Linear kernels for edge deletion problems to immersion-closed graph classes

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LIRMM and University of Warsaw

Journées du GT COA 2016

Joint work with Archontia Giannopoulou (TU Berlin), Michał Pilipczuk (University of Warsaw), Dimitrios M. Thilikos (LIRMM–CNRS), and Marcin Wrochna (University of Warsaw).

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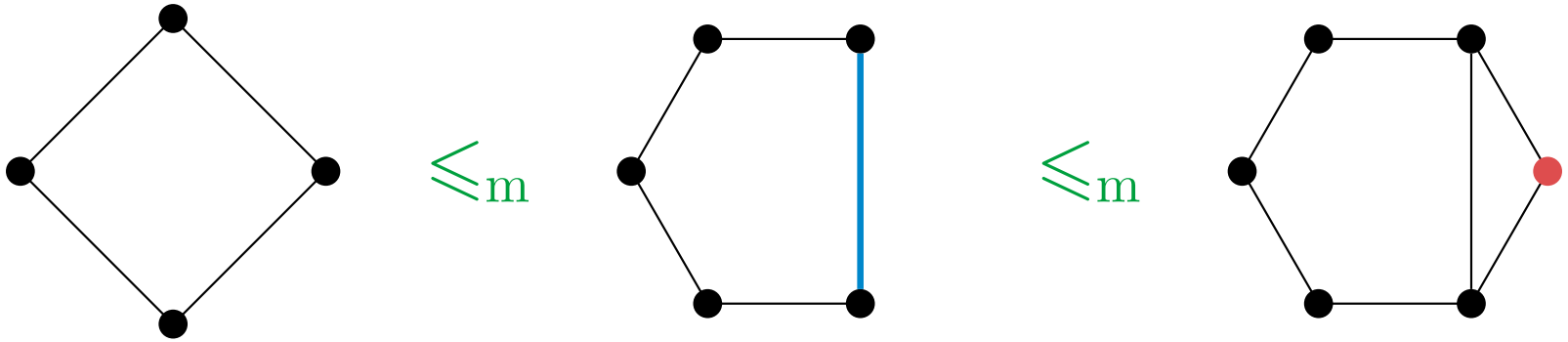
Many natural problems are deletion problems: Vertex Cover, Feedback Vertex Set, Feedback Arc Set, etc..

Minors and immersions

H is a *minor* of G if it can be obtained by deleting vertices or edges and contracting edges.



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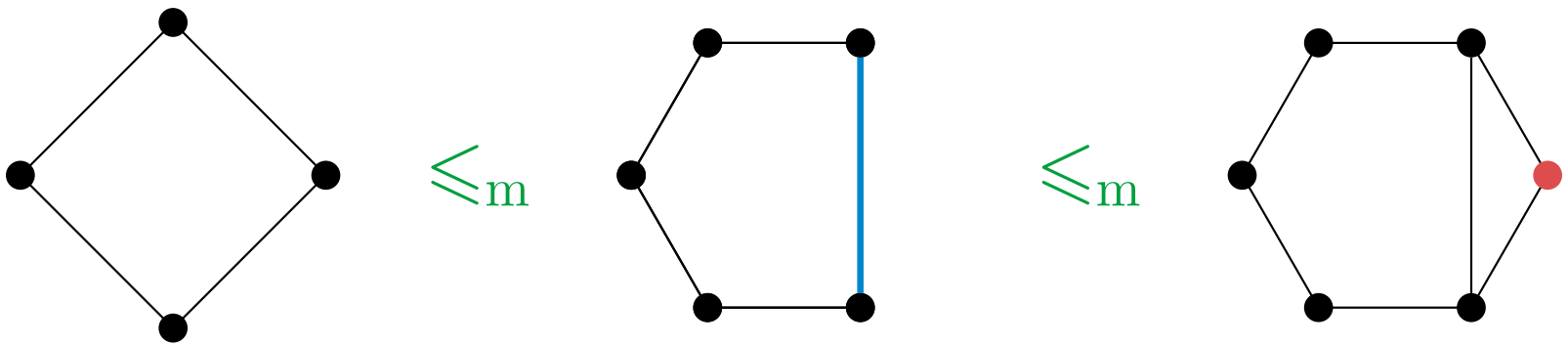

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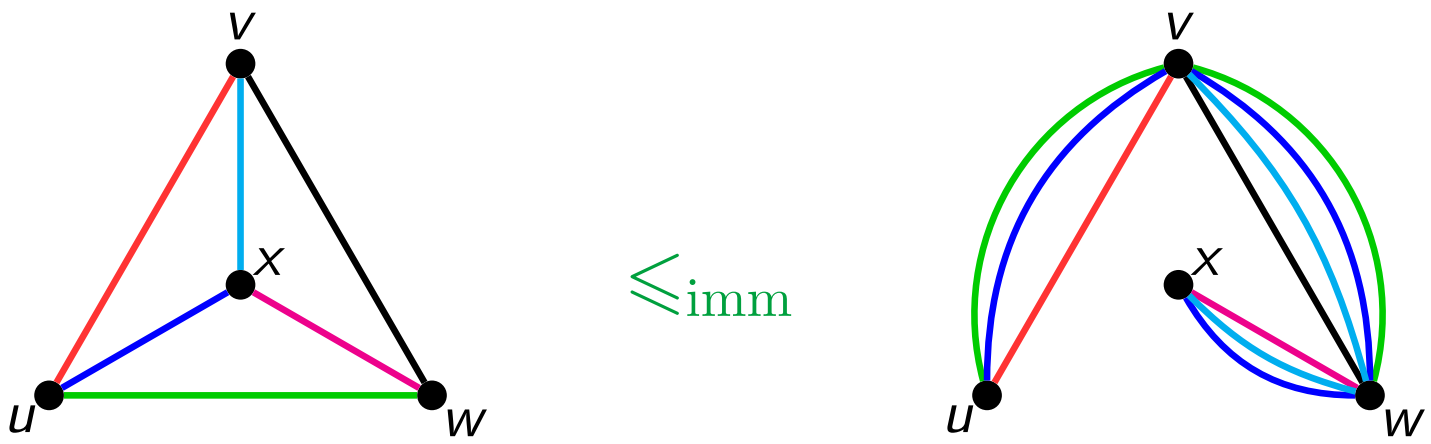

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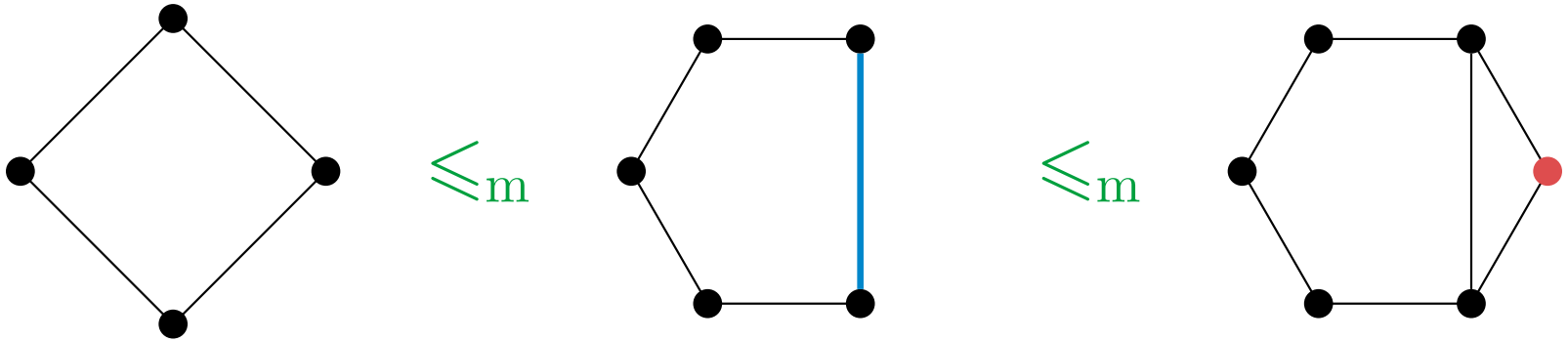

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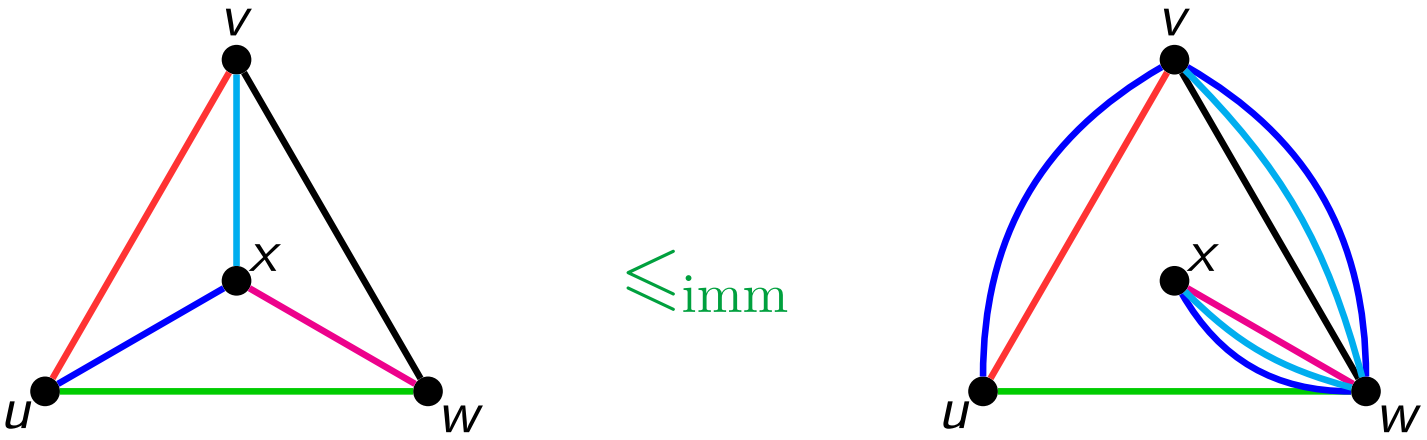

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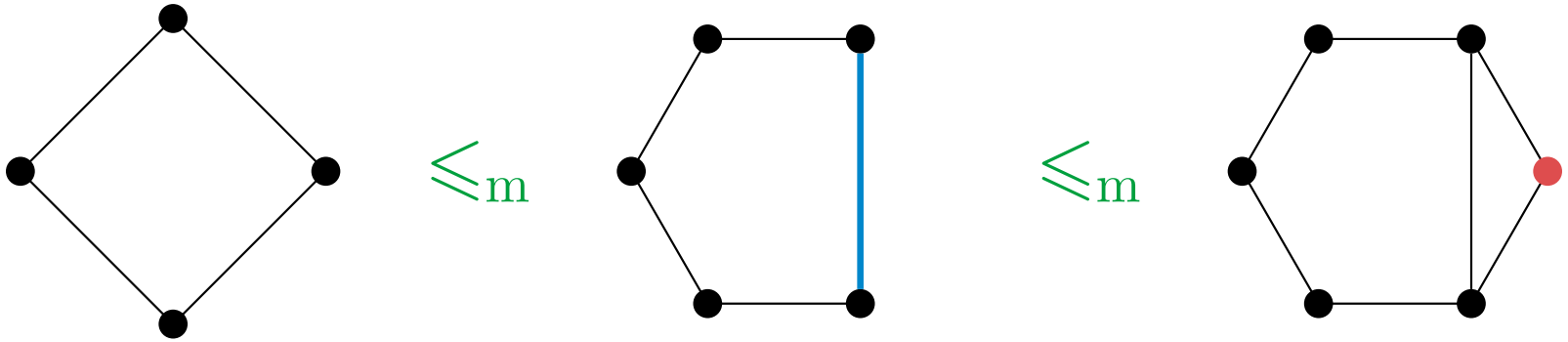

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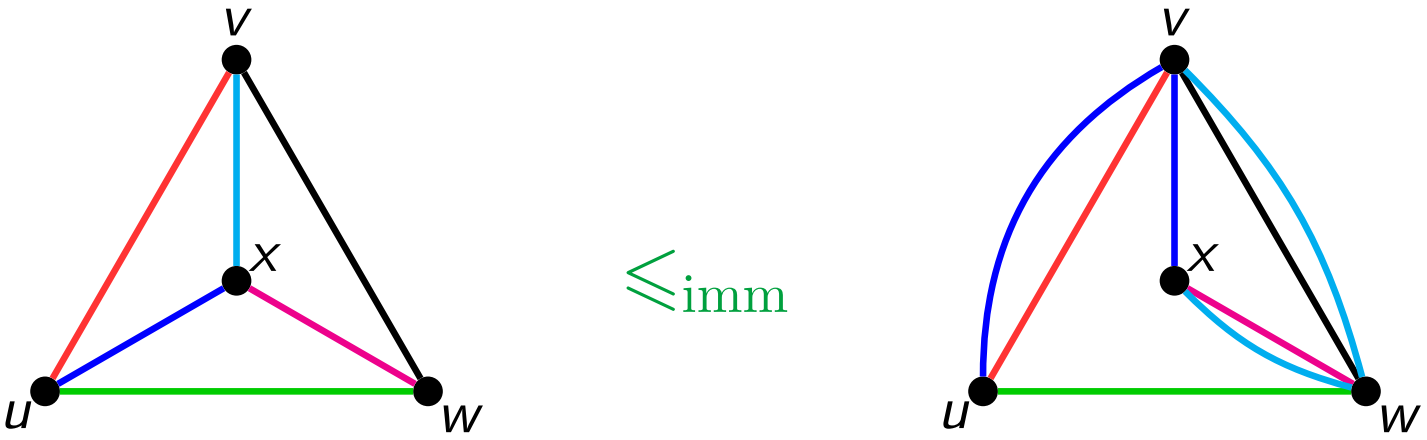

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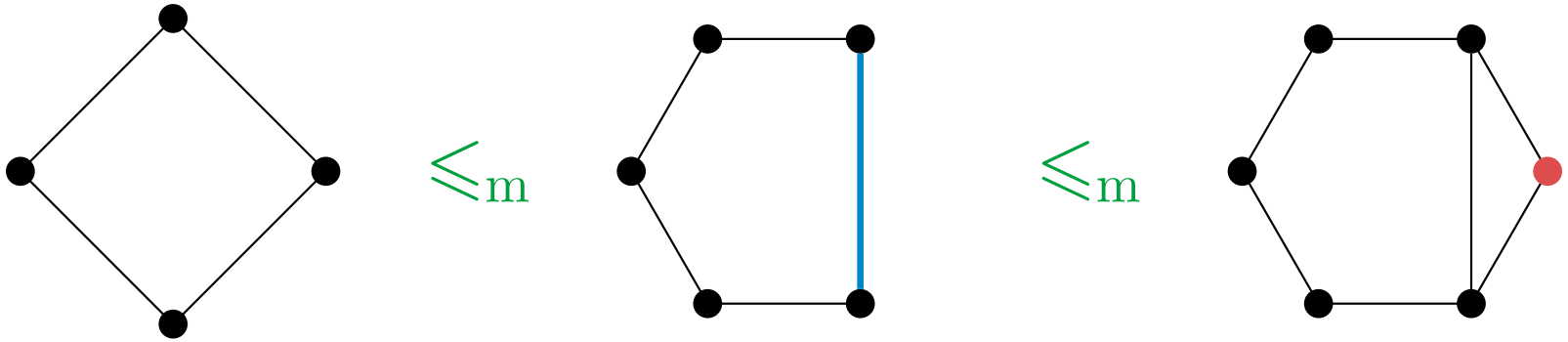

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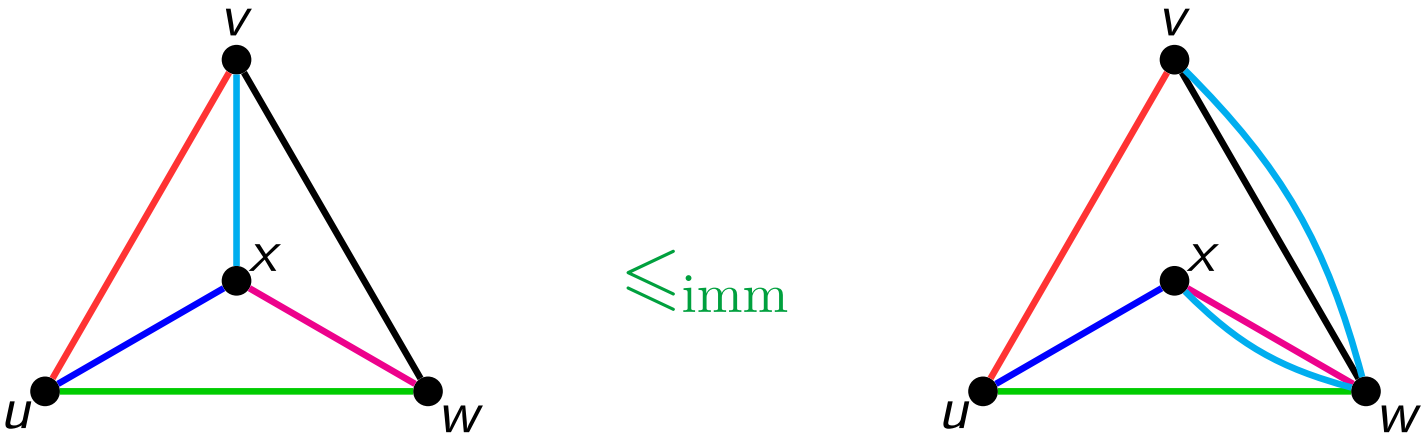

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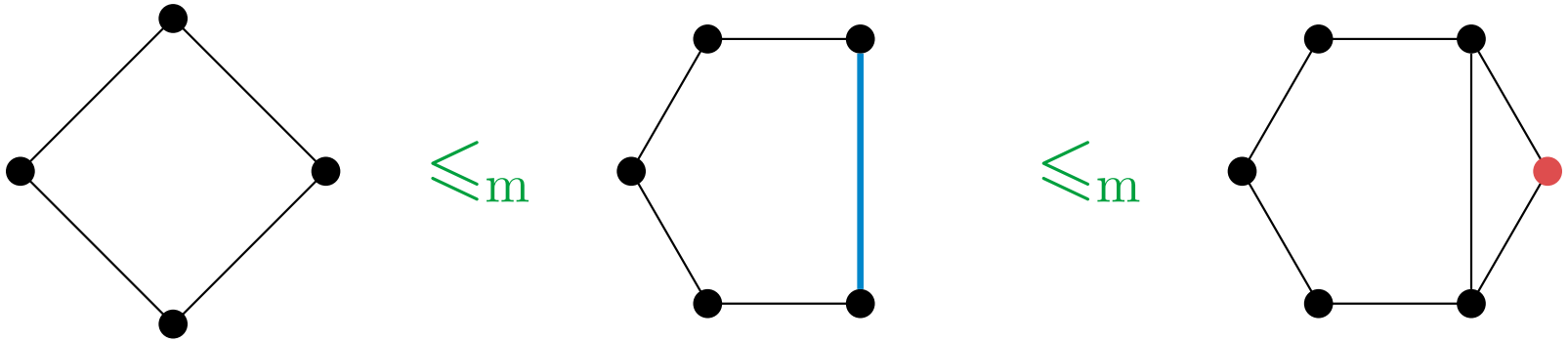

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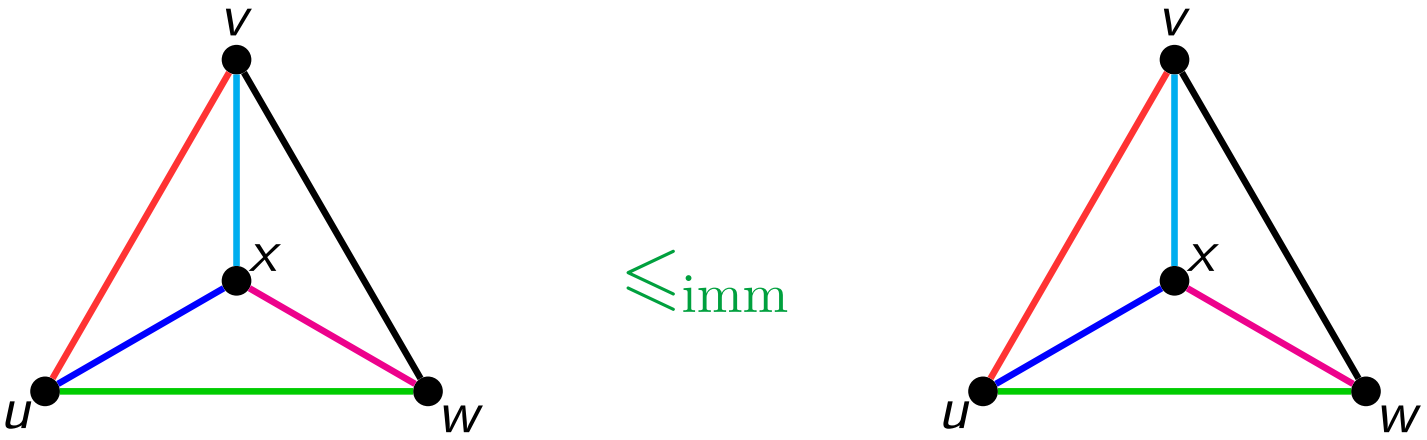
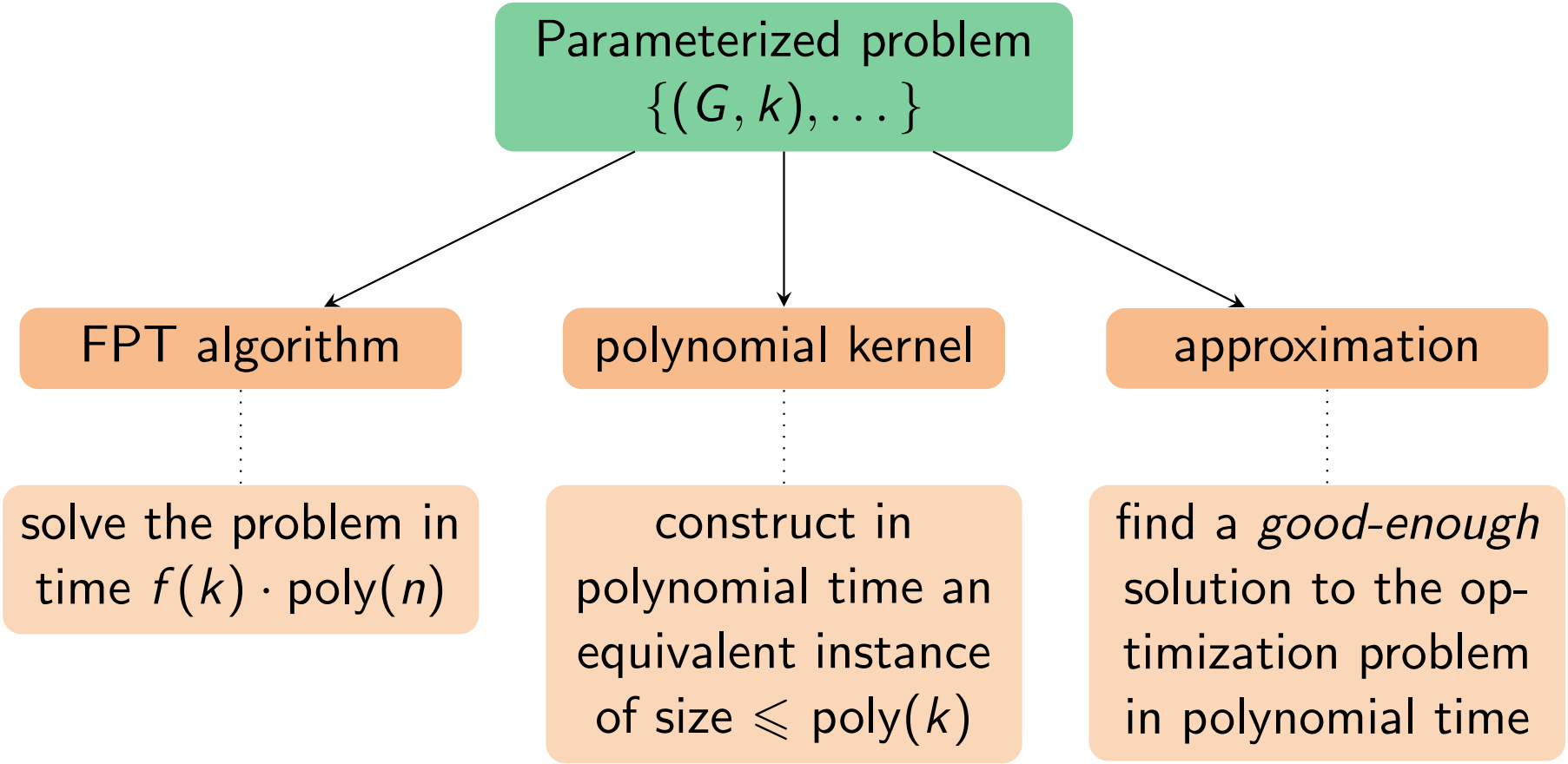


The targets


Parameterized problem
$\{(G, k), \dots\}$
FPT algorithm
polynomial kernel
approximation
solve the problem in time $f(k) \cdot \mathrm{poly}(n)$
construct in polynomial time an equivalent instance of size $\leqslant \mathrm{poly}(k)$
find a *good-enough* solution to the optimization problem in polynomial time

Meta-theorems on deletion problems

$\mathcal{F}$: finite family of connected graphs that contains a planar graph.

Theorem [Fomin, Lokshtanov, Misra, Saurabh, FOCS 2012]

$\mathcal{F}$-Minor-Deletion (vertex removal) admits a

- randomized constant factor approximation running in polynomial time;
- $O(k^{c_{\mathcal{F}}})$-kernel that can be computed in polynomial time;
- single exponential FPT algorithm.

Meta-theorems on deletion problems

$\mathcal{F}$: finite family of connected graphs that contains a planar subcubic graph.

Theorem (Giannopoulou, Pilipczuk, Thilikos, R., Wrochna, 2016+)

$\mathcal{F}$-IMMERSION-DELETION (edge removal) admits a

- constant factor approximation that runs in polynomial time;
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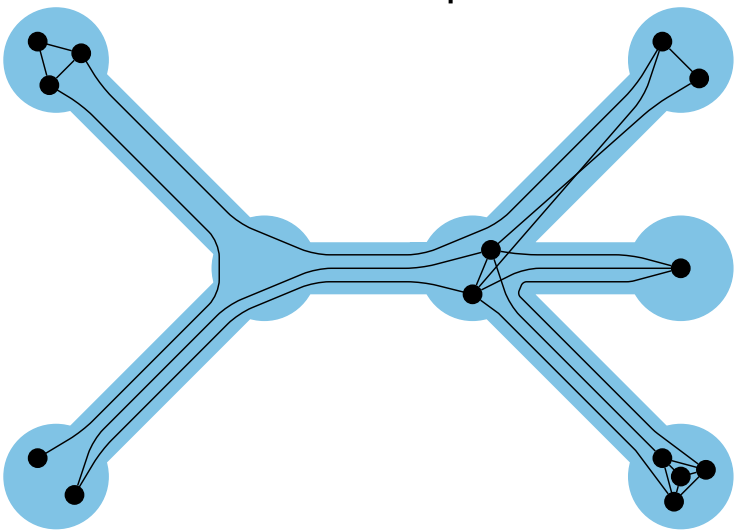
The tools for dealing with immersions

Tree-cut decomposition:



Associated parameter: **tcw**
(tree-cut width)

Small **tcw** implies:

- small bags;
- *thin* edges;
- small number of *thick* neighbors.

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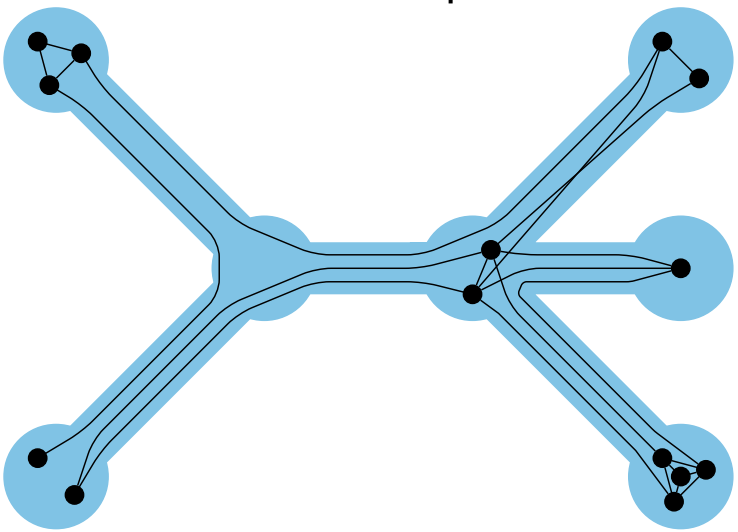

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Theorem [Wollan, JCTB 2015] + [Chuzhoy STOC 2015+]

For every planar subcubic graph H on h edges,

$$H \not\leq_{\mathrm{imm}} G \quad \Rightarrow \quad \mathbf{tcw}(G) = O(h^{30}).$$

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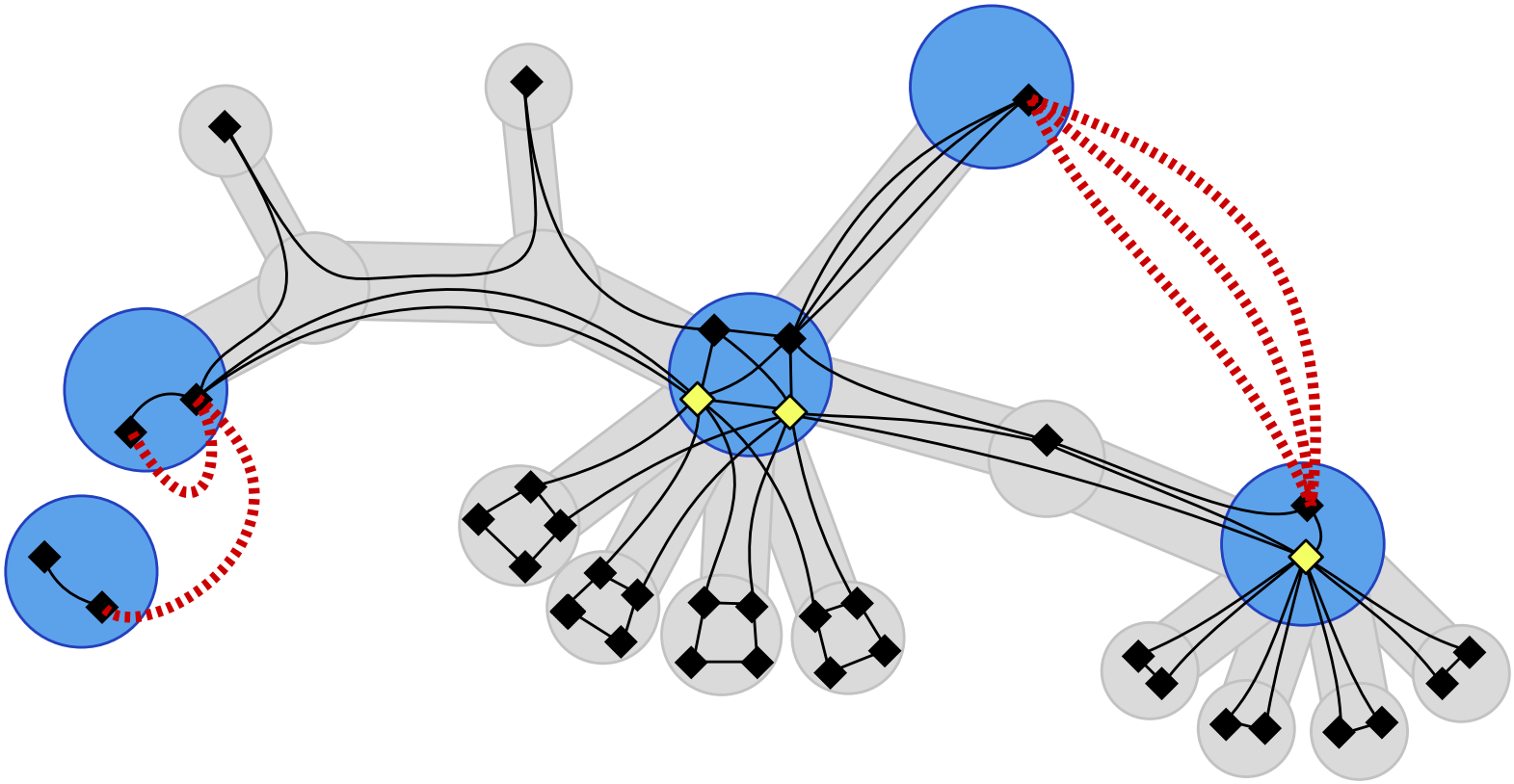

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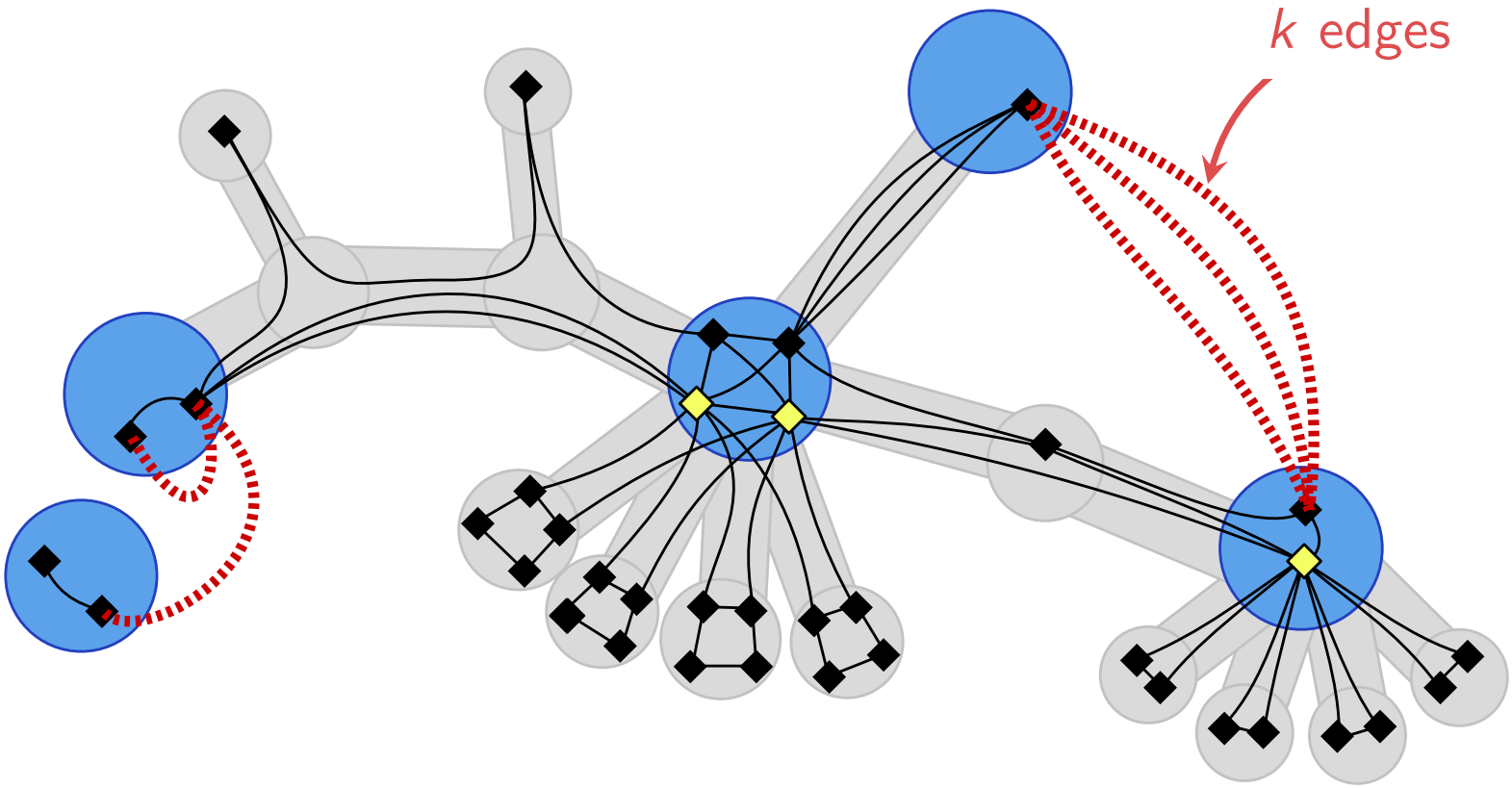

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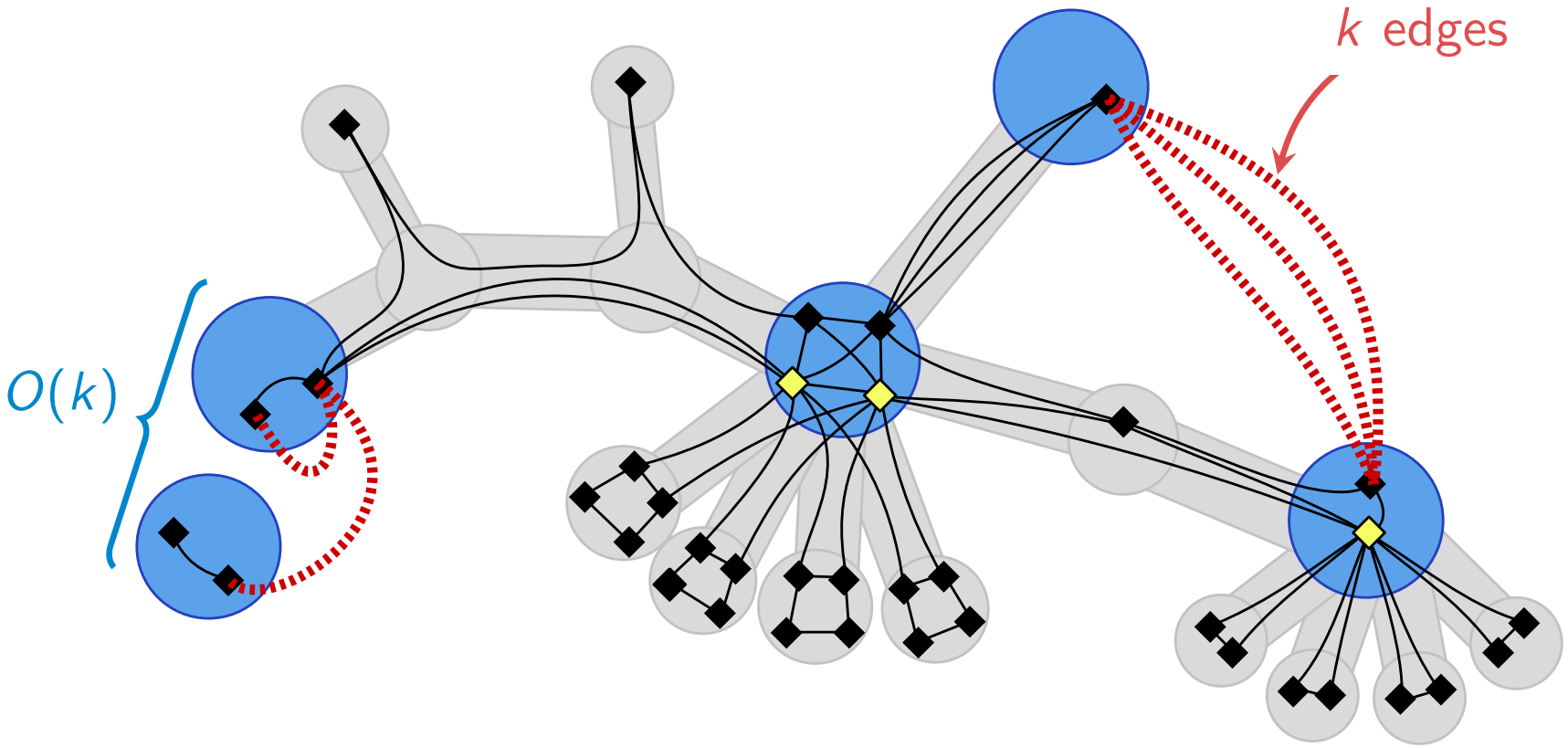

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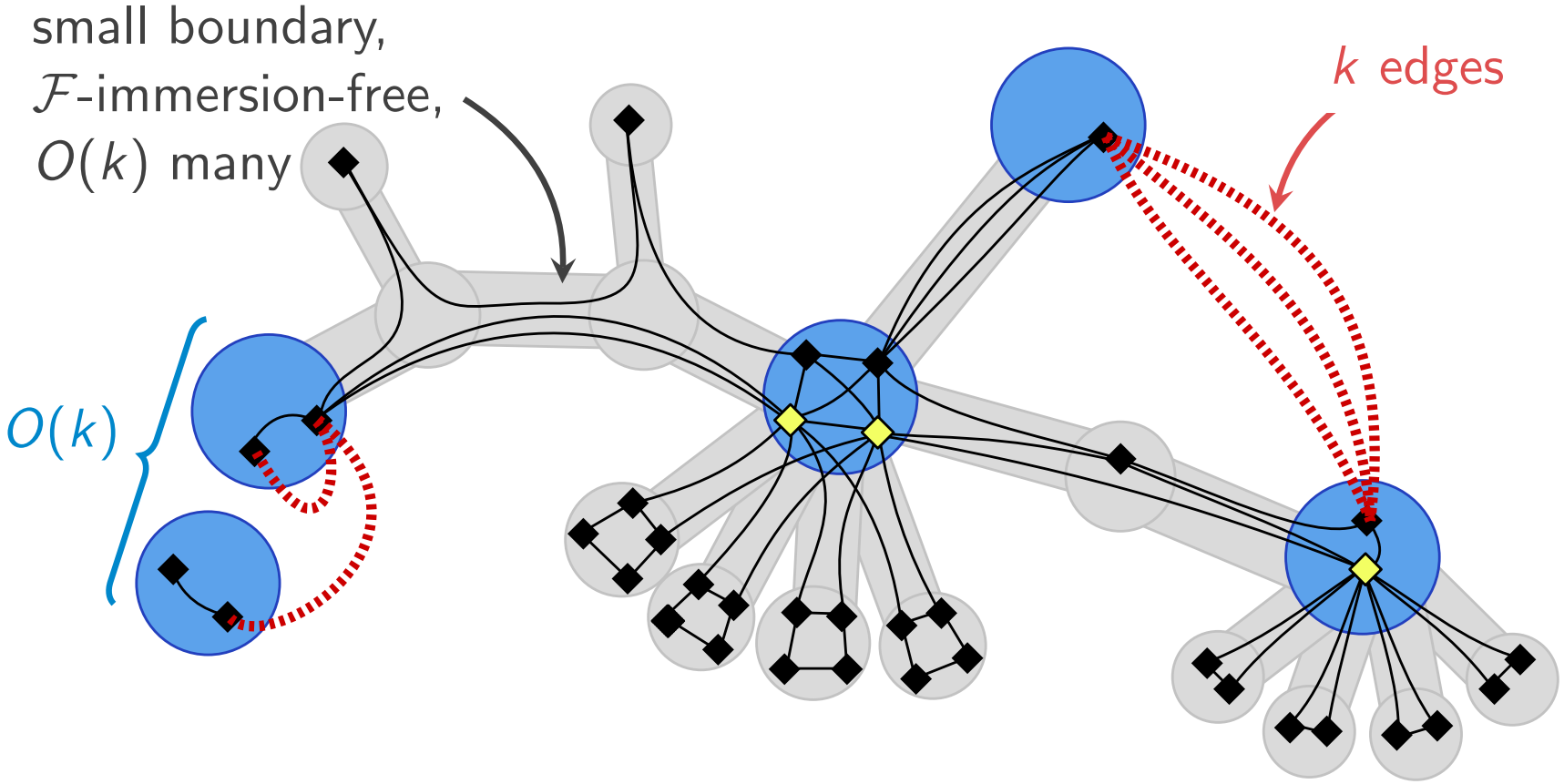

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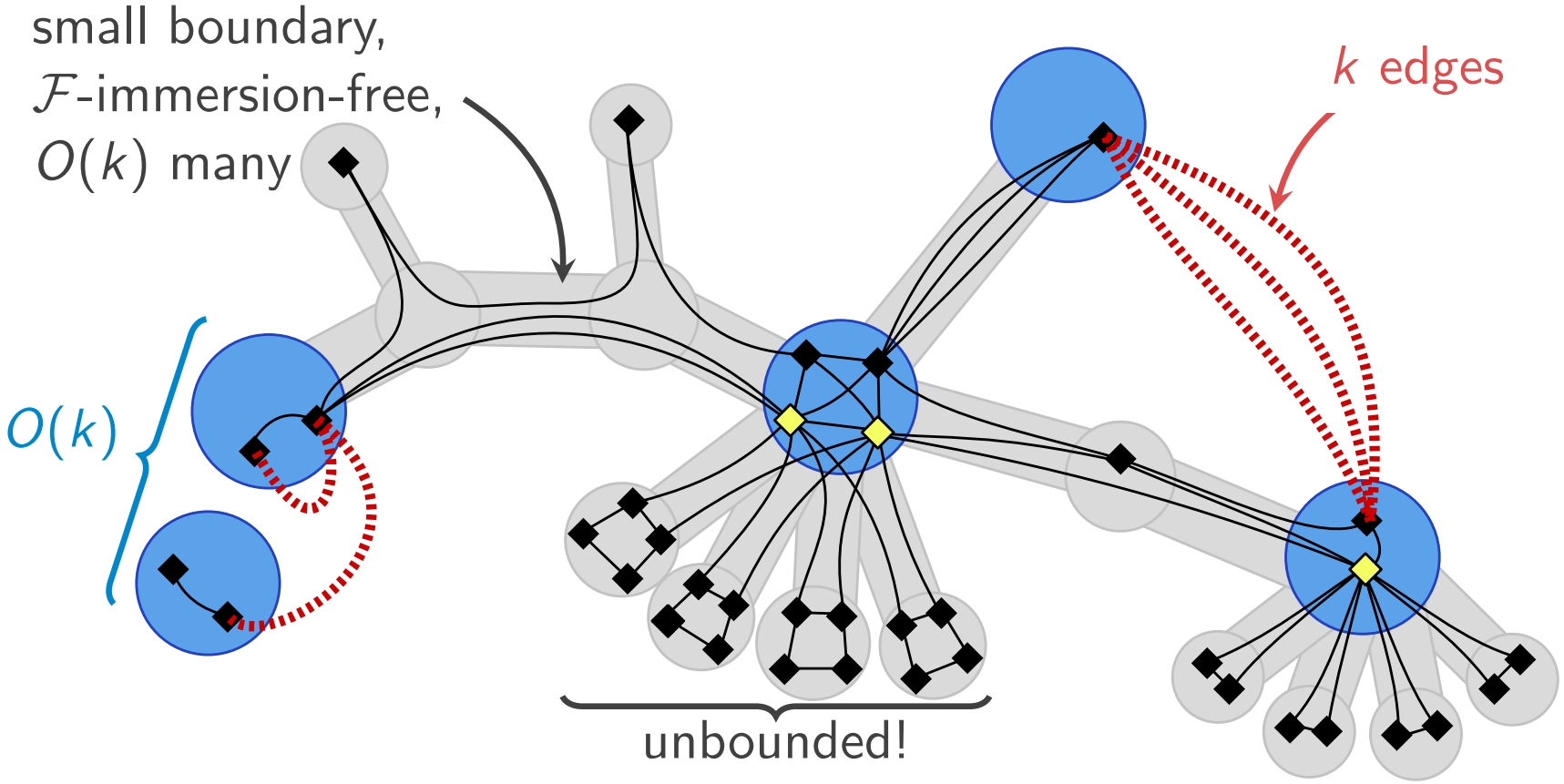

Replacing protrusions

A protrusion is part of the graph that

- has a small boundary;
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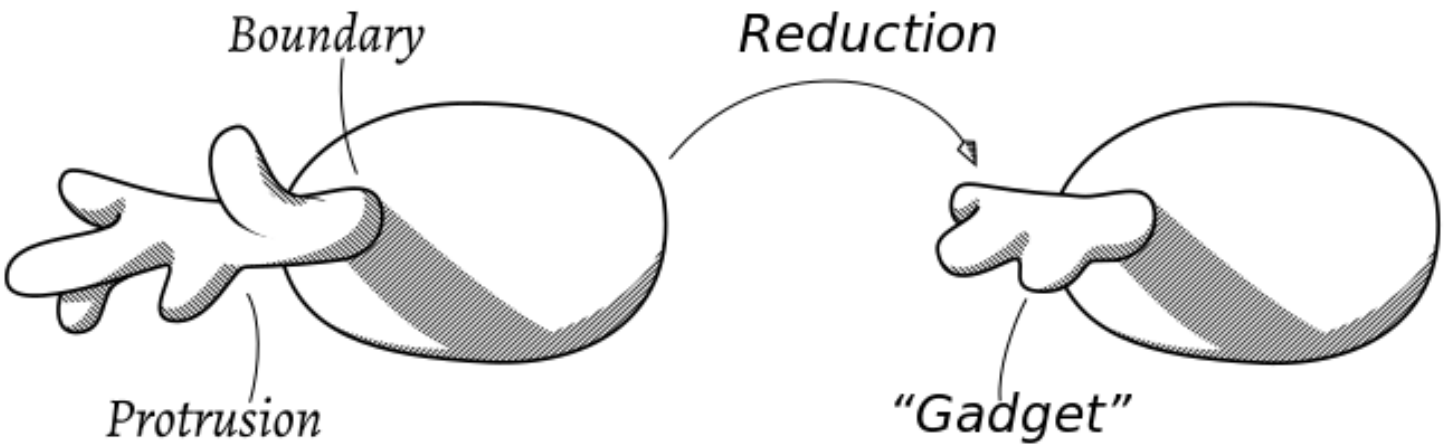

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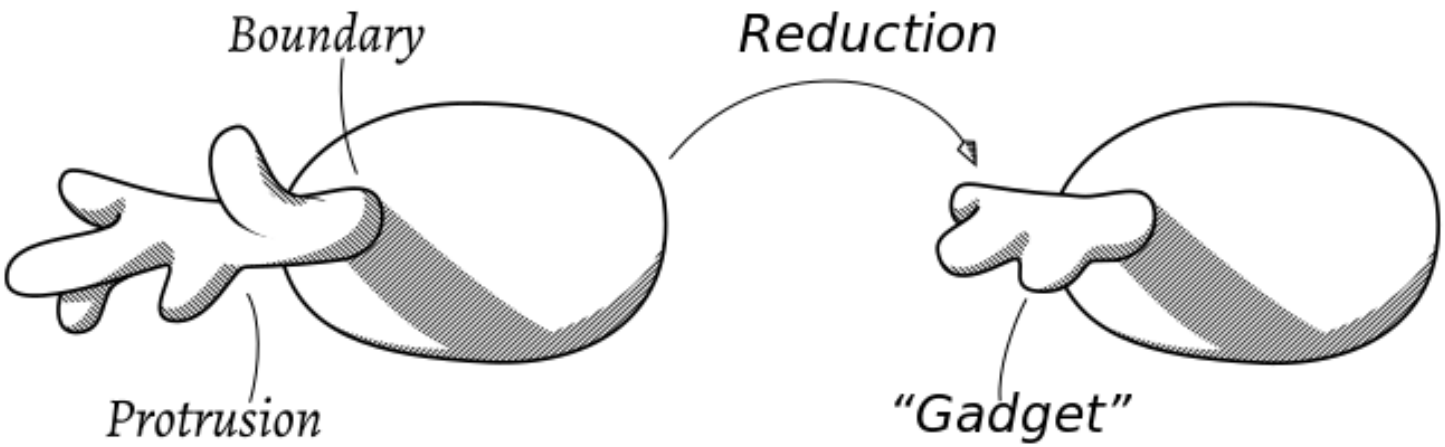

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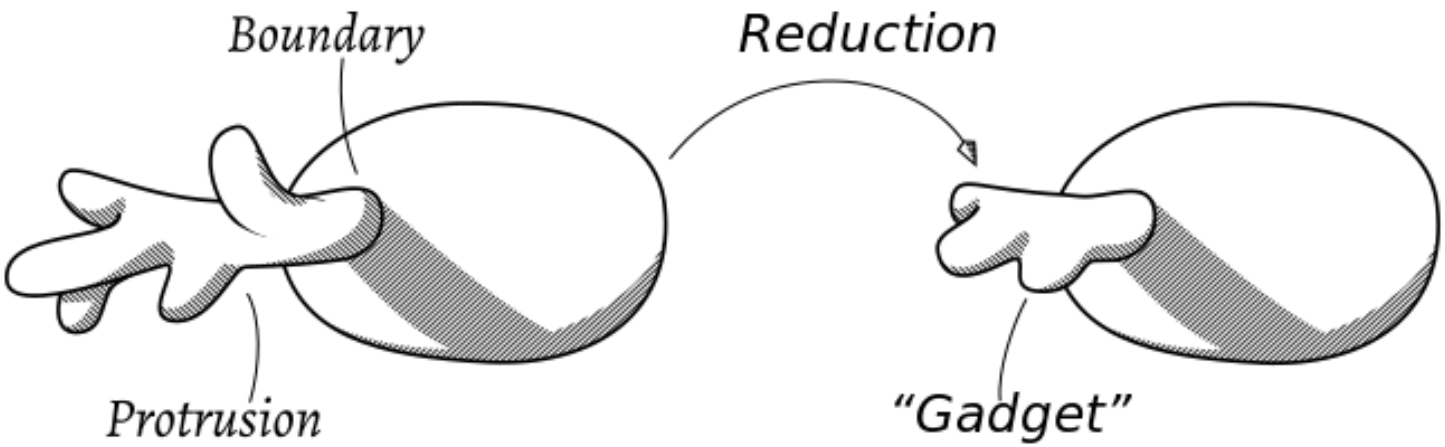

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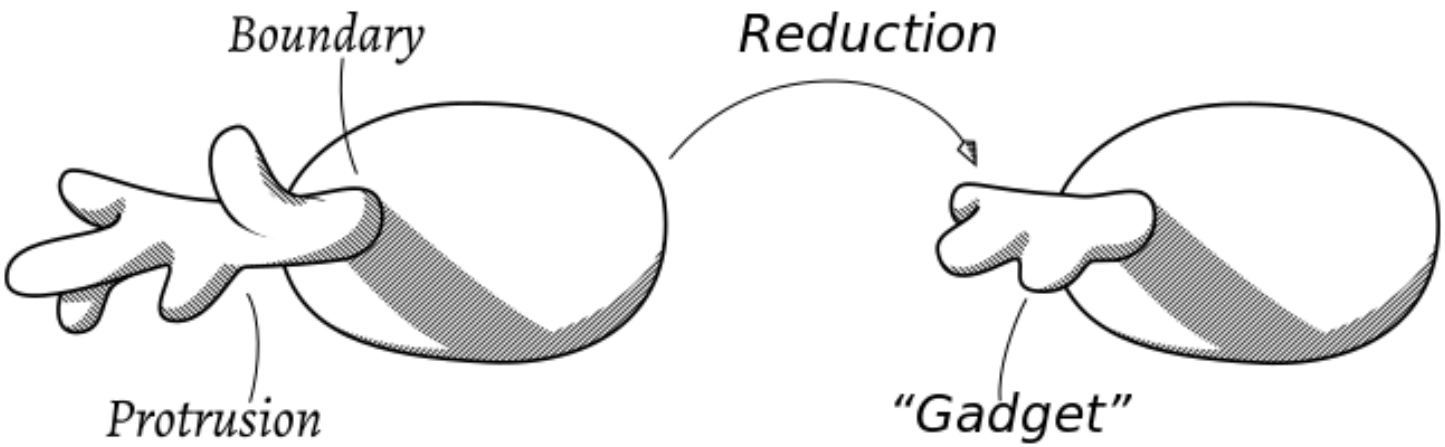

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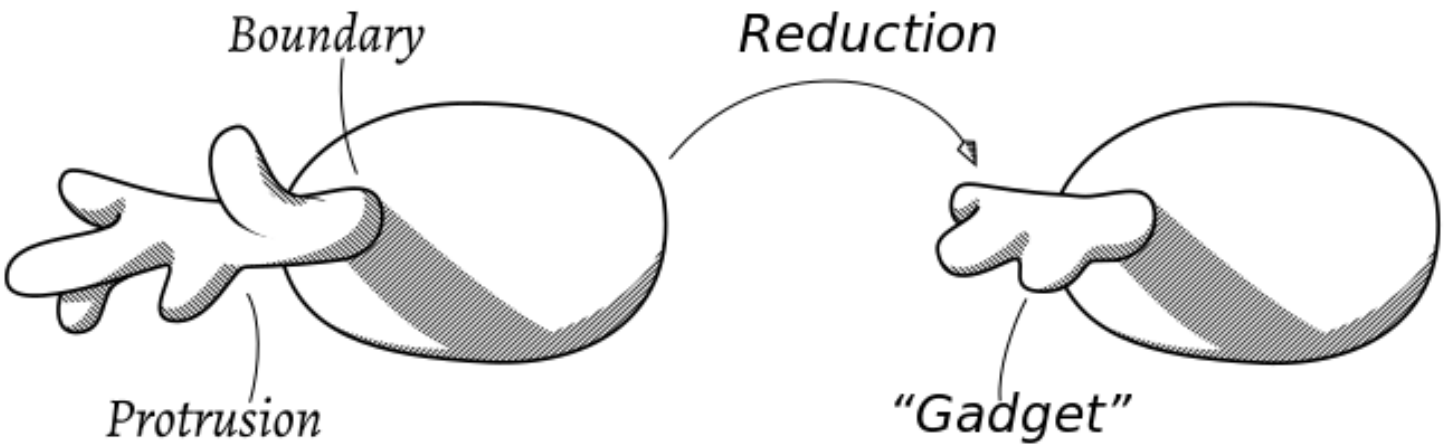

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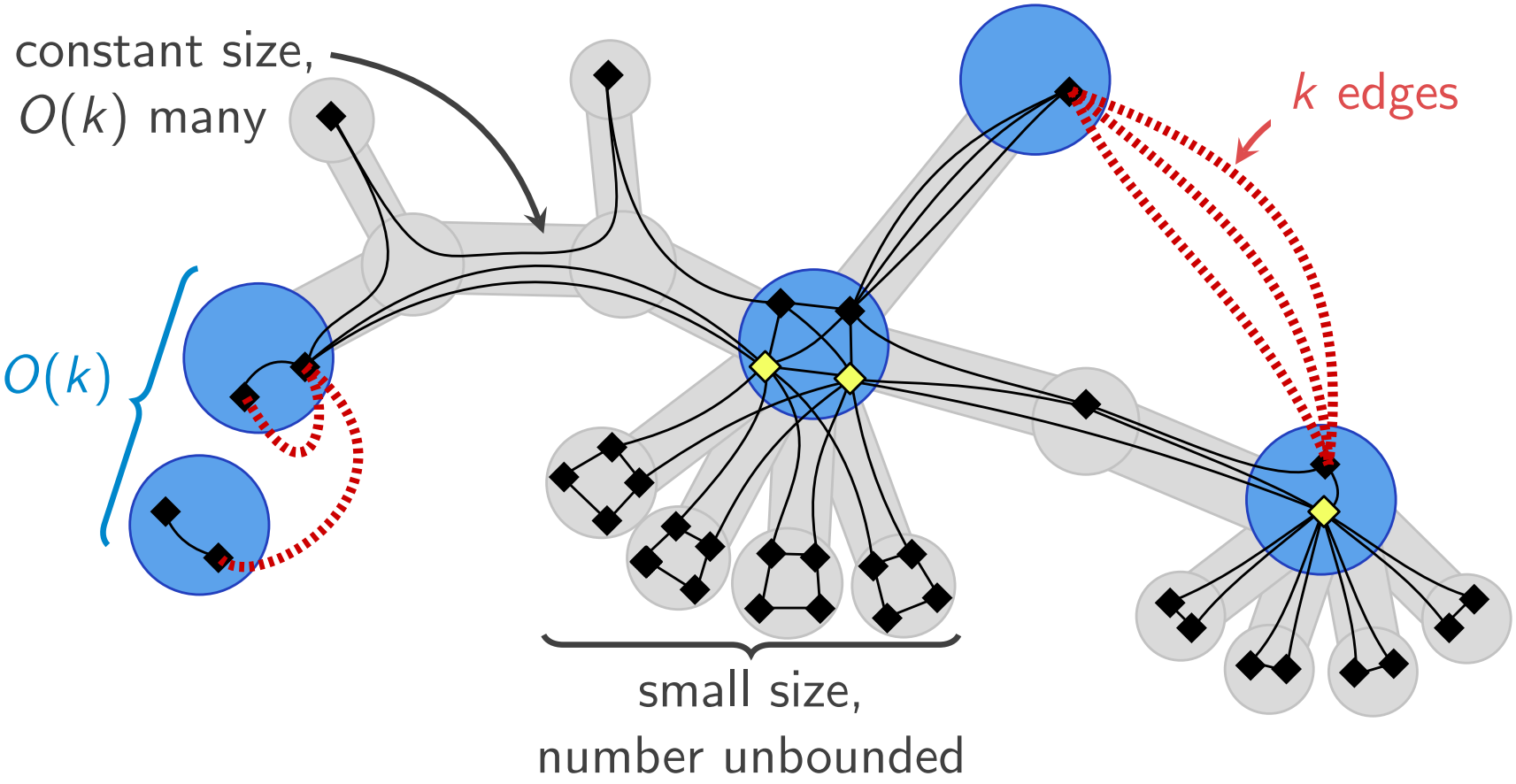

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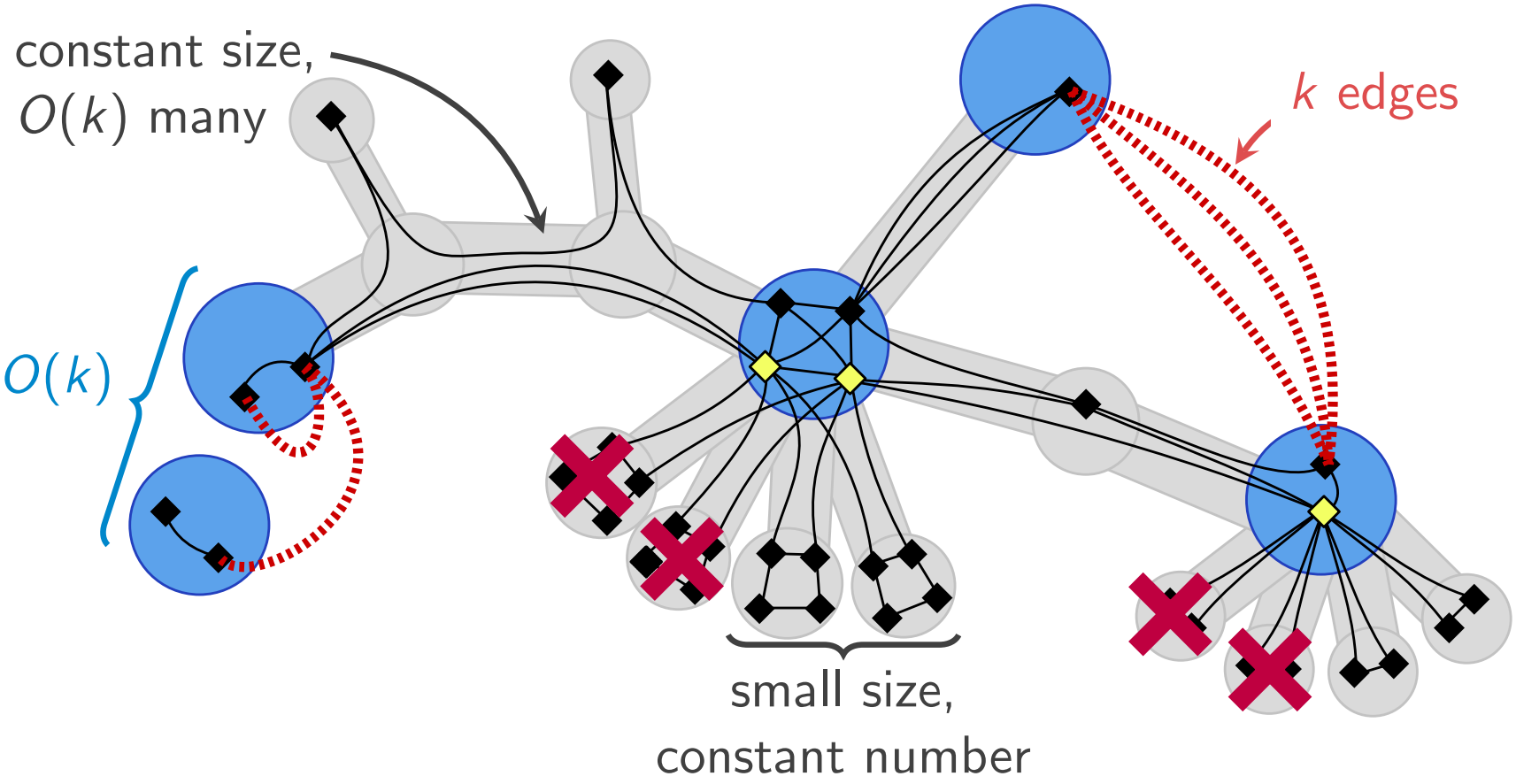

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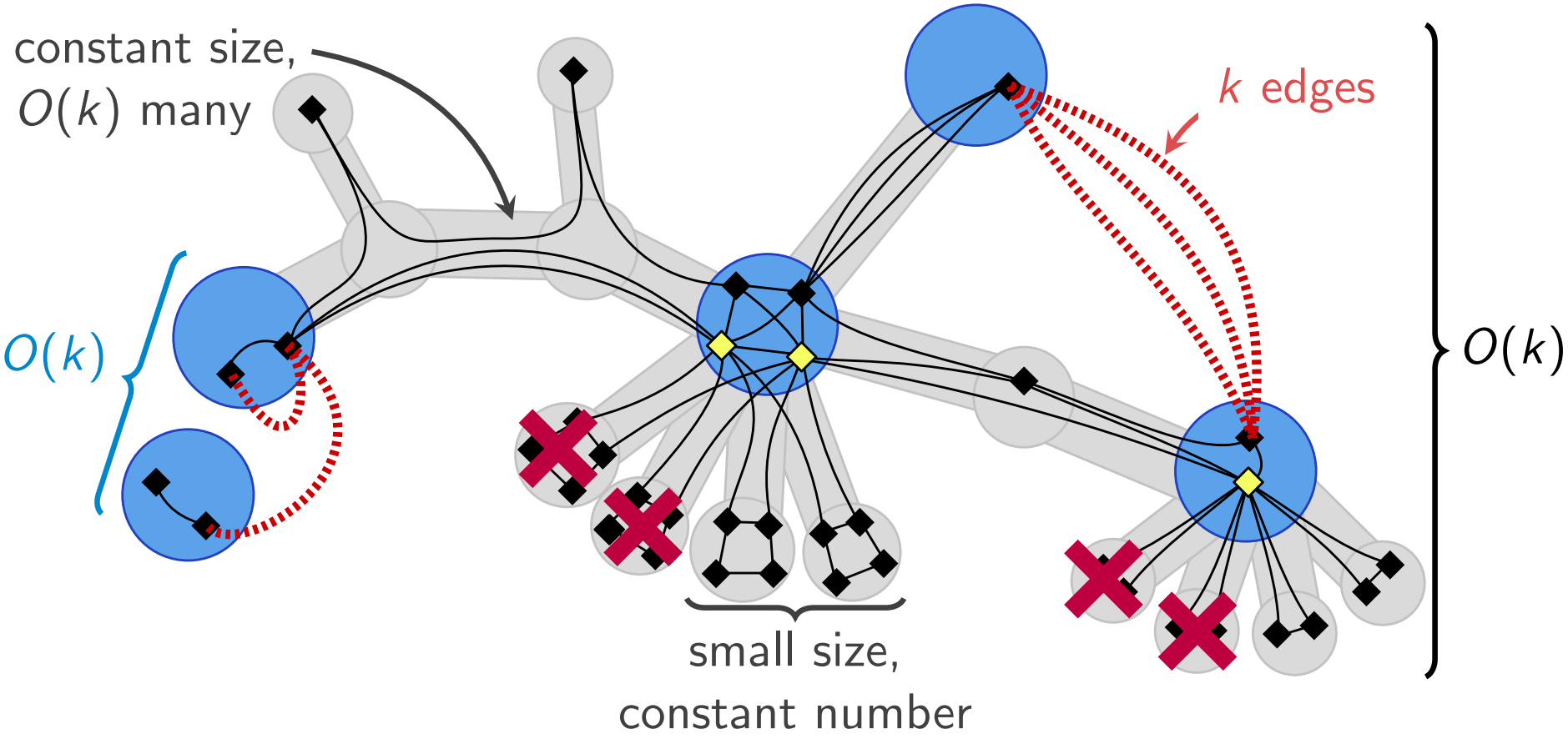

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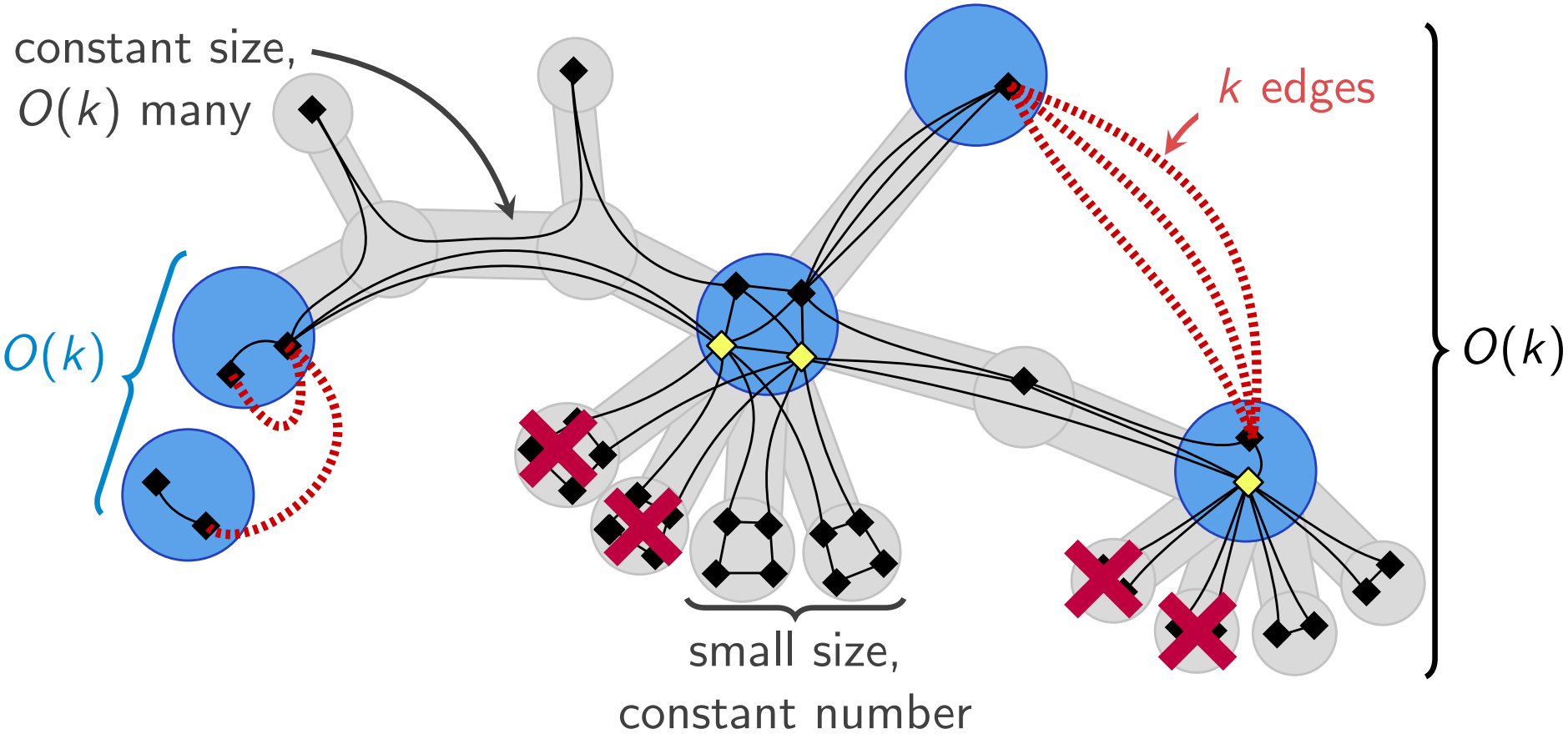

Lemma

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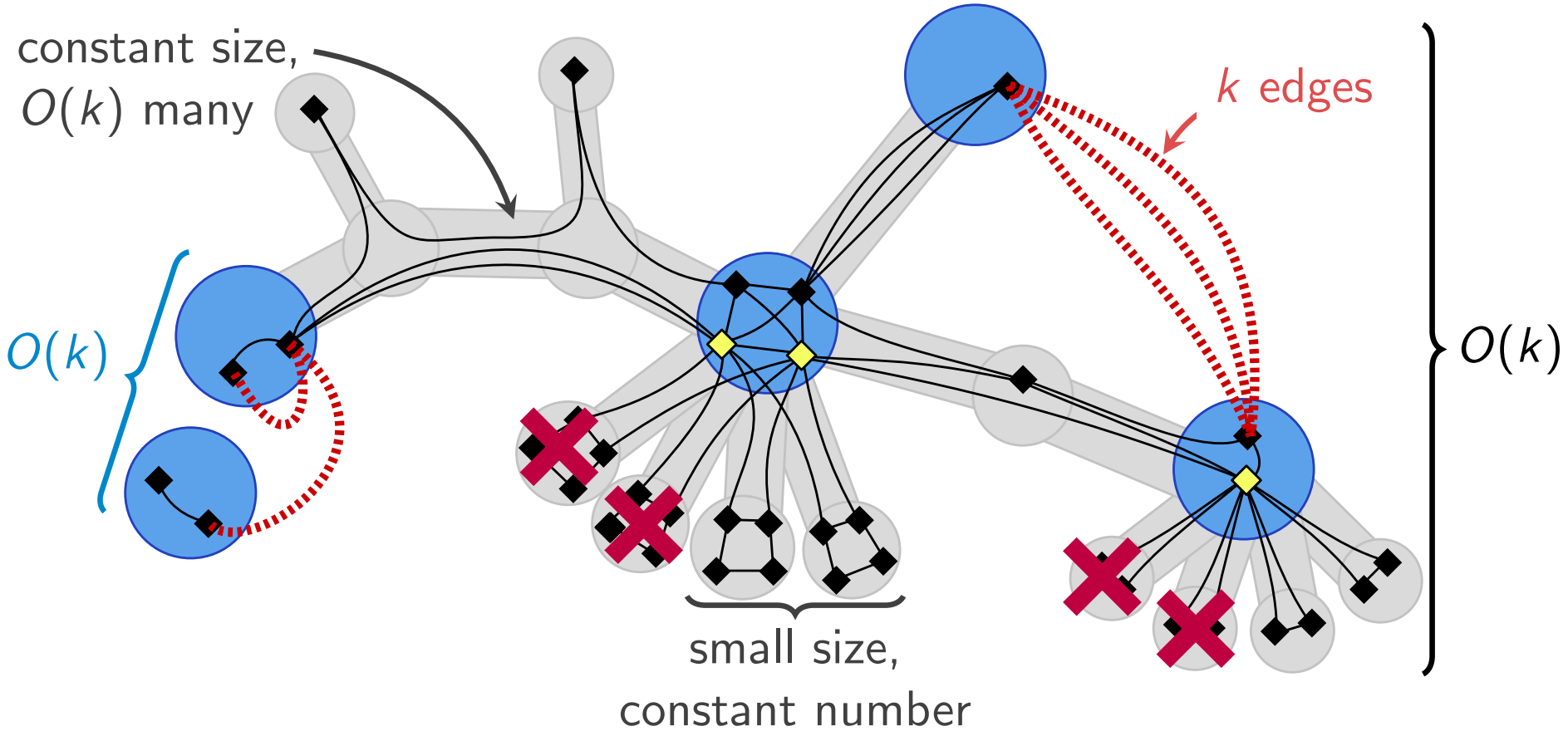

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$\rightsquigarrow$ *iterate and get an approximation.*

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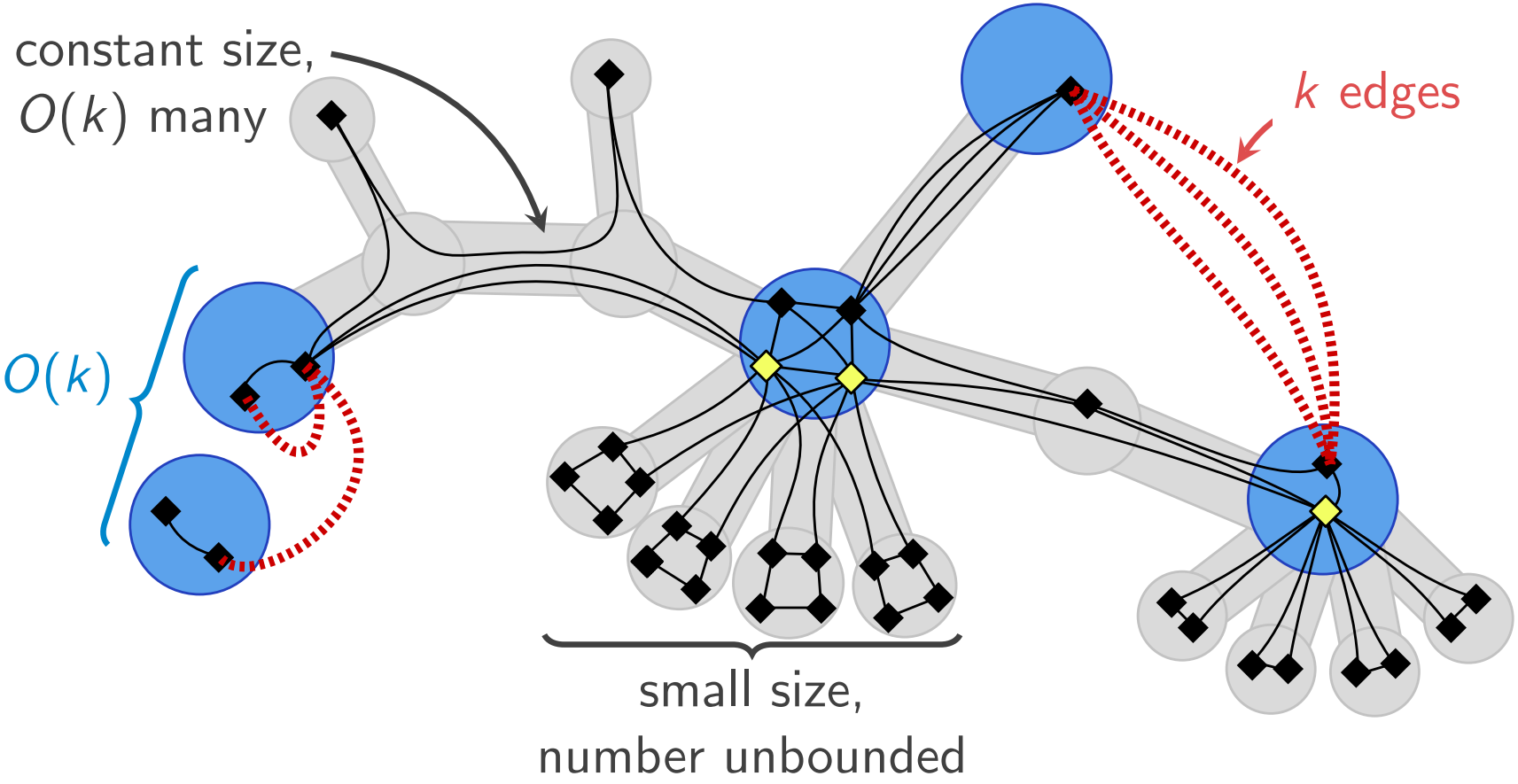
constant size,
$O(k)$ many
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small size,
number unbounded

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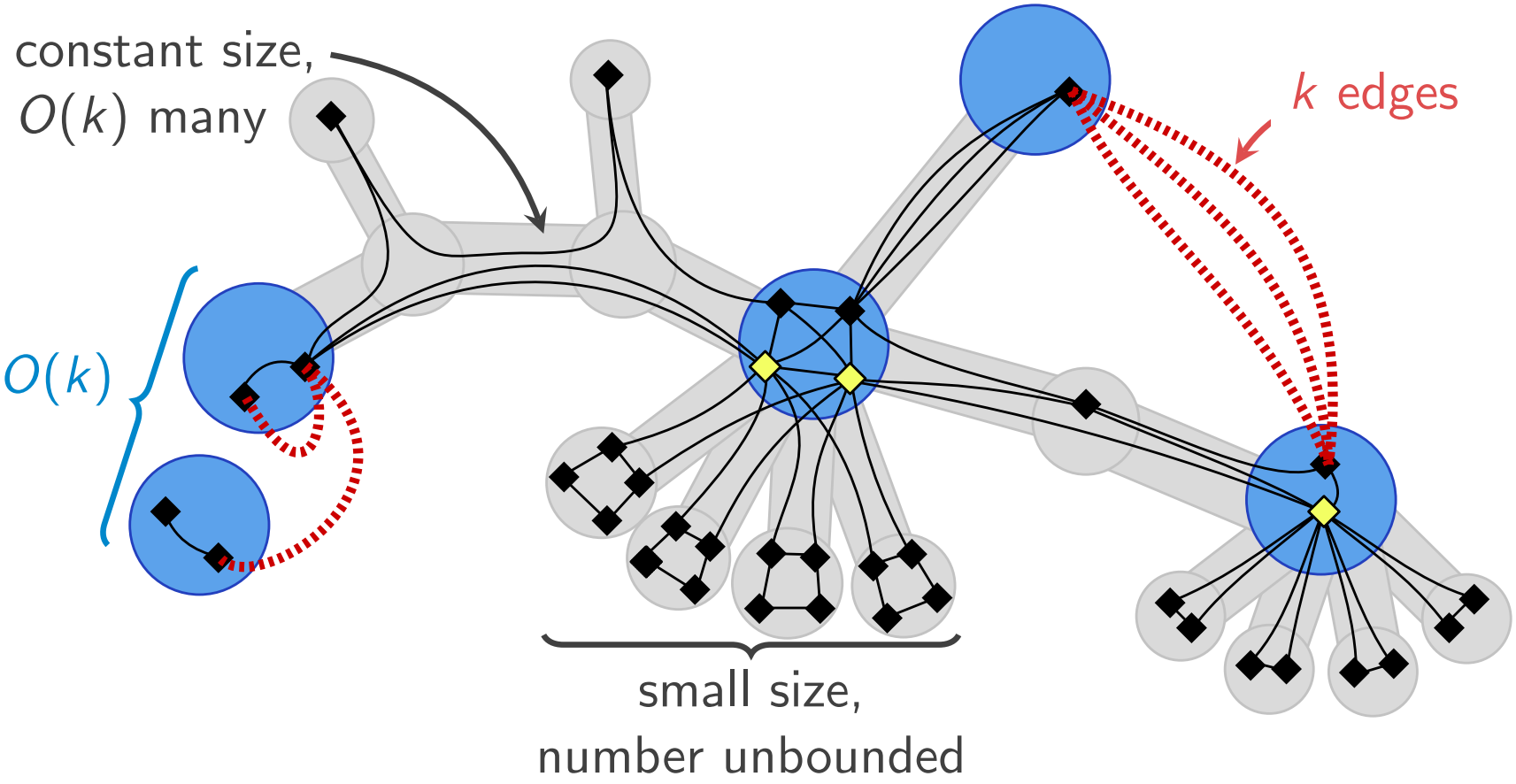

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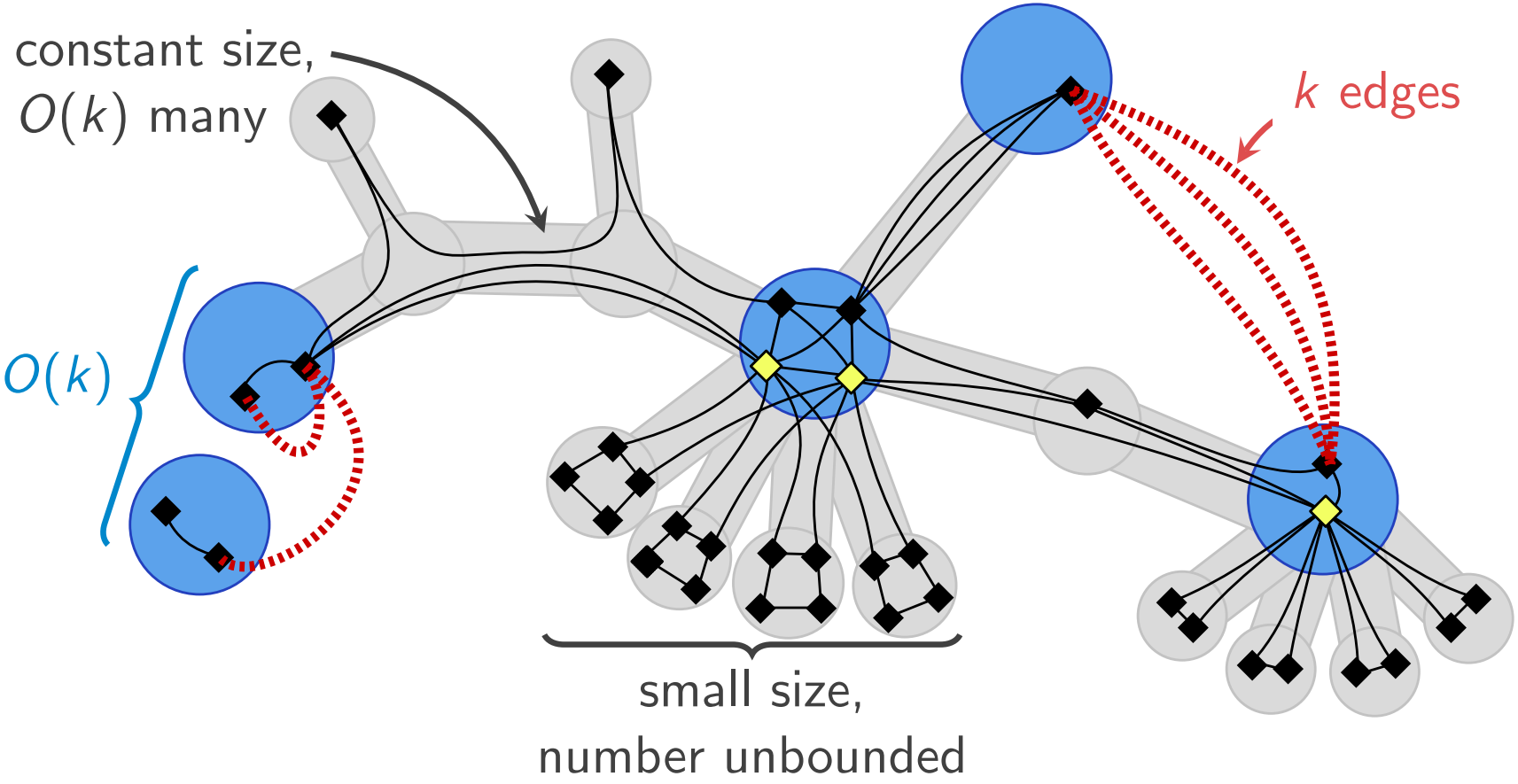

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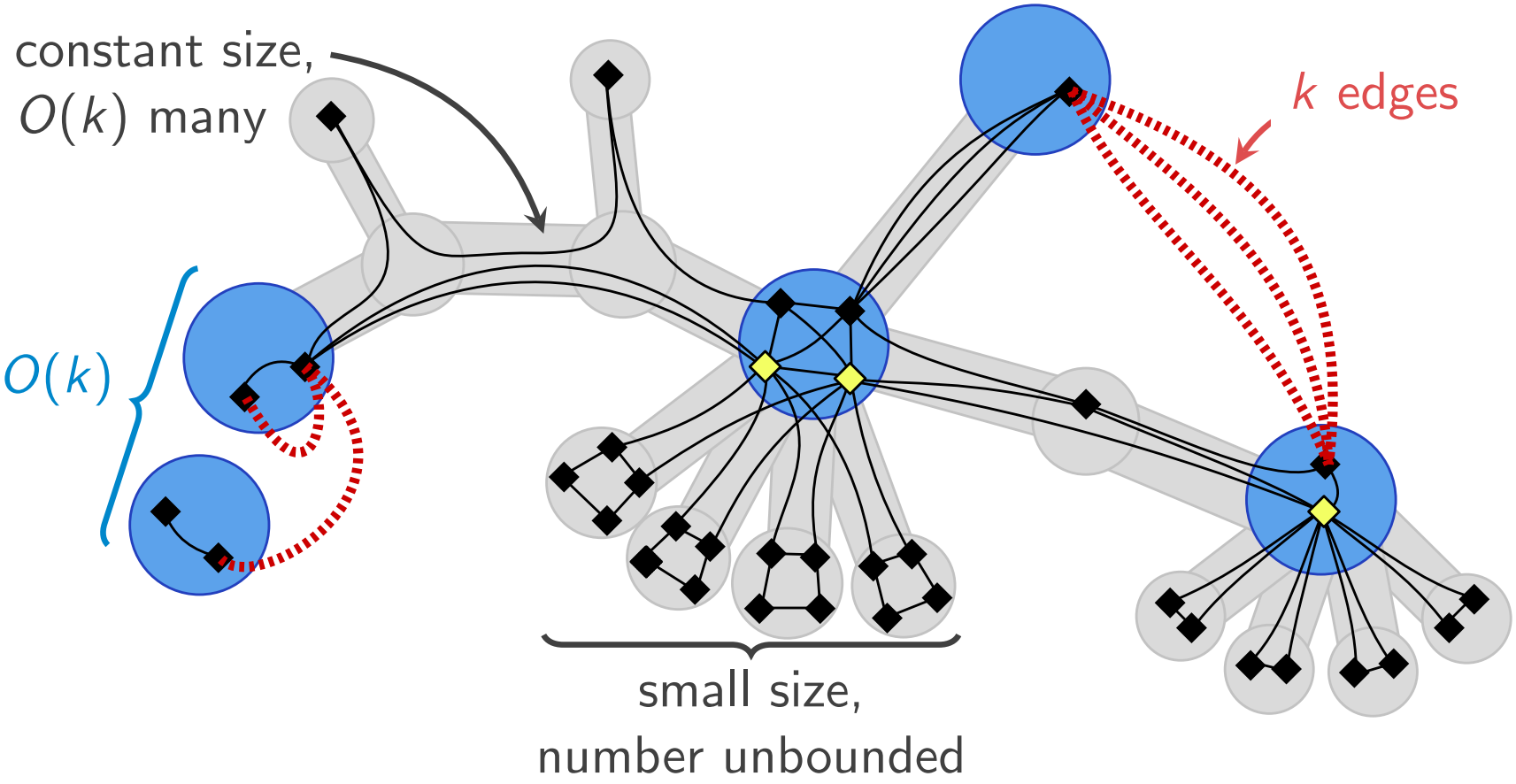

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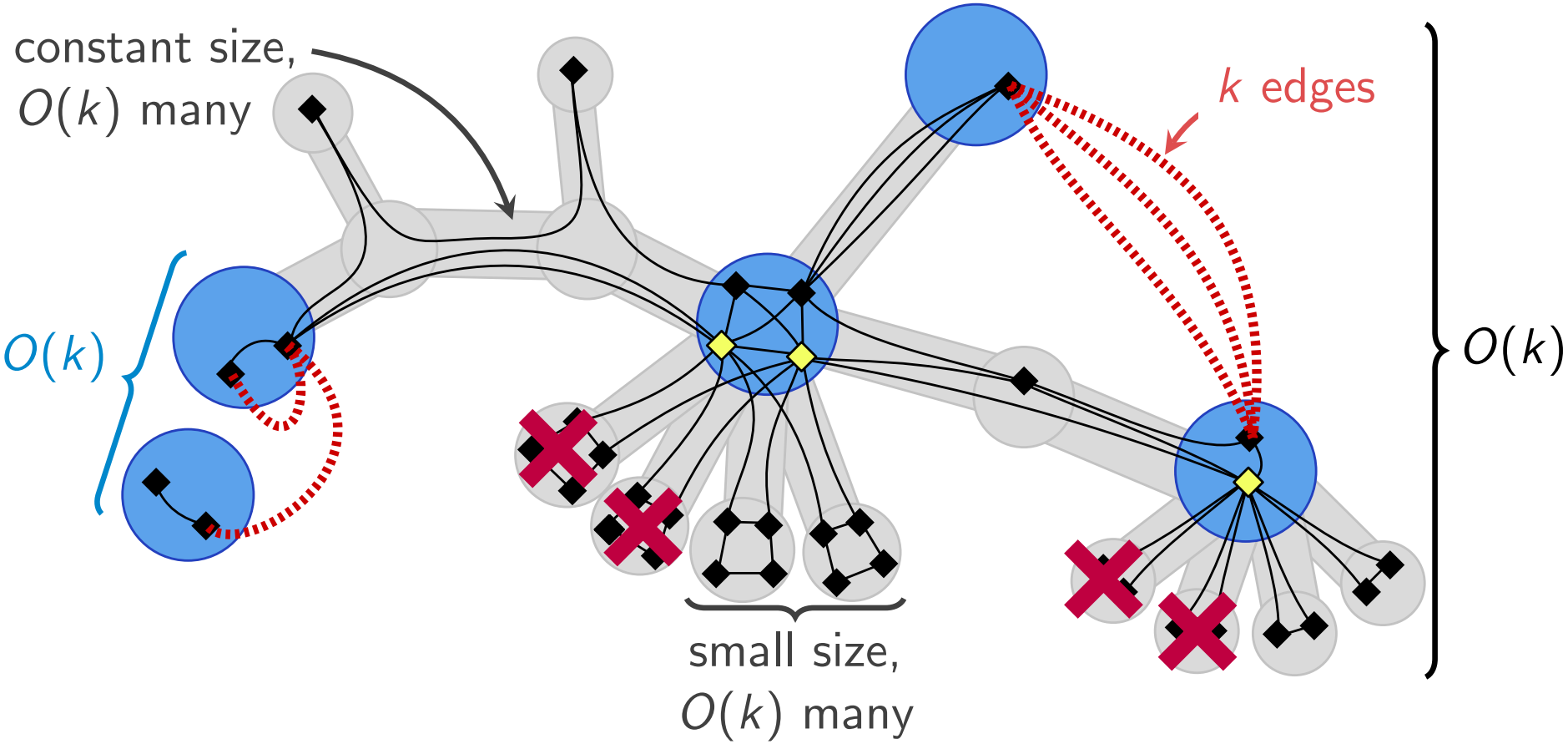

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Proof: the kernel we construct is an immersion of the original graph.

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For every $r \in \mathbb{N}$, the problem **p**-AT-MOST-r-EDGE-DELETION *admits:*

- *a constant factor approximation;*
- *a linear kernel.*

Application to immersion closed parameters

Let **p** be a parameter that is:

- closed under immersion and disjoint union;
- large on walls.

For instance: cutwidth, carving-width, tree-cut width, etc..

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Mirrors the result of Fomin et al. on TREEWIDTH-AT-MOST-r-VERTEX-DELETION.

Recap

When graphs of $\mathcal{F}$ are connected and one of them is planar subcubic:

- constant factor approximation and linear kernel for $\mathcal{F}$-Immersion Deletion;
- linear bound on obstructions size for k-edge-distance to $\mathcal{F}$.

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What do we learn?

- Immersion-Deletion $\neq$ Minor-Deletion
- tree-cut width might be the right parameter to study immersions.

Further work

1. lift the connectivity requirement;
2. consider the case where $\mathcal{F}$ contains does not contain a planar subcubic graph;
3. give a constructive bound on the number of obstructions.

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Thank you for your attention!