Multigraphs without large bonds are wqo by contraction

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January 2015, CIRM, Luminy.

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∃? infinite sequence of $\preceq$-incomparable graphs?

=: infinite antichain

If no such sequence exists, $\preceq$ is a well-quasi-order (**wqo**).

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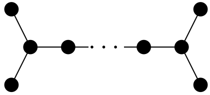

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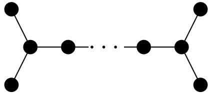

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Open problems: induced immersions and strong immersions.

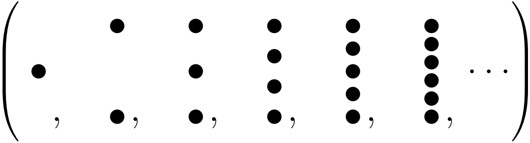
Back to contractions

Are (finite) graphs well-quasi-ordered by $\preceq$?

Back to contractions

Are (finite) graphs well-quasi-ordered by $\preceq$? No!

$\mathcal{A}_{\overline{K}} =$



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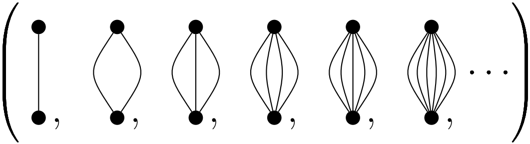

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   - [Thomas '85] $(\mathsf{Excl}_{\mathrm{im}}(K_4), \leqslant_{\mathrm{im}})$ is a **wqo**;
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    - ...
2. look at antichains (= sequences on incomparable elements):
    - [Ding '09] $\exists$ canonical antichain for $\leqslant_{\mathrm{s}}$, but not for $\leqslant_{\mathrm{is}}$;
    - [Lozin & Mayhill '10] $\exists$ canonical antichain for $\leqslant_{\mathrm{is}}$ on unit interval graphs and on bipartite permutation graphs.

Well-quasi-ordering graphs without big bonds

Theorem (Kamiński, R., Trunck)

A $\preceq$-closed class $(\mathcal{H}, \preceq)$ is **wqo** *iff it contains finitely many graphs of*

$$\mathcal{A}_{\overline{K}} = (K_1, \overline{K_2}, \overline{K_3}, \dots)$$
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$\mathcal{G}_{p,k}$: graphs with $\leqslant p$ connected components and no bond of order $\geqslant k$.

(bond: minimal edge cut)

Theorem (Kamiński, R., Trunck)

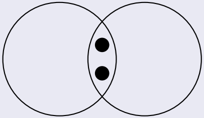
For every $p, k \in \mathbb{N}$, the class $\mathcal{G}_{p,k}$ is well-quasi-ordered by $\preceq$.

The structure of 2-connected graphs

Theorem (Tutte '61)

Every 2-connected graph has a tree decomposition

- *of adhesion 2*



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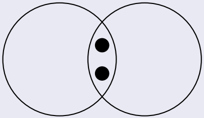
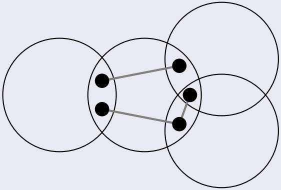


- *where every torso is either 3-connected, or a cycle*



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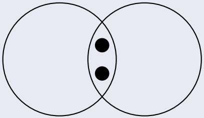

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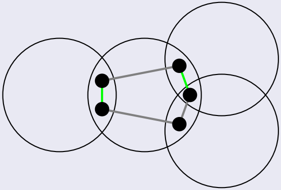

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a root bag whose torso is 3-connected/cycle

$+$

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attach

edge-rooted 2-c graphs

to the edges of an

edge-rooted 3-c graph/cycle.

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- to construct them:

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- type: "type H" if the torso H of the root is 3-c, "cycle" otherwise.

How many types?

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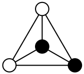
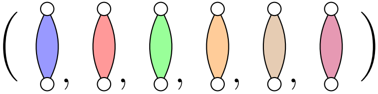
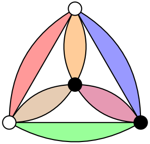
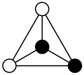
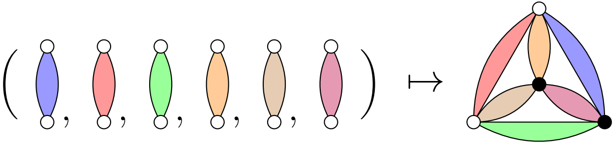
Assume this type is K_4.

Monotonicity

$H =$ 

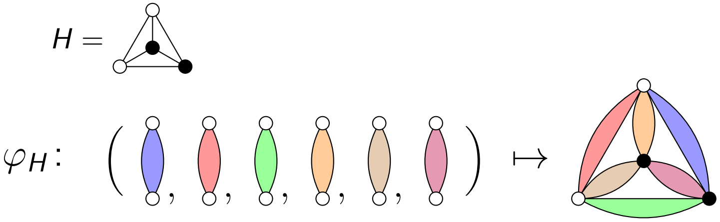
$\varphi_H :$  $\mapsto$ 

$H =$ 

$\varphi_H :$ 

Lemma

If $(\mathcal{H}, \preceq)$ *is a* **wqo**, *then so is* $(\varphi_H(\mathcal{H}^6), \preceq)$.



Lemma

If $(\mathcal{H}, \preceq)$ is a **wqo**, *then so is $(\varphi_H(\mathcal{H}^6), \preceq)$.*

Why? Every monotone function sends a **wqo** onto a **wqo**.

$$x \leqslant y \Rightarrow \psi(x) \leqslant \psi(y)$$

The proof

Goal: show that 2-c graphs without big bonds are **wqo** by $\preceq$.

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Then:

- from 2-c to connected;
- from connected to bounded number of connected components.

Looking at antichains

$\mathcal{A}$ canonical antichain of $(\mathcal{S}, \preceq)$

$\Updownarrow$ def

$\forall$ $\preceq$-closed $\mathcal{J} \subseteq \mathcal{S}$, $\mathcal{J}$ has an infinite antichain iff $\mathcal{J} \cap \mathcal{A}$ is infinite.

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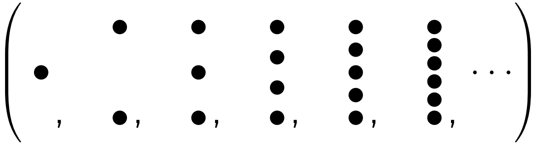

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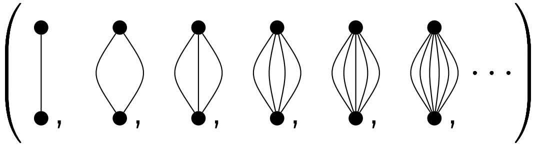

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Theorem (Kamiński, R., Trunck)

Every antichain $\mathcal{A}$ of (finite) graphs under $\preceq$ is canonical iff:

$$\begin{aligned} \mathcal{A} = &\{\textit{all but finitely many graphs from } \mathcal{A}_\theta\} \\ &\cup \{\textit{all but finitely many graphs from } \mathcal{A}_{\overline{K}}\} \\ &\cup \{\textit{a finite number of other graphs}\} \end{aligned}$$

(in particular there are canonical antichains for $\preceq$)

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2. looking at antichains:
   - characterization of canonical antichains under $\preceq$;
   - $\exists$ canonical antichains under $\preceq$.

Thank you!