

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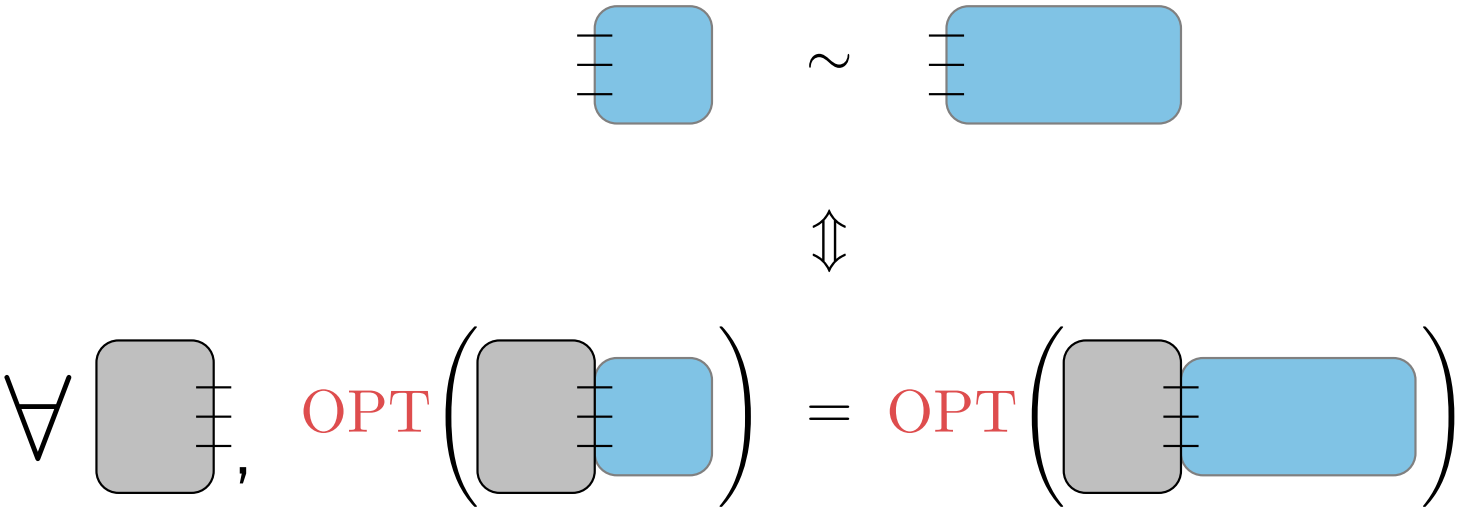

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- for each protrusion there is an equivalent protrusion of constant size;
- we can find and *compress* protrusions in polynomial time.

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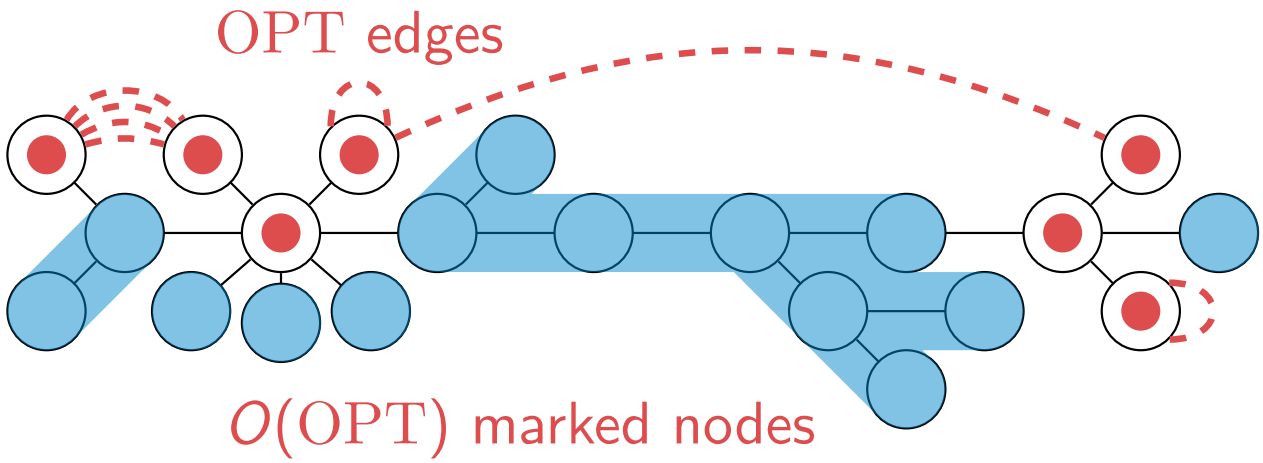

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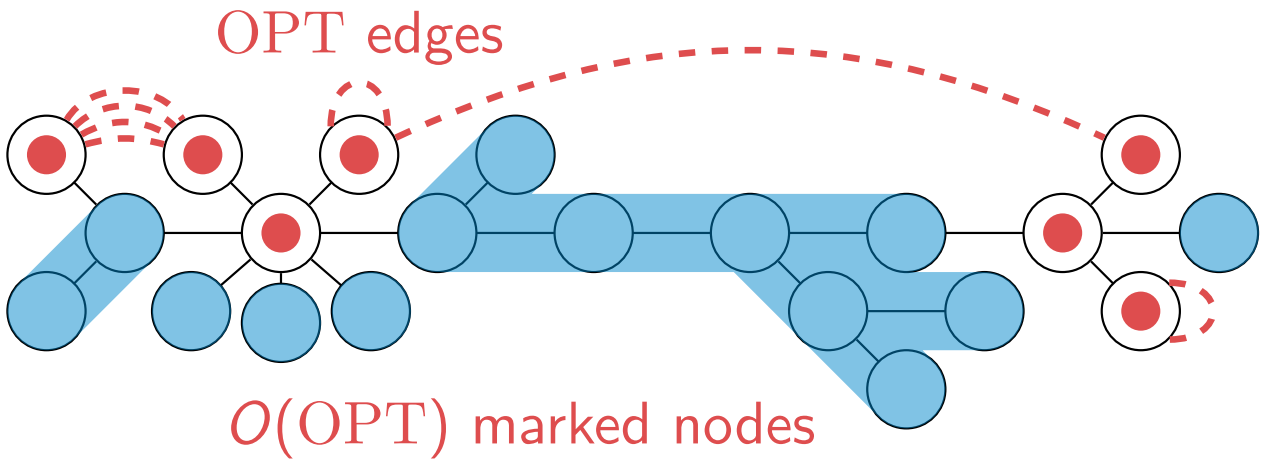

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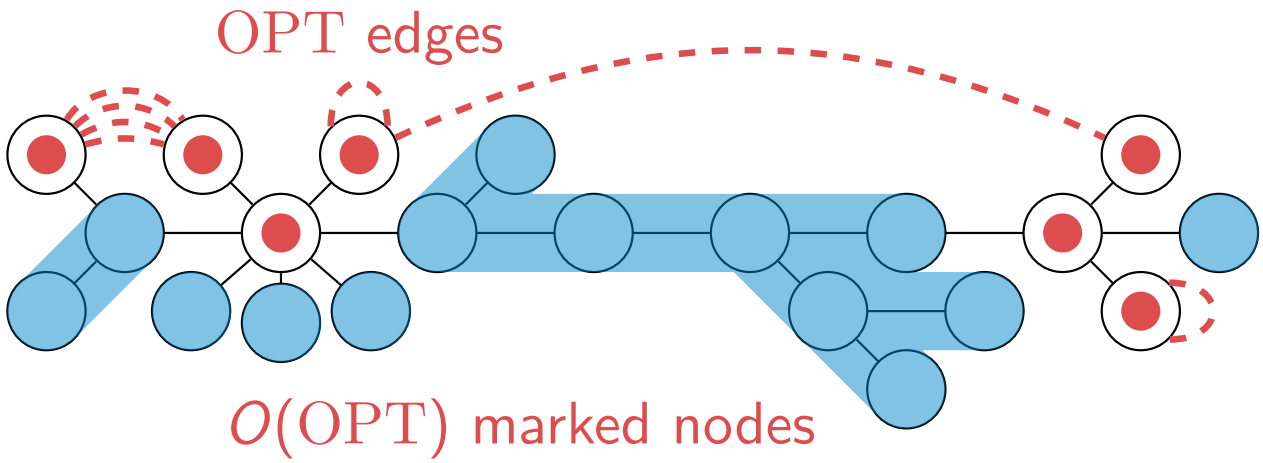

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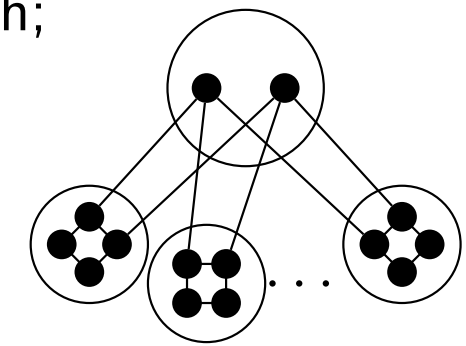

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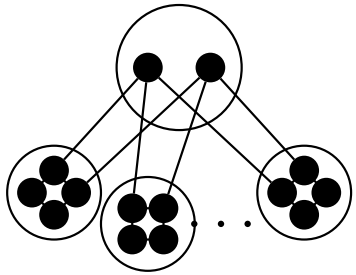

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We can find in polytime $Z \subseteq E(G)$ s.t., for some constant c,

$$\mathrm{OPT}(G \setminus Z) < \mathrm{OPT}(G) \qquad |Z| \leq c \cdot (\mathrm{OPT}(G) - \mathrm{OPT}(G \setminus Z)).$$

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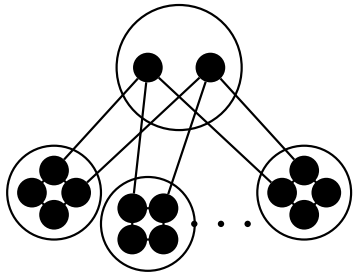

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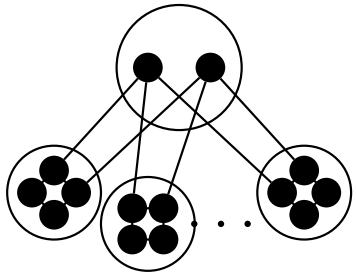

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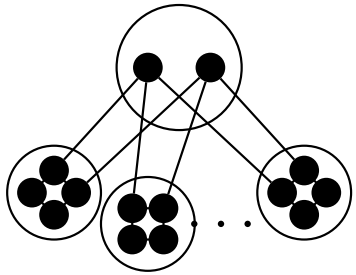

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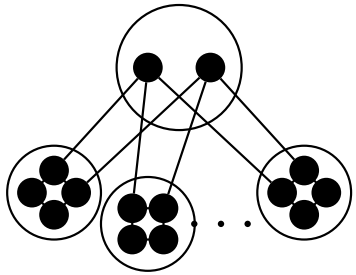

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Iteratively removing this set of edges gives the desired approximation.

The approximation algorithm

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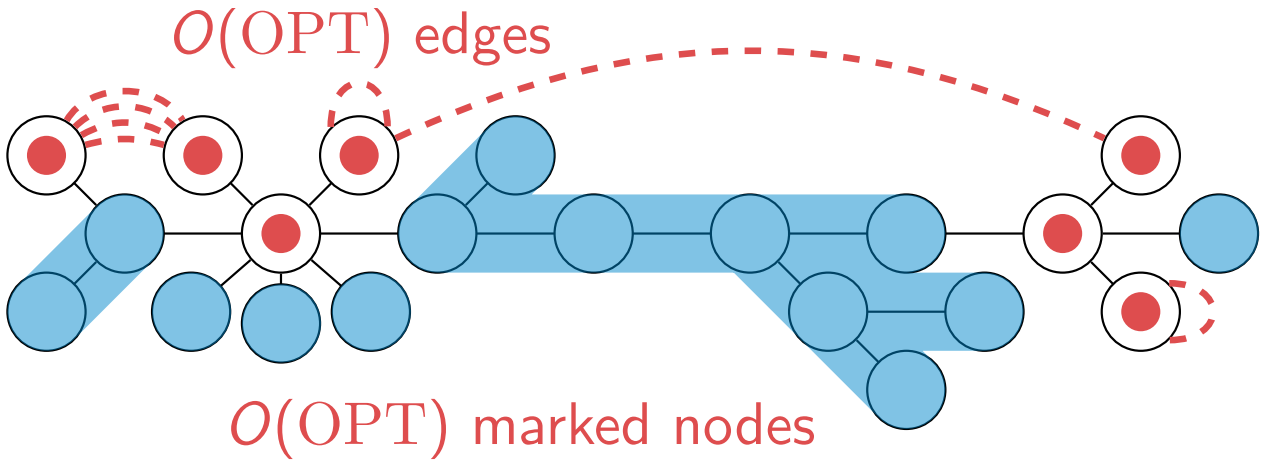
5. output $Z \cup Z'$.

Kernelization

We consider an instance (G, k) with an approximate solution.



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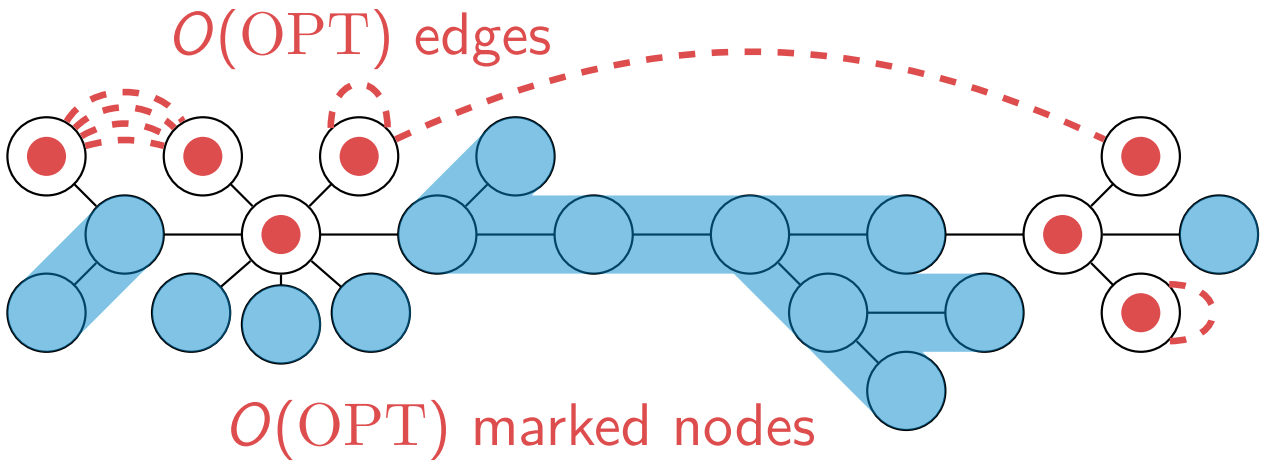

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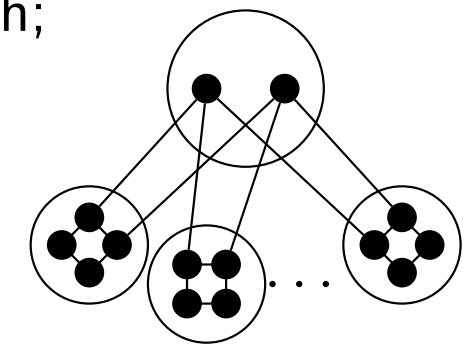
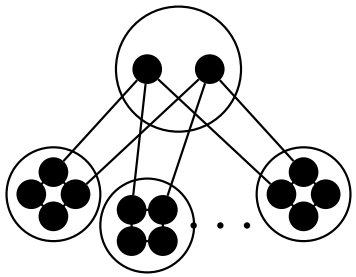


Dealing with bouquets (again)



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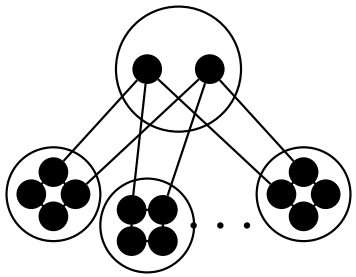

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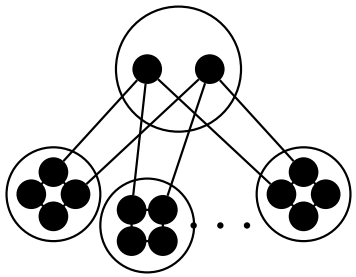

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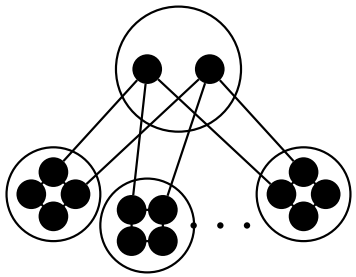

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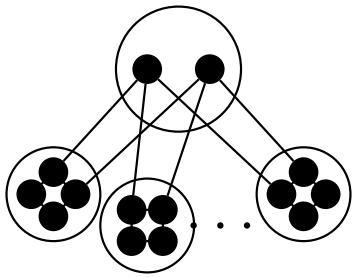

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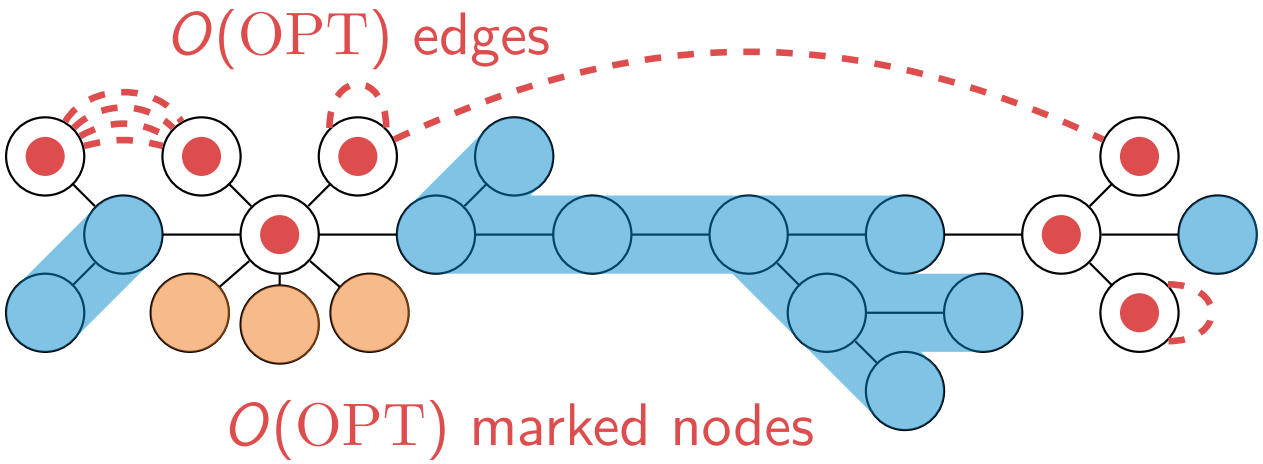
After reducing protrusions and cleaning bouquets



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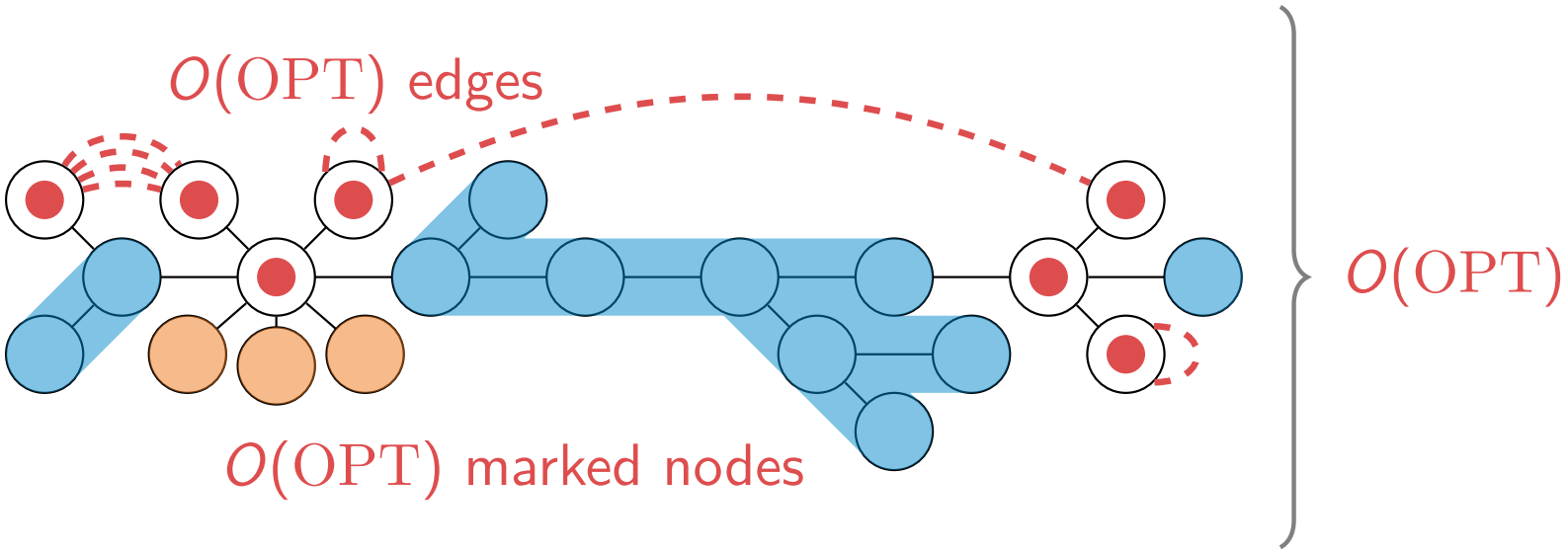

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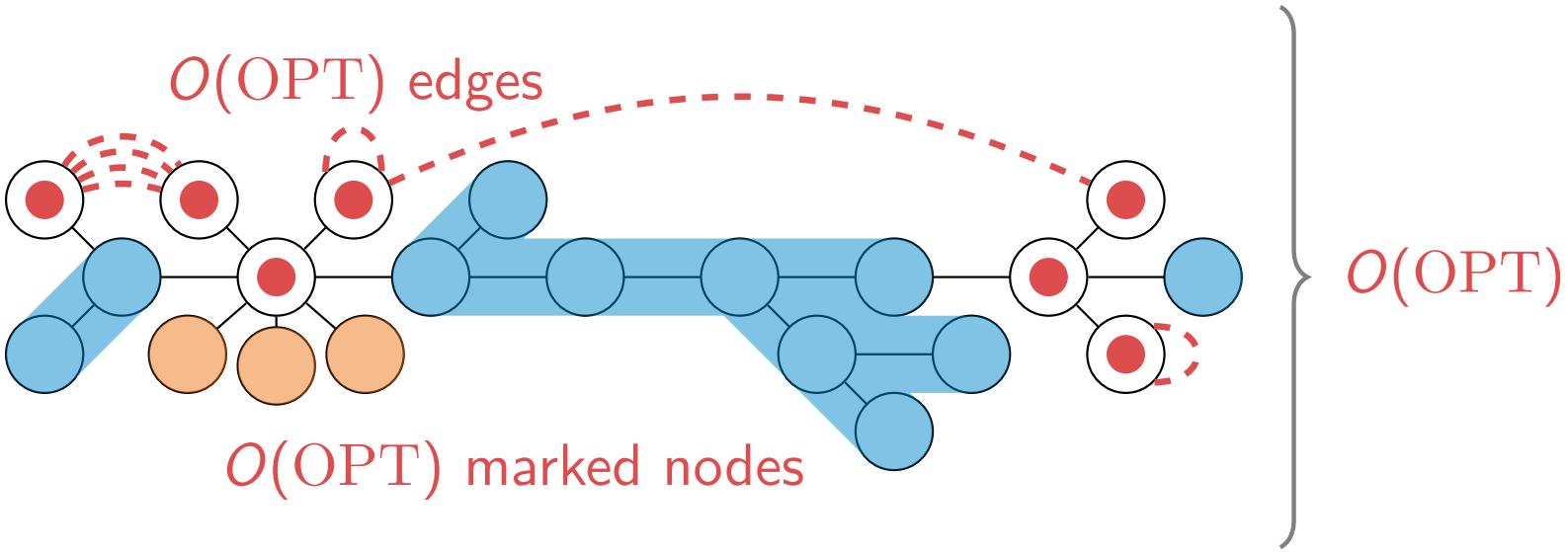

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The kernel:

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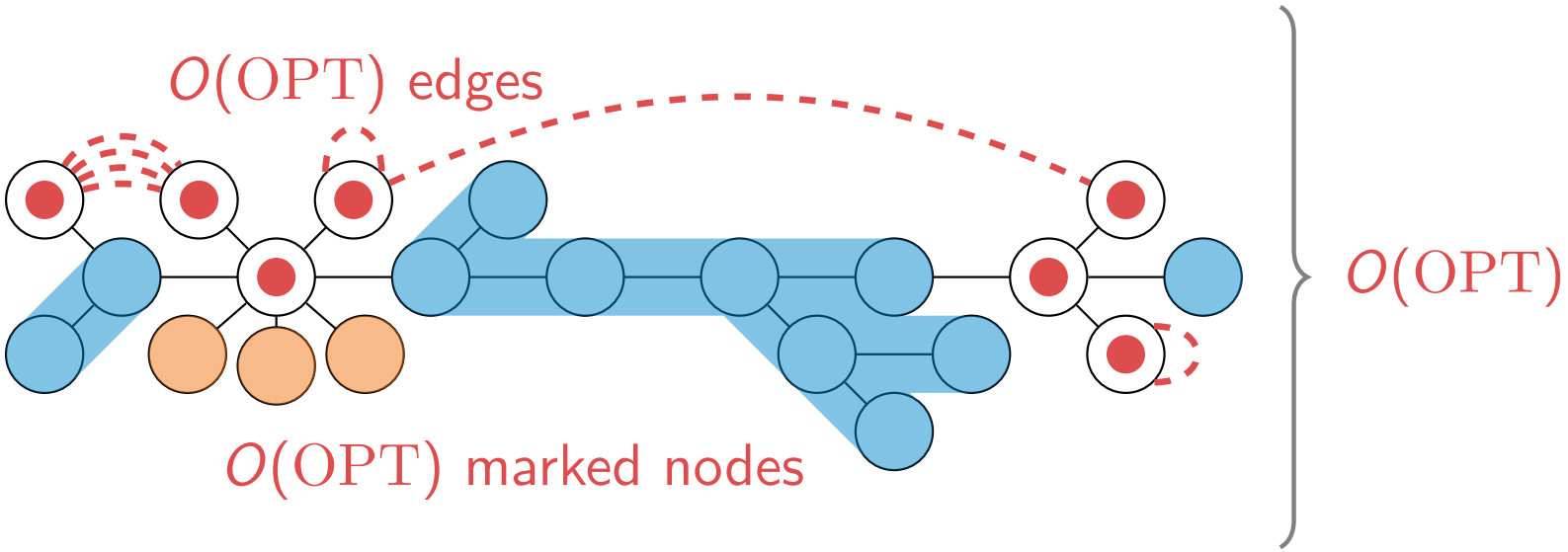

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- constant size;
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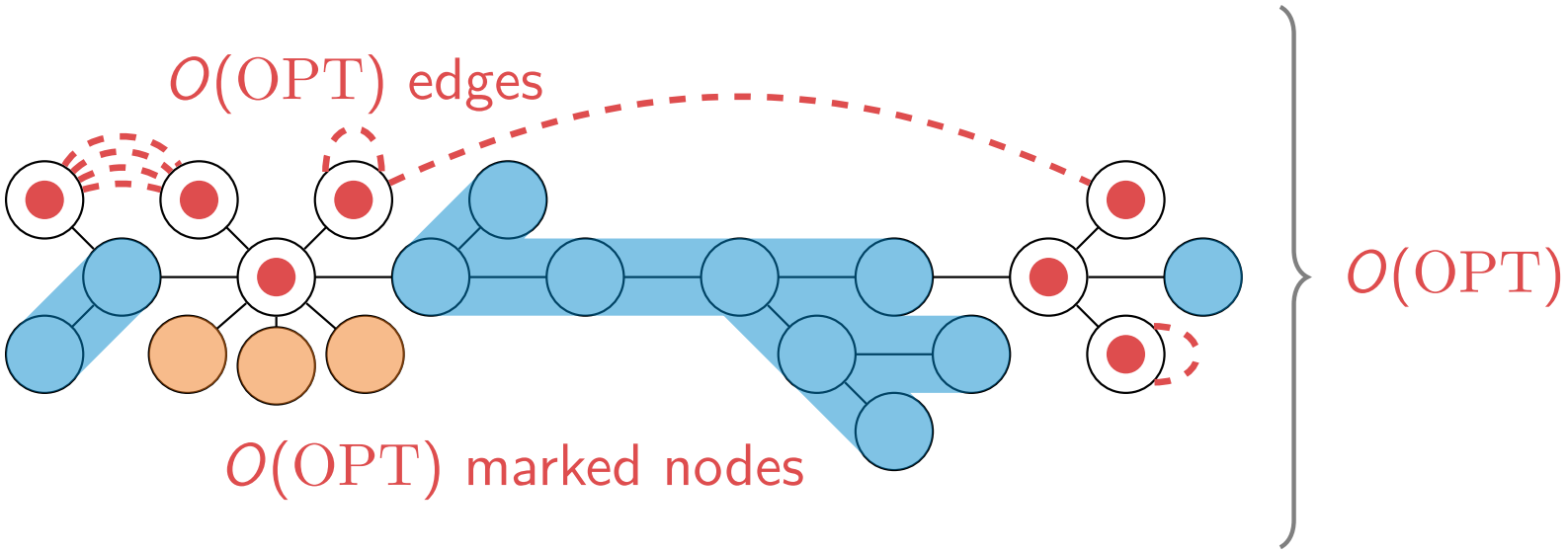

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Application to obstructions

F: connected planar subcubic

Kernelization result (reminder)

Given an instance (G, k) of F-Immersion Deletion, we can compute in polynomial time an equivalent instance of size $O(k)$.

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- as G is minimal, $G = G'$.

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Idea: graphs G with small $\mathbf{p}(G)$ exclude a planar subcubic graph.

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F-Minor-Deletion

Input: a graph G and an integer k;
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Same approach, different issues.

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When F is connected planar subcubic, F-Immersion Deletion has:

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Thank you for your attention!