Packing and covering immersions expansions of planar subcubic graphs

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Joint work with Archontia Giannopoulou (University of Warsaw), O-joung Kwon (Hungarian Academy of Sciences) and Dimitrios M. Thilikos (LIRMM).

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Duality between packing and covering a familly of objects in a graph.

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Theorem (Menger, 1927)

For every graph G, every $S, T \subseteq V(G)$, and every $k \in \mathbb{N}$, G has k vertex-disjoint (S, T)-paths, or a set of at most $k-1$ vertices meeting all (S, T)-paths.

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Theorem (Erdős and Pósa, 1965)

For every $k \in \mathbb{N}$, every graph has k vertex-disjoint cycles, or $O(k \log k)$ vertices meeting all cycles.

The Erdős-Pósa property

$\mathcal{H}$ has the EP-property if $\exists f : \mathbb{N} \to \mathbb{N}$ s.t. $\forall G,\ \forall k \in \mathbb{N}$,

- G has k vertex-disjoint subgraphs isomorphic to graphs in $\mathcal{H}$; or
- $\exists X \subseteq V(G)$ s.t. $G \setminus X$ is $\mathcal{H}$-free and $|X| \leqslant f(k)$.

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Packing and covering numbers:

- $\mathbf{pack}_{\mathcal{H}}(G)$: max # of v-disjoint sgr. of G isomorphic to a graph in $\mathcal{H}$;
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Theorem (Erdős-Pósa, revisited)

The class of cycles has the EP-property, with gap $O(k \log k)$.

Results and applications related to the EP-property

Applications:

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A huge amount of results and variants:

- acyclic patterns (forests, cuts, etc.);
- triangles (Tuza problems);
- cycles and generalizations (with modularity constraints, with prescribed vertices, etc.);
- patterns related to containment relations (minors, immersions, etc.);
- edge variant.

Minors and majors

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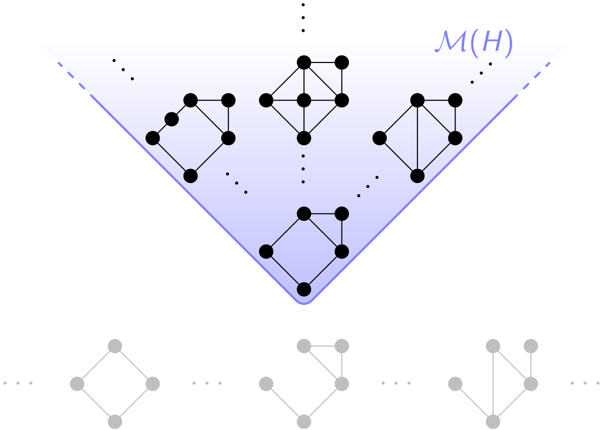
- major of H: any graph having H as a minor.



Robertson and Seymour's Erdős-Pósa extension

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$\mathcal{M}(H)$ has the Erdős-Pósa property iff H is planar.

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Consequence of:

Theorem (Grid exclusion theorem, ibid.)

$\exists f : \mathbb{N} \to \mathbb{N}$ *s.t.* $\forall H$ *planar*, $\forall G$, $\mathbf{tw}(G) \geqslant f(|H|) \Rightarrow H \leqslant_{\mathrm{m}} G$.

($\forall H$ planar, all the graphs of *large* treewidth contain H as a minor)

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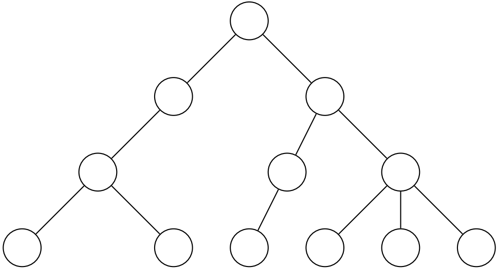

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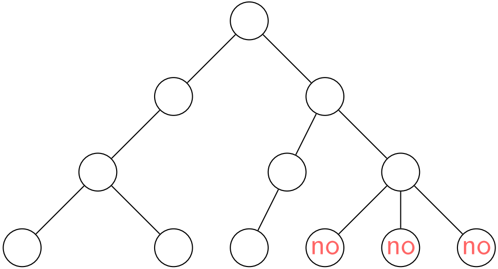

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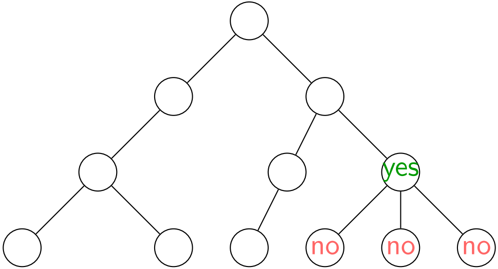

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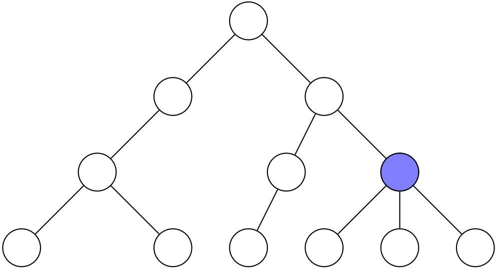

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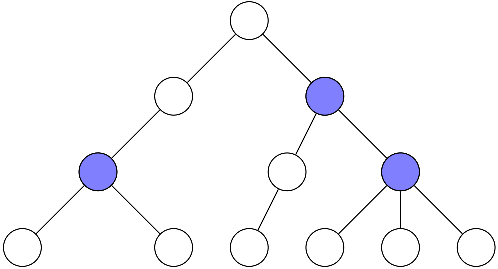

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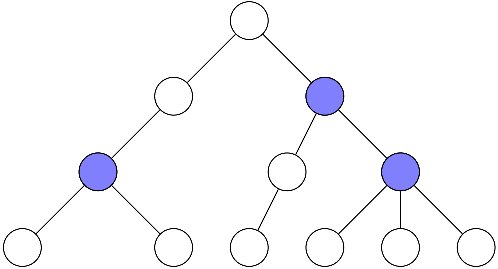

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$\rightarrow$ cover of size $O(k \cdot \mathbf{tw}(G)) = O(k \cdot f(k \cdot |H|))$.

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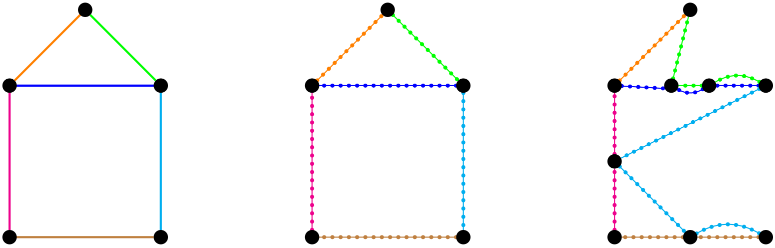
- immersions: this talk.

Immersions expansions



H Subdivision Immersion expansion

- subdivision: edges of H correspond to vertex-disjoint paths;
- immersion expansion: edges of H correspond to edge-disjoint paths.

$\mathcal{I}(H)$: expansions of H.

Erdős-Pósa for immersion expansions

Theorem (Giannopoulou, Kwon, R., Thilikos, 2016)

$\forall H$ connected planar subcubic, $\exists f : \mathbb{N} \to \mathbb{N}$ s.t. $\forall k \in \mathbb{N}$, every graph contains

- *k edge-disjoint expansions of H; or*
- *$f(k)$ edges meeting all expansions of H.*

In other words: expansions immersions of planar subcubic graphs have the edge-Erdős-Pósa property.

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- treewidth $\rightarrow$ tree-cut width (**tcw**);
- tree-decomposition $\rightarrow$ tree-cut decomposition.

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$\exists f : \mathbb{N} \to \mathbb{N}$, $\forall H$ planar subcubic, every graph of tree-cut width $\geqslant f(|H|)$ contains an expansion of H.

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$\exists f : \mathbb{N} \to \mathbb{N}$, $\forall H$ *planar subcubic, every graph of tree-cut width* $\geqslant f(|H|)$ *contains an expansion of* H.

Consequence: if $\mathbf{tcw}(G) \geqslant f(k|H|)$ then $k \cdot H \leqslant_{\mathrm{imm}} G$.

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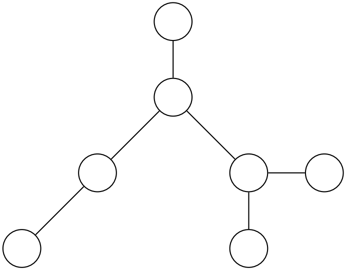

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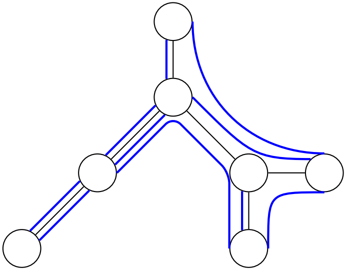

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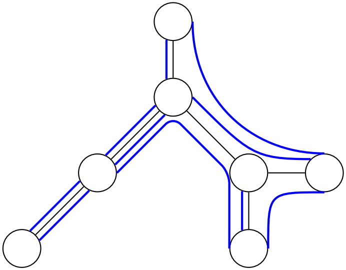

Tree-cut decomposition:

- near-partition of the vertices;
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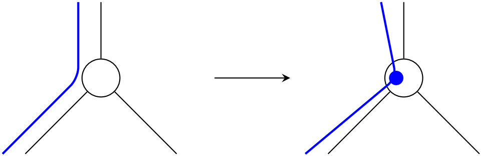
- width depends on bag size, max # of *thick* edges around a node, and max cut along an edge of the tree.

From tree-cut decompositions to tree-partitions

We modify G as follows:



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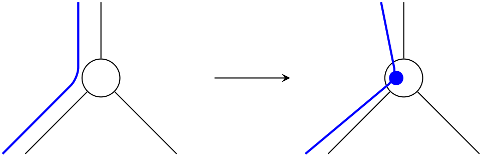

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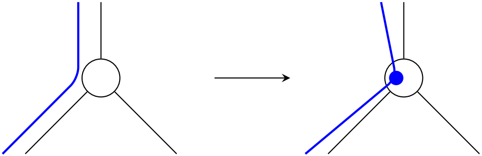

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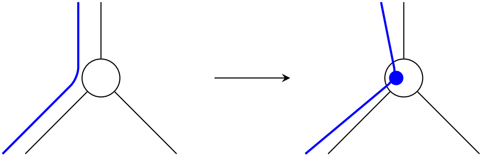

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Lemma (Giannopoulou, Kwon, R., Thilikos, 2016)

There are graphs G' and H' with $\mathbf{tpw}(G') = O(\mathbf{tcw}(G)^2)$ s.t

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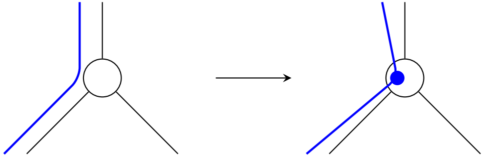

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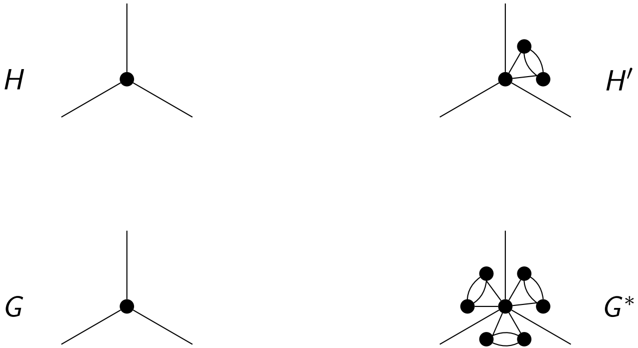
Erdős-Pósa for H' and G' implies Erdős-Pósa for H and G.

Technicalities



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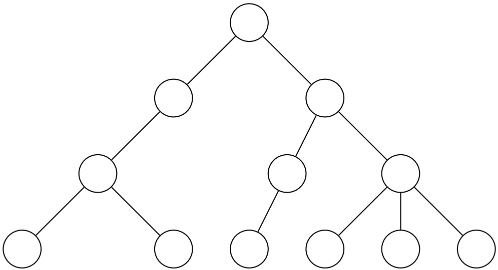

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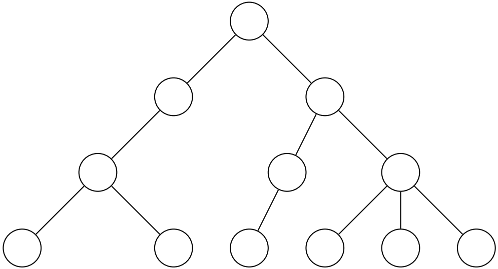

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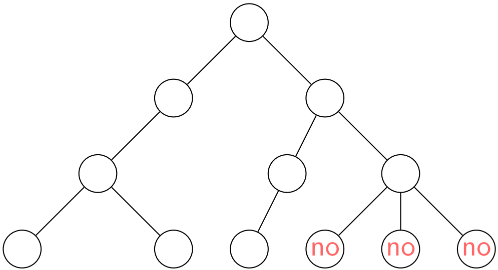

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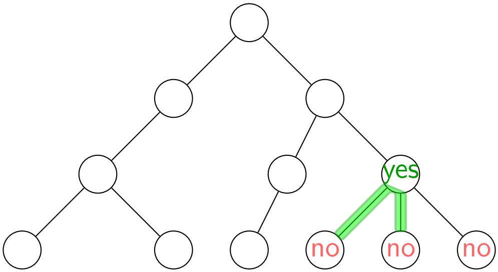

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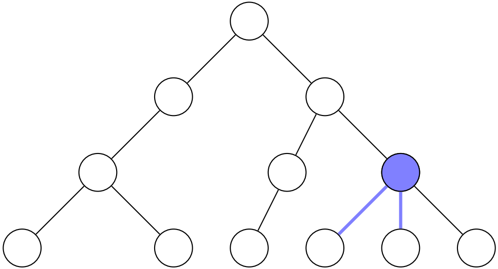

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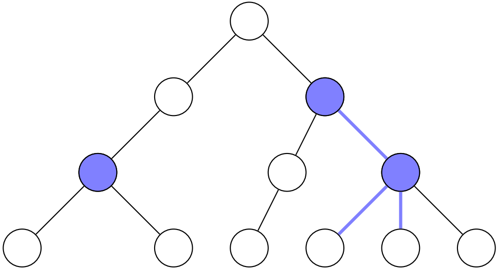

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Graphs of bounded tree-partition width

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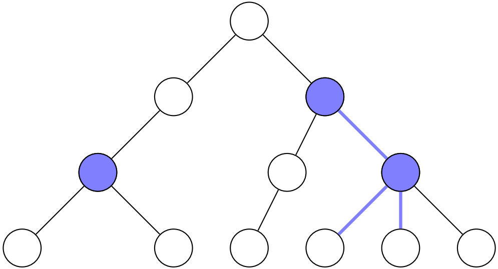
Goal: if $\mathbf{epack}_{\mathcal{I}(H)}(G) < k$ then $\mathbf{ecover}_{\mathcal{I}(H)}(G) \leqslant h(k)$.



1. choose a maximum-depth bag s.t. the "graph below" has a major of H;
2. remove its content (edges) + incident edges and loop.

$\rightarrow$ cover of size $O(k \cdot \mathbf{tpw}(G) \cdot |E(H)|)$.

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2. otherwise, there is an *equivalent* pair of graphs (H', G') with $\mathbf{tpw}(G') = O(f(k)^2)$;
3. we apply the classic trick on (H', G') and get the result for (H, G).

The vertex case: easier

We can also prove the vertex variant.

Theorem (Giannopoulou, Kwon, R., Thilikos, 2016)

$\forall H$ connected planar subcubic, $\exists f : \mathbb{N} \to \mathbb{N}$ s.t. $\forall k \in \mathbb{N}$, every graph contains

- *k vertex-disjoint expansions of H; or*
- *$f(k)$ vertices meeting all expansions of H.*

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Proof:

- if $\mathbf{tw}(G)$ is large then G contains a subdivision of a large wall;
- otherwise we apply the classic trick on an optimal tree-decomposition of G.

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Lemma (copying [Thomassen 1988])

There are infinitely many trees with $\Delta > 3$, the expansions of which do not have the edge/vertex-EP property.

Related work: restricting the class of host graphs

Theorem (Liu 2015)

$\forall H$, $\exists f : \mathbb{N} \to \mathbb{N}$ *s.t.* $\forall k \in \mathbb{N}$, *every 4-edge-connected graph* G *contains*

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Note: there are 3-edge connected graphs, the expansions of which do not have the edge-EP property.

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Thank you for your attention!