Induced minors and well-quasi-ordering

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Induced minors

$\leqslant_{\mathrm{im}}$: induced minor relation

- vertex deletion;
- edge contraction;
- but no edge deletion.

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$=$: infinite antichain

If not, $\leqslant_{\mathrm{im}}$ is a well-quasi-order (**wqo**).

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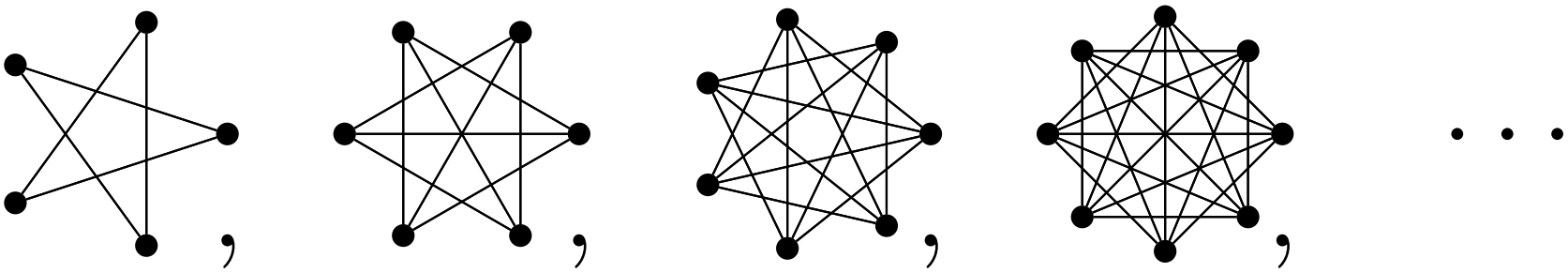
Examples:

- sequences over a finite alphabet with the subsequence order;
- finite graphs with the minor relation;
- ...
- $(\mathcal{P}(\mathbb{N}), \subseteq)$ is not a wqo.

Induced minors and well-quasi-ordering

(finite graphs, $\leqslant_{\mathrm{im}}$) is not a wqo:



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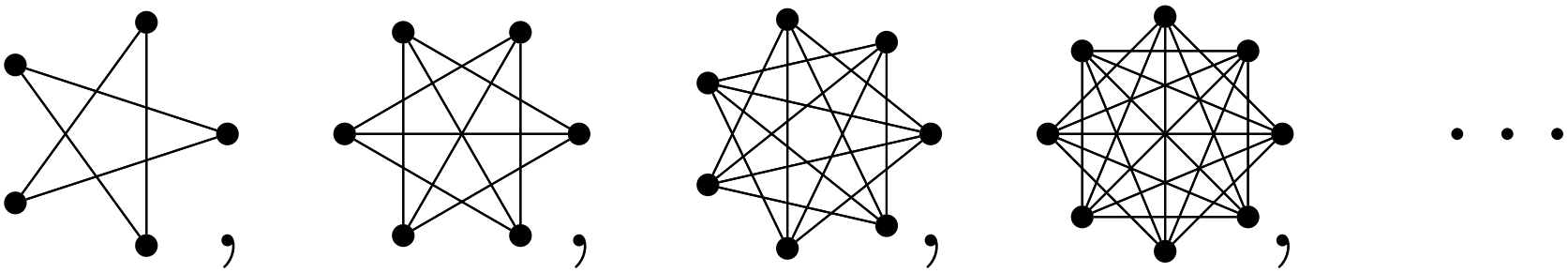
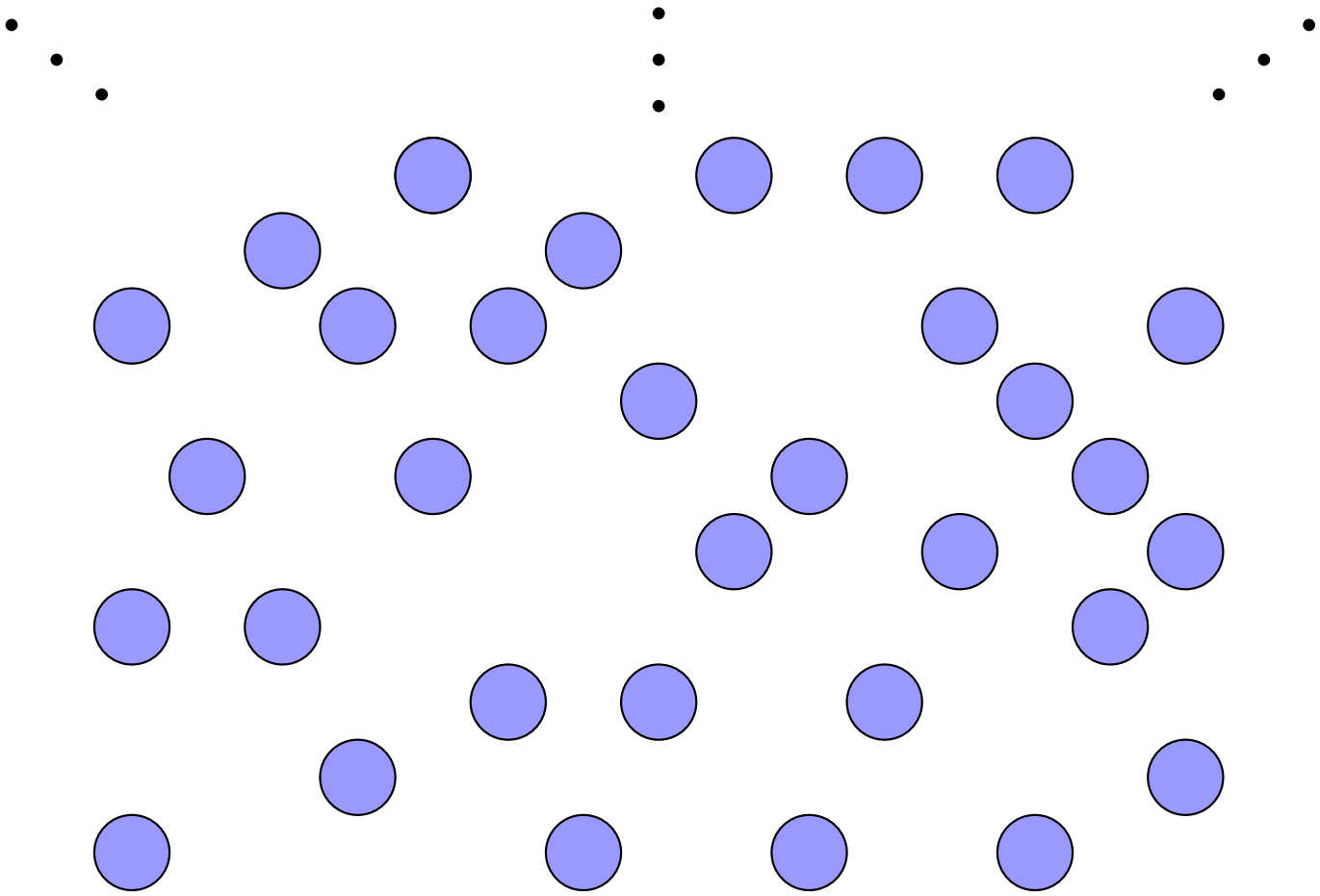


Which subclasses are wqo by $\leqslant_{\mathrm{im}}$?

$\mathrm{Ex}(H)$ = H-induced minor-free graphs

Excluding a graph



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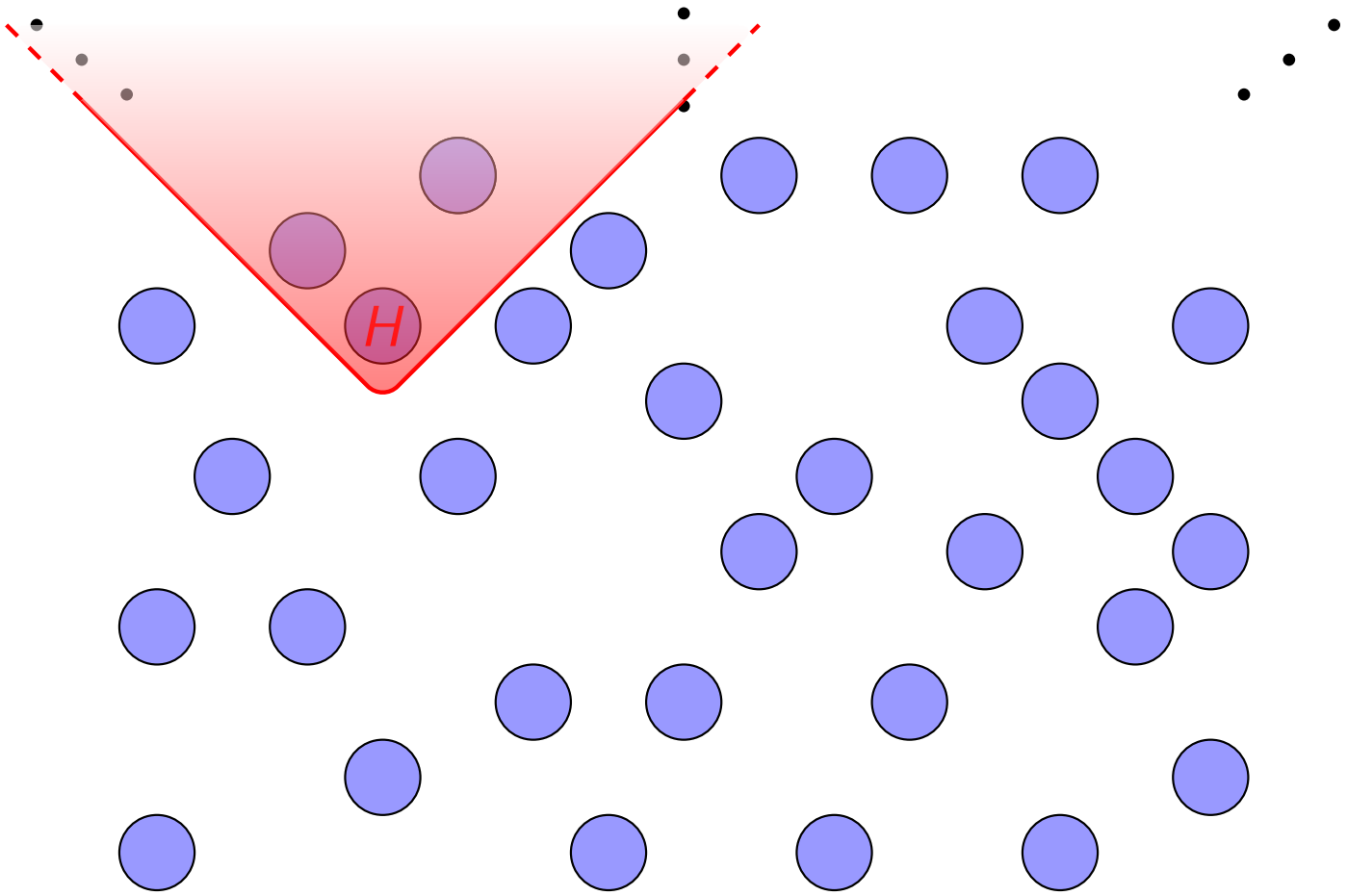

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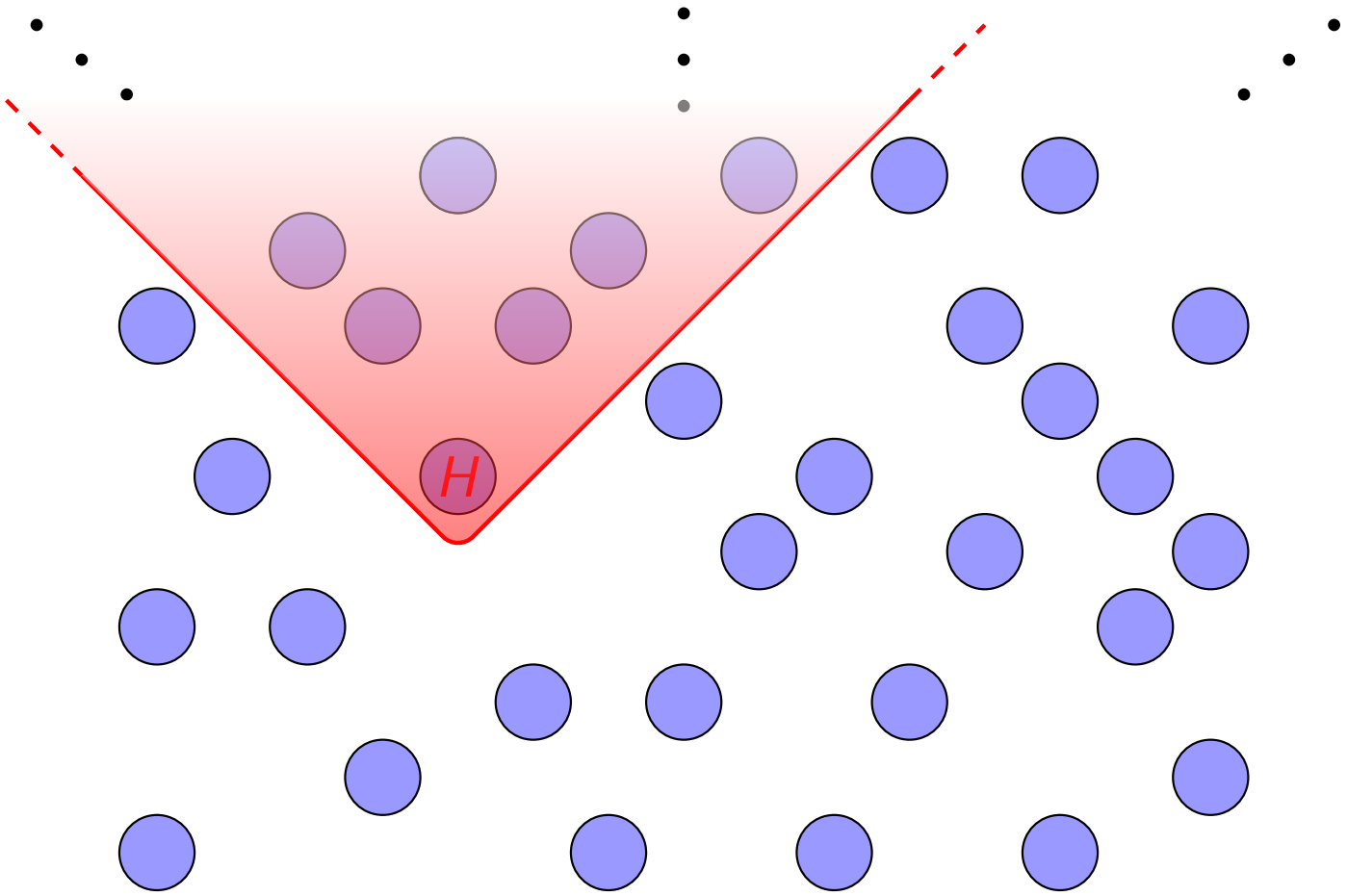

Well-quasi-ordering subclasses

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K_4-induced minor-free graphs are wqo by $\leqslant_{\mathrm{im}}$.

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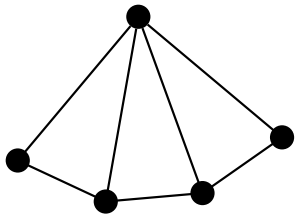
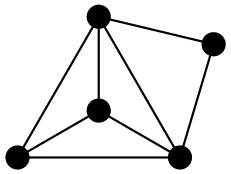
Theorem (Błasiok, Kamiński, R., Trunck '15+)

H-induced minor-free graphs are wqo by $\leqslant_{\mathrm{im}}$ iff H is induced minor of Gem or $\hat{K}_4$.

$\mathrm{Gem} =$ 

$\hat{K}_4 =$ 

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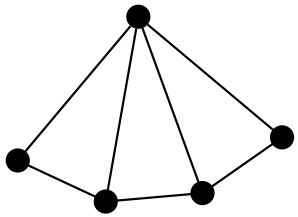
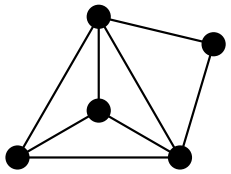
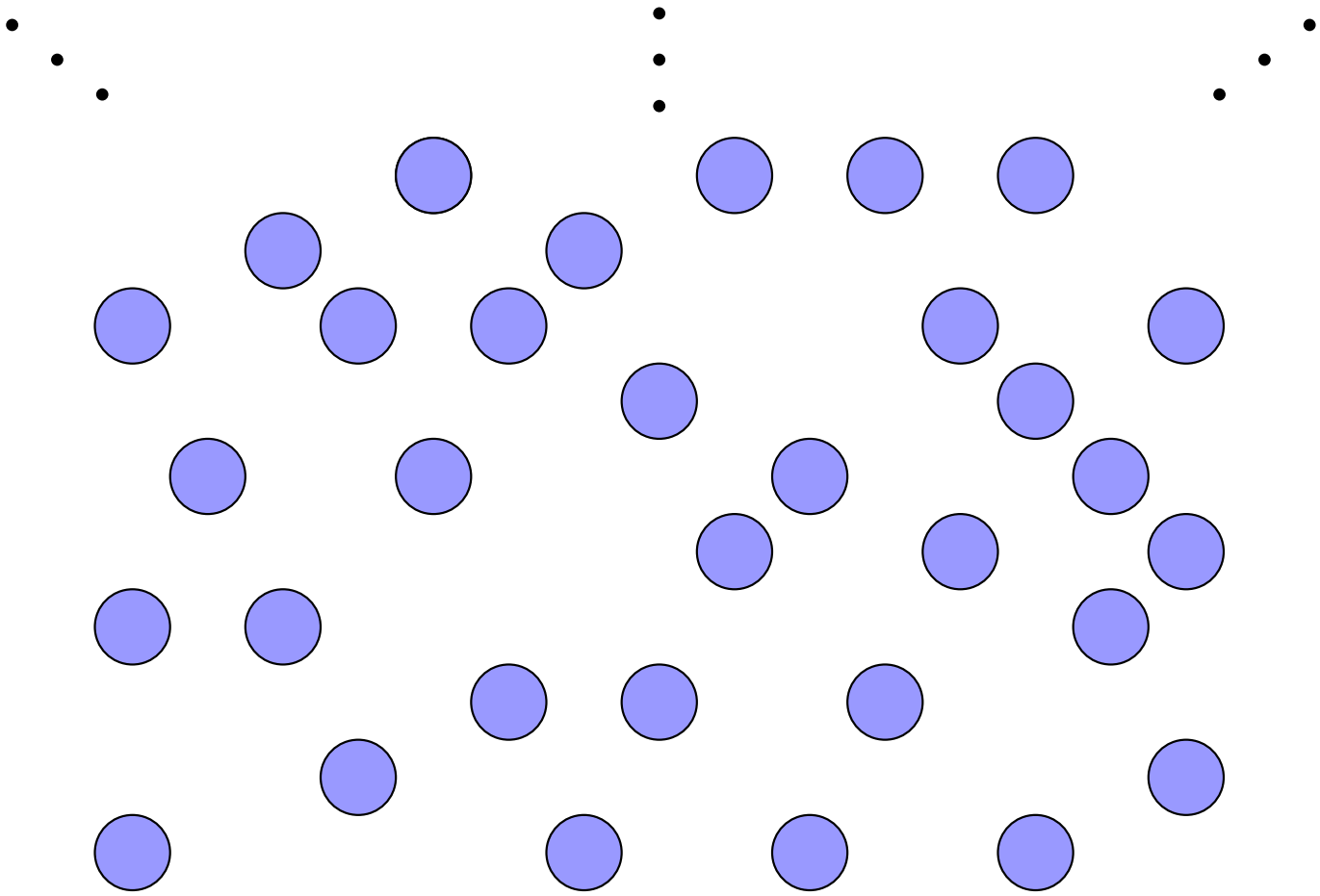
$\mathrm{Gem} =$ 

$\hat{K}_4 =$ 

Similar dichotomies exist for:

- induced subgraphs [Damaschke, '90];
- subgraphs [Ding, '92];
- topological minors [Liu, Thomas '14+?];
- multigraph contractions [Błasiok, Kamiński, R., Trunck '15+].

Assuming that $(\mathrm{Ex}(H), \leqslant_{\mathrm{im}})$ is a wqo...



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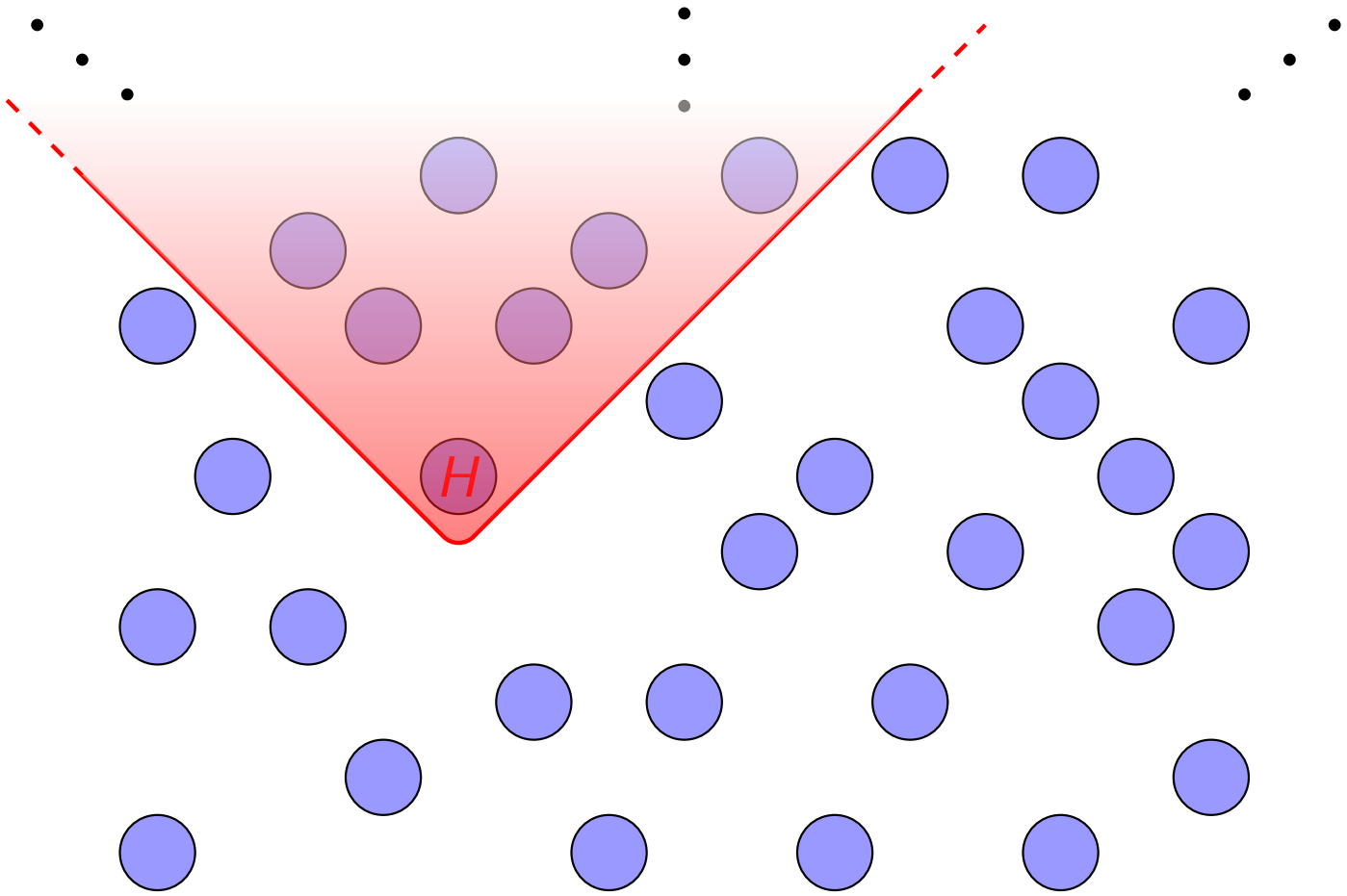

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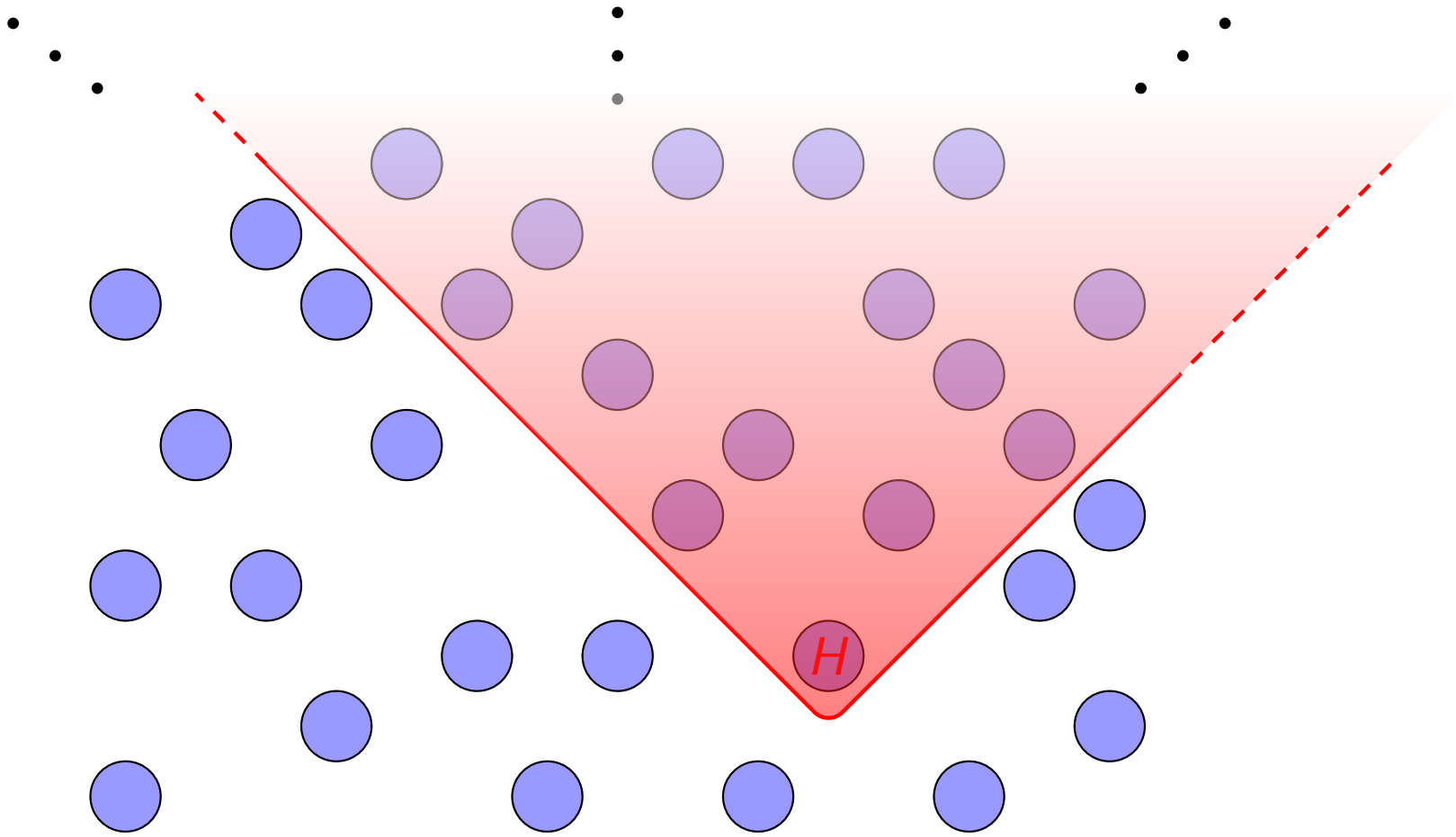

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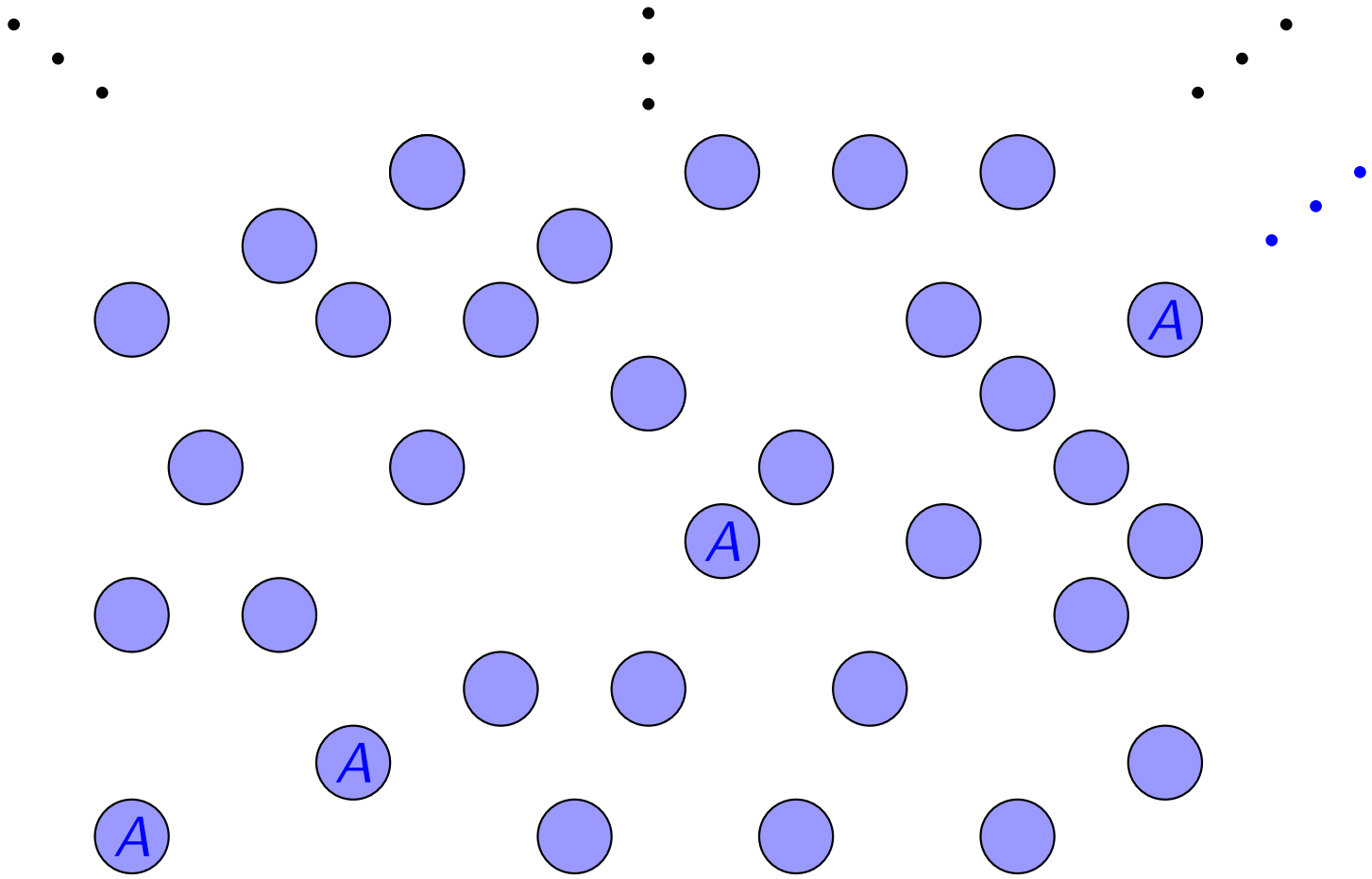

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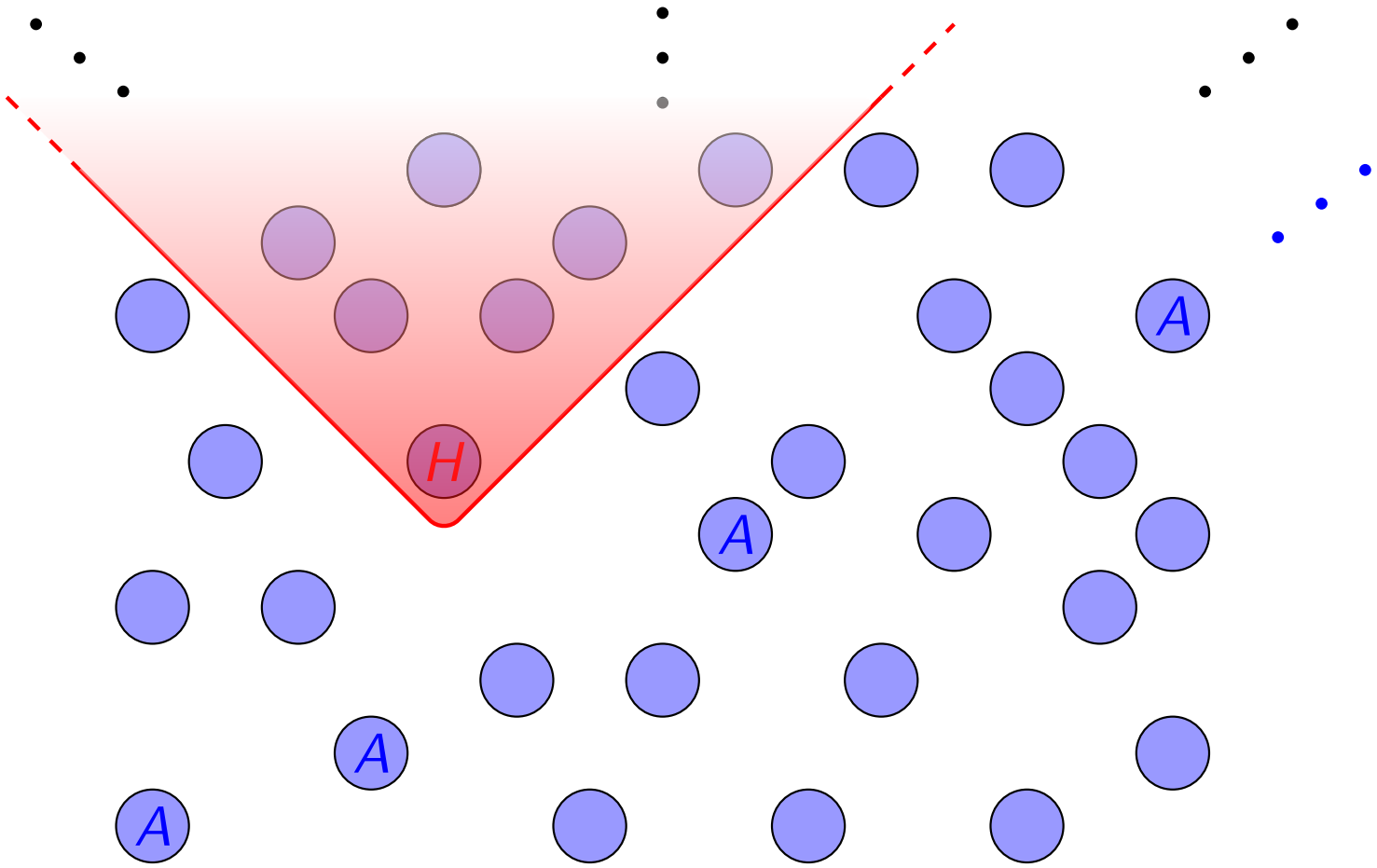

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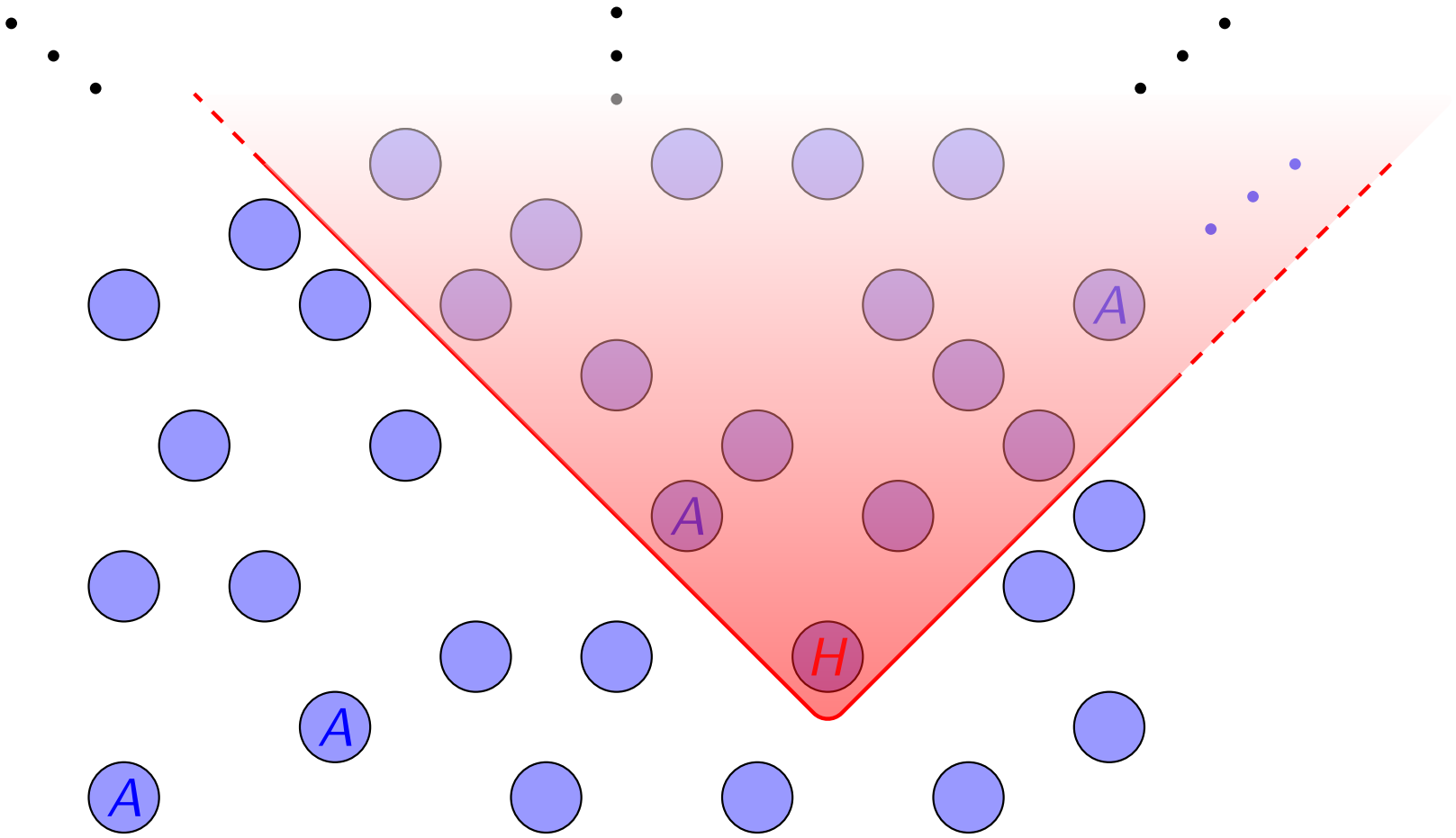

If $(\mathrm{Ex}(H), \leqslant_{\mathrm{im}})$ is a wqo, what is H like?

Remark

If $\mathcal{S}$ is an infinite antichain, H must be $\leqslant_{\mathrm{im}}$ of most of its terms.

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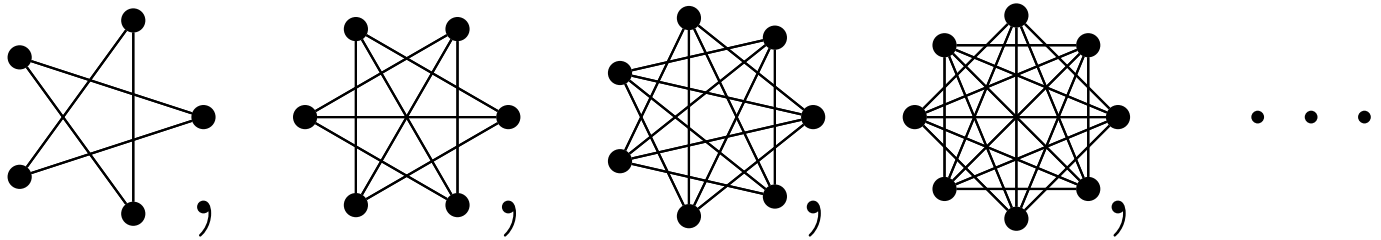

1. $\bar{H}$ is a linear forest;

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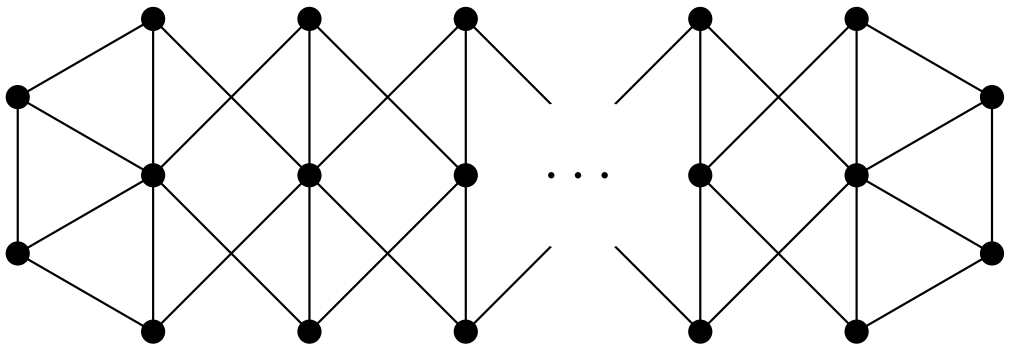

$\not\leqslant_{\mathrm{m}} K_5, K_5^-$

[Matoušek, Nešetřil, Thomas '88]

1. $\bar{H}$ is a linear forest;
2. $K_5, K_5^- \not\leqslant_{\mathrm{m}} H \quad \rightsquigarrow \quad 5 \cdot K_1 \not\leqslant_{\mathrm{m}} \bar{H}$ and $3 \cdot K_1 + K_2 \not\leqslant_{\mathrm{m}} \bar{H}$;

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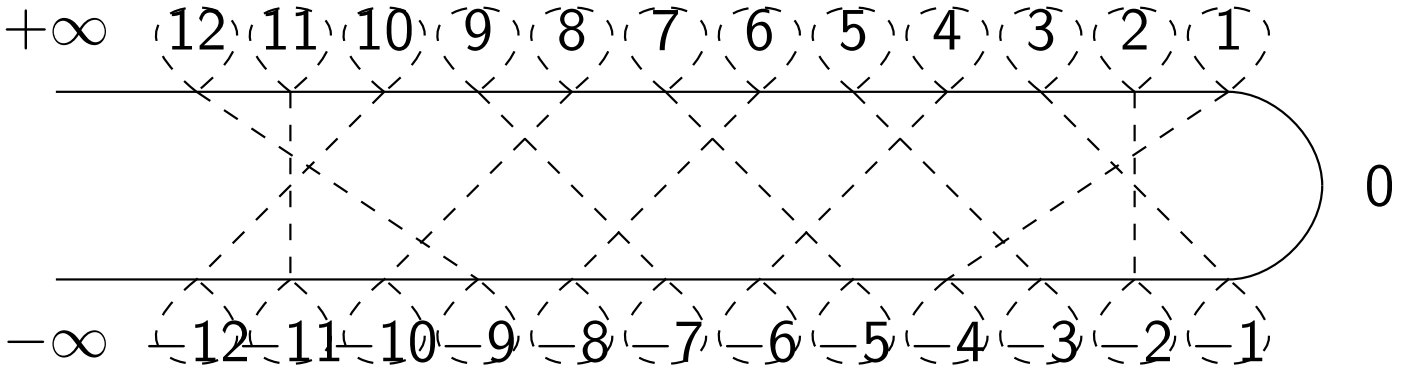

[Ding '98] (interval graph)

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3. $C_4 \not\leqslant_{\mathrm{m}} H \quad \rightsquigarrow \quad 2 \cdot P_2 \not\leqslant_{\mathrm{m}} \bar{H}$.

What can we deduce about $\bar{H}$?

Possible values for $\bar{H}$:

$\lvert V(H)\rvert \setminus cc(\overline{H})$	1	2	3	4
1	K_1	$\emptyset$	$\emptyset$	$\emptyset$
2	K_2	$2 \cdot K_1$	$\emptyset$	$\emptyset$
3	P_3	$K_2 + K_1$	$3 \cdot K_1$	$\emptyset$
4	P_4	$P_3 + K_1$	$K_2 + 2 \cdot K_1$	$4 \cdot K_1$
5	$\emptyset$	$P_4 + K_1$	$P_3 + 2 \cdot K_1$	$\emptyset$
6	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

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4	P_4	K_3^+	K_4^-	K_4
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If $(\mathrm{Ex}(H), \leqslant_{\mathrm{im}})$ is wqo, then $H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} \hat{K}_4$.

Reverse direction

If $H \leqslant_{\mathrm{im}} \mathrm{Gem}$ or $H \leqslant_{\mathrm{im}} \hat{K}_4$, is $(\mathrm{Ex}(H), \leqslant_{\mathrm{im}})$ a wqo?

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Lemma

$(\mathrm{Ex}(\mathrm{Gem}), \leqslant_{\mathrm{im}})$ *is a wqo.*

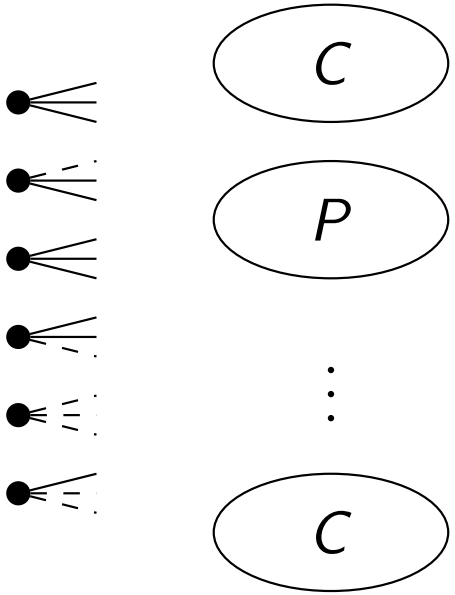
Lemma

$(\mathrm{Ex}(\hat{K}_4), \leqslant_{\mathrm{im}})$ *is a wqo.*

Decomposition of Gem-induced minor-free graphs

Theorem

If $\mathrm{Gem} \not\leq_{\mathrm{im}} G$ *and* G *is 2-c, then* G *is like the figure below.*



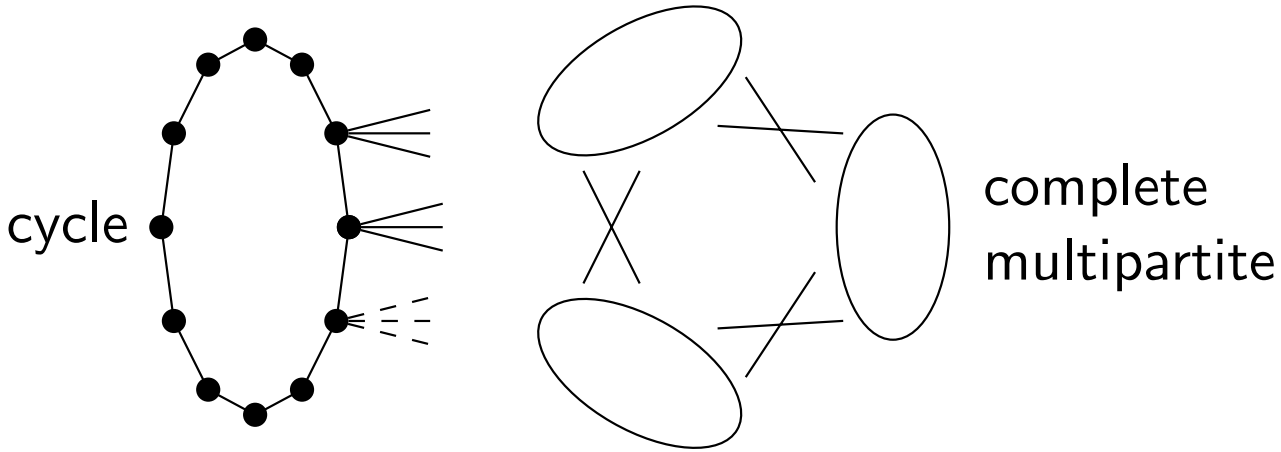
C: cograph
P: path

Decomposition of $\hat{K}_4$-induced minor-free graphs

Theorem

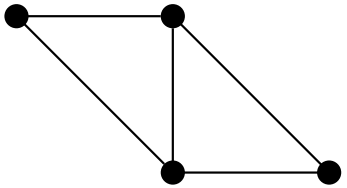
If $K_4 \not\leq_{\mathrm{im}} G$ and G is 2-c, then

- *either $K_4 \not\leq_{\mathrm{im}} G$;*
- *or G is a subdivision of a graph on at most 9 vertices;*
- *or G is like the figure below.*



From structure to wqo

Example: subdivisions of some fixed graph.



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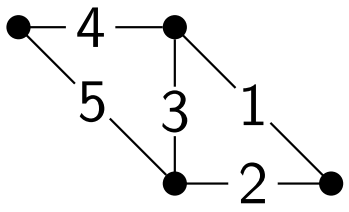
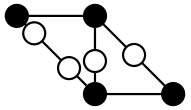


1. choose an encoding of graphs as strings (or tuples, or trees)
   e.g. # of subdivisions for each edge, in some chosen order;

$\mathrm{enc}($  $) = (1, 0, 1, 0, 2)$

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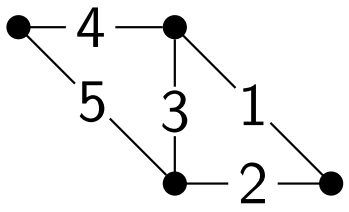

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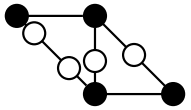
   $\mathrm{enc}($  $) = (1, 0, 1, 0, 2)$

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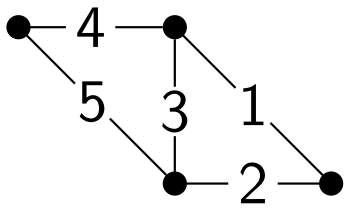

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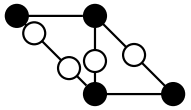
$\mathrm{enc}($  $) = (1, 0, 1, 0, 2)$

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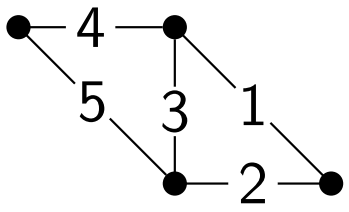

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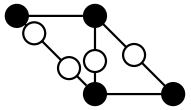
$\mathrm{enc}($  $) = (1, 0, 1, 0, 2)$

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4. that's it!

Summing up

Theorem (Błasiok, Kamiński, R., Trunck '14+)

$(\mathrm{Ex}(H), \leqslant_{\mathrm{im}})$ *is wqo iff* $H \leqslant_{\mathrm{im}} \mathrm{Gem}$ *or* $H \leqslant_{\mathrm{im}} \hat{K}_4$,

"$\Rightarrow$" infinite antichains;

"$\Leftarrow$" decompositions, encodings, labels and classic wqo results.

Theorem (Błasiok, Kamiński, R., Trunck '14+)

Decomposition theorem for graphs of:

- $\mathrm{Ex}(\hat{K}_4)$;
- $\mathrm{Ex}(\mathrm{Gem})$.

Further research

- canonical antichains;
- excluding two (or more) graphs;
- other containment relations.

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Thank you!