Separators & expansion

Jean-Florent Raymond

17th JCALM, Montpellier, March 2015

Talk based on: *Strongly sublinear separators and polynomial expansion*, Dvořák and Norin, 2015.

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Theorem (Plotkin, Rao, and Smith, 1994)

$\forall G$, *polynomial expansion* $\Rightarrow$ *strongly sublinear separators.*

Theorem (Dvořák and Norin, 2015)

$\forall G$, *strongly sublinear separators* $\Rightarrow$ *polynomial expansion.*

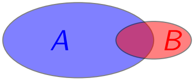
Reminder of the previous talk:

- r-minor of G: obtained by contracting disjoint balls of radius $\leqslant r$ of a subgraph of G;
- $\nabla_r(G) = \max\left\{\frac{|E(G')|}{|V(G')|},\ G' \text{ is a } r\text{-minor of } G\right\}$
- G has bounded expansion if there is a function f such that

$$\forall r \in \mathbb{N},\ \nabla_r(G) \leqslant f(r)$$

- G has polynomial expansion if f is polynomial.

Separators

- separator (A, B): 

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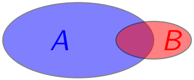
- separator (A, B): 

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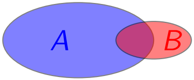
- separator (A, B): 

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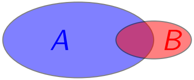
- separator (A, B): 

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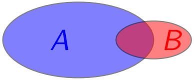
- separator (A, B): 

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Example: planar graphs have $c\bullet^{\frac{1}{2}}$-separators (for some constant c).

Why do we care about (small) separators?

- they give structural information;
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Theorem (Dvořák and Norin, 2014)

$\forall c \geqslant 1, \beta \in [0, 1)$, *for every graph* G,
G has $c \bullet^{\beta}$ separators $\Rightarrow \forall H \subseteq G,\ \mathbf{tw}(H) \leqslant 105c|V(H)|^{\beta}$.

Sublinear separators and polynomial expansion, once again

Recall: G has $c\bullet^{\delta}$-separators $\equiv$ every $H \subseteq G$ has a balanced separator of order $\leqslant c|V(H)|^{\delta}$.

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$\forall d \geqslant 0$, *for every graph* G,

$$G \text{ has expansion } O(r^d) \Rightarrow G \text{ has } c\bullet^{1-\frac{1}{4d+3}}\text{-separators}$$

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$$G \text{ has } c\bullet^{1-\delta}\text{-separators} \Rightarrow G \text{ has expansion } O(r^{5/\delta^2}).$$

A few words about expanders

α-expander G: at least $\alpha|A|$ vertices of $G \setminus A$ are adjacent to A, for every $A \subseteq V(G),\ |A| \leqslant |V(G)|/2$.

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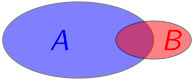

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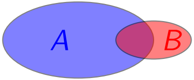

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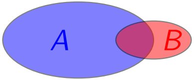

wlog $|B \setminus A| \leqslant |V(H)|/2$, then $|A \cap B| \geqslant \alpha|B \setminus A|$.
we then use the fact that (A, B) is balanced and $|A \cap B| \leqslant \mathbf{tw}(G) + 1$ to conclude.

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Lemma (Dvořák and Norin, 2015)

For every $\varepsilon \in (0,1]$ and every t large enough,
if G has $\geqslant f(\varepsilon)t^4|V(G)|^{1+\varepsilon}$ edges
then $\exists H \subseteq G$ subcubic s.t. $|V(H)| \leqslant f'(\varepsilon)t$ and $\mathbf{tw}(H) \geqslant \frac{t}{25}$.

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Hence $\nabla_r(G) = O\left(r^{5/\delta^2}\right)$.

G has polynomial expansion.

Strongly sublinear separators force low density

Lemma (2.)

For every G graph, if G has $c\bullet^{1-\delta}$-separators,
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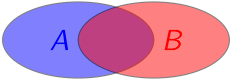

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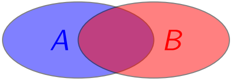

$$
\begin{aligned}
|E(G)| &\leqslant |E(G[A])| + |E(G[B])| \\
&\leqslant f_\delta(c)|A|\left(1 - \frac{1}{\log|A|}\right) + f_\delta(c)|B|\left(1 - \frac{1}{\log|B|}\right) \\
&\leqslant f_\delta(c)n\left(1 - \frac{1}{\log n}\right)
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The densities of bounded-depth minors

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4. F' has $c\bullet^{1-\delta}$-separators.

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6. $\mathbf{tw}(F) \leqslant 105c|V(F')|^{1-\delta}$ and $\mathbf{tw}(F)$ is *large*: contradiction.

Strongly sublinear separators in bounded-depth minors

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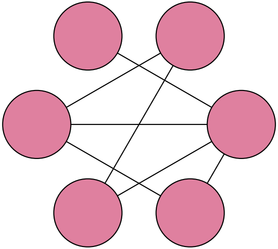

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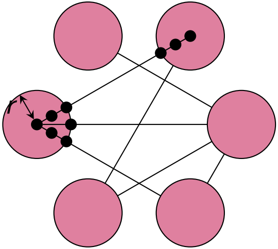

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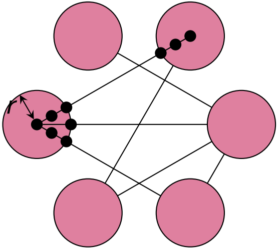

Every edge of F

- meets two balls;
- contributes to $2r$ vertices.

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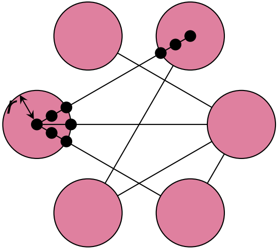

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- meets two balls;
- contributes to $2r$ vertices.

H is a minor of some $H' \subseteq G$ s.t. $|V(H')| \leqslant 2r|E(H)| + |V(H)|$.

Lemma (1.)

For every $r \geqslant 1$, if G has $c\bullet^{1-\delta}$-separators,
then every r-minor H of G has $O_{c,\delta}(r^{4/\delta})\bullet^{1-\frac{5\delta}{6}}$-separators.

Goal: show that H has a sublinear balanced separator.

1. we proved that $\forall \varepsilon \in (0,1],\ |E(H)| \leqslant O_{c,\delta,\varepsilon}(r^{4/\delta})|V(H)|^{1+\varepsilon}$.

2. H is a minor of $H' \subseteq G$ with $|V(H')| \leqslant 2r|E(H)| + |V(H)|$.

Strongly sublinear separators in bounded-depth minors

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Rk: if $\mathcal{C}$ is subgraph-closed then every $G \in \mathcal{C}$ has $c\bullet^{1-\delta}$ separators.

The end

Conjecture

$\exists k > 0,\ \forall c \geqslant 1, \delta \in (0,1]$,
if a graph G has $c\bullet^{1-\delta}$ separators, then its expansion is $O(r^{k/\delta})$.

Currently: $O(r^{5/\delta^2})$.

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Thank you for your attention!